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EXCESS VALUES FOR COOPERATIVE GAMES WITH TRANSFERABLE UTILITIES AND DOUBLE CONSISTENT ALLOCATION METHODS

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For cooperative games with transferable utilities (TU games) excess functions $e : \mathbb{R}^2 \to \mathbb{R}^1$ whose values $e(x(S), v(S))$, $S \subset N$ are relative negative utilities of coalitions S with respect to their payoffs $x(S) = \sum_{i \in S} x_i$ are defined. The excess values for the class of two-person games are defined as those giving to both players equal excess values. An extension of this definition to the class of all TU games is given.
For surplus sharing problems as a particular class of TU games, the excess values turned out to be parametric methods which are allocation-consistent. However, allocation consistency may not coincide with game theoretic consistency on the class of surplus sharing problems.
Necessary and sufficient conditions on the excess functions under which both definitions of consistency – for the allocation methods and for TU game solutions – coincide on the class of surplus sharing problems are given.

Key words: Allocation problem, Allocation method, Surplus sharing problem, TU game solution, Excess function, Excess value, Consistency.

JEL classification: C71, D63.

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1 Introduction

An *allocation problem* is a problem of distributing a single homogeneous divisible good among a variable set of agents under the condition that each agent has a positive claim. If the sum of claims of all agents is greater than the total amount of good then an allocation problem is a *cost sharing* or *rationing* problem. In the opposite case, it is a *surplus* or *profit sharing* problem. A rule of the distribution is called an *allocation method (rule).*

Allocation methods are characterized by some desirable properties which they should satisfy. Young (1987) proved that every symmetric, continuous, and consistent allocation method has a parametric representation and optimizes an additively separable objective function.

Parametric methods for cost allocation problems are generated by numerical *standards* (Young (1994)) which are real-valued functions associating with every claim and amount of good a negative utility of the amount for each agent. The standards have a 'hydraulic' interpretation invented by M.Kaminski (2000).

"Every agent gets one vessel from a system of connected vessels that are linked to a central reservoir through a system of pipes. The amount of water in a central reservoir is equal to the amount of good to be rationed. The allocation is obtained by opening the main sluice-gate and letting the water flow down and fill the vessels. Since the vessels are connected, the water levels in all vessels will be equal to one another." (Kaminski (2000), p.132). This water level is equal to the corresponding standard of every agent. Observe that in this interpretation the vessels may have parts of zero volume (infinitely tight pipes). This cannot happen if the method is strictly monotonic in the total amount of good. In this case, a parametric method equalizes the standards of all agents. Consistency of such a method means that the standards are unchanged after some agents leave with their shares of good.

The cost and surplus allocation problems may be considered as special classes of positive TU games. The characteristic function values of the games are defined either by summation of claims belonging to the agents of the corresponding coalitions (for surplus sharing problems) or by the positive parts of the differences between the total amount to be allocated and the sums of claims belonging to the agents of the complementary coalition (for cost sharing problems).

Solutions for TU games corresponding to allocation problems may be considered as allocation methods as well (see e.g. Moulin (1985)). The solutions are also characterized by their properties, among which, as for the allocation methods, the most fundamental are symmetry or/and anonymity, and consistency.

Recall that consistency of TU game solutions depends on the definitions of the reduced games. Some solutions are consistent in one definition of the reduced games and are not consistent in others. Moreover, some definitions of the reduced games (e.g. the one used in the definition of *linear consistency* due to Ruiz et al.(1998), when applied to the class of allocation problems, do not coincide with the definition of allocation consistency.

However, even if both definitions coincide (e.g. for Davis–Maschler (1965) definition of reduced games), some consistent allocation methods considered as solutions to the special classes of TU games mentioned above may not be extendable to the whole class of TU games with the preservation of the corresponding TU game consistency. Therefore, the question arises: given a consistent allocation method, does the definition of the reduced game exist such that the method could be extended to a consistent (w.r.t. this definition) solution to the whole class of TU games or to the class of the positive ones?

In this, paper the answer to this question has been obtained for surplus sharing problems.

Since there is a one-to-one correspondence between two-person allocation problems and two-person positive TU games, allocation methods for these problems coincide with TU game single-valued solutions (values) for two-person games. Therefore, parametric methods generate the solutions of two-person games equalizing the corresponding numerical standards, which are, in terms of TU game theory, nothing but excess functions measuring negative utilities of players and coalitions by their payoffs. Excess functions generate a class of excess TU game solutions. Every excess solution is invariant under transformations of games and of the corresponding solution vectors not changing the excess values. In particular, the well-known excess function equal to the difference between the values of the characteristic function and of the corresponding payoff of a coalition generates excess values which are translation covariant.

The present paper investigates the relationship between parametric methods for surplus sharing problems and consistent excess values for positive TU games.

On the first stage, a class of the reduced games defining consistency of TU game solutions is characterized axiomatically. The key tool providing the characterization of the reduced games is Gorman's overlapping theorem (Gorman(1968)) about separability of social welfare functions.

All consistent (up to some technical conditions) excess values in this definition of consistency are found. Every consistent excess value minimizes an additively separable objective function. The Lagrange technique leads to the condition of equalizing the sums in coalitions containing any fixed player of some functions depending on the corresponding values of the excess function.

Then, on the second stage, among these solutions, we find those which are extensions of strictly monotonic parametric allocation methods, i.e. coincide with them on the class of TU games corresponding to surplus sharing problems.

On the class of surplus sharing problems they coincide with the one-parametric family of consistent and decentralizable methods due to Moulin (1987).

The paper is organized as follows. In section 2 we give the definition of solutions and some of their properties both for allocation problems and for cooperative games. An interrelation between them is briefly discussed. In section 3 the definition of excess solutions for TU games is given and a correspondence between these solutions and parametric allocation methods is established. Section 4 is devoted to the definition of consistency for TU game excess solutions. The general form of consistent excess values is found.

The main result of section 5 shows that some mixtures of the egalitarian and of proportional allocation methods (Moulin (1987)) for surplus sharing problems are unique, satisfying symmetry, continuity, and consistency in both allocation and game-theoretic senses. Concluding remarks describe the open problems connected with the conjecture that strict monotonicity of allocation methods and of characteristic functions of the reduced games, used in the paper, may not be necessary.

2 Definitions and comparisons of allocation methods and TU game solutions

Let N be any finite set. An *allocation problem* or a problem *with claims,* is a pair $\langle x, T\rangle$, where $x \in \mathbb{R}^N_{++}$ is the *claim*, $T > 0$ the *total* to be allocated. If $T \leq \sum_{i\in N} x_i$, then the allocation problem is a *cost sharing* problem, and if $\sum_{i\in N} x_i \leq T$, then the allocation problem is called a *surplus* or *profit-sharing* problem.

Definition 1 A *solution* of $\langle x, T\rangle$ is a vector $t \in R^N_+$ such that
1)$\sum_i t_i = T$, and
2) $0 \leq t_i \leq x_i$. for $T \leq \sum_{i\in N} x_i$, and $x_i \leq t_i \,\forall i$ for $T \geq \sum_{i\in N} x_i$. .
An *allocation method* is a function F that assigns to every allocation problem a unique solution vector $t = F(x; T)$.

The condition 2) is called the *core property.* If it does not hold, then the solution (allocation method) is called *unconstrained.*

It is easy to note that the allocation problems can be considered as a particular class of TU games. In fact a *TU cooperative game* is a pair $\langle N, v\rangle$, where N is a finite set of players, $v : 2^N \to \mathbb{R}^1$ is a characteristic function with a convention $v(\emptyset) = 0$. The values $v(S)$, $S \subset N$ represent powers of coalitions. For cost allocation problems the characteristic function is defined as follows:
$v(S) = (T - \sum\limits_{j\in N\setminus S} d_j)_+ = \max\{0, T - \sum\limits_{j\in N\setminus S} d_j)\}$, $S \subset N$. In fact, the coalition S can guarantee itself the amount remained after satisfaction all other players by their claims if the amount is positive.

For surplus sharing problems the claims may be considered as the amounts which the agents guarantee themselves separately and all coalitions different from the grand one are flat: $v(S) = \sum_{j\in S} d_j$, and $v(N) = T$ as for cost allocation problems.

Surplus sharing problems can be also considered as TU bargaining problems whose disagreement points coincide with claims and the bargaining sets are half spaces $\{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i \leq T\}$.

Denote by $\mathcal{G}_c$ and $\mathcal{G}_s$ the classes of TU games coresponding to cost and surplus sharing problems respectively. Note that the both classes are subclasses of *positive* TU games, i.e. games with positive range of the characteristic functions.

Now we are going to compare the definition of cooperative game solutions with that of solutions to allocation problems. For this purpose recall the definition of TU game solutions:

Let $\mathcal{N}$ be a universal set of players, $\mathcal{G}_{\mathcal{N}}$, $(\mathcal{G}_+)$ be the classes of all (positive) TU games whose player sets are contained in $\mathcal{N}$, $\mathcal{G}_{\mathcal{N}} \subset \mathcal{G}$ be the class of all TU games with the set of players N:

$$\langle N, v\rangle \in \mathcal{G}_{\mathcal{N}} \iff N \subset \mathcal{N},\ v : 2^N \to \mathbb{R}, v(\emptyset) = 0.$$

Let $\Gamma = \langle N, v\rangle \in \mathcal{G}_{\mathcal{N}}$ be an arbitrary game,

$$X(\Gamma) = \{y \in \mathbb{R}^N \mid \sum_{i\in N} y_i \leq v(N)\},$$

be the set of feasible payoff vectors of the game Γ, and

$$Y(\Gamma) = \{y \in \mathbb{R}^n \mid \sum_{i\in N} y_i = v(N)\}$$

be the set of efficient (Pareto optimal) payoff vectors.

For any $x \in \mathbb{R}^N$, $S \subset N$ denote by x_S the projection of x on the space $\mathbb{R}^S$, and $x(S) := \sum_{i\in S} x_i$, with a convention $x(\emptyset) = 0$. Throughout the paper we shall write $v(i)$ instead of $v(\{i\})$ and $v(S \cup i)$ instead of $v(S \cup \{i\})$.

Definition 2 *A solution* for a class $\mathcal{G}' \subset \mathcal{G}_{\mathcal{N}}$ is a mapping σ, assigning to each game $\Gamma \in \mathcal{G}'$ a subset $\sigma(\Gamma) \subset X(\Gamma)$ of its payoff vectors. A solution is *efficient* if $\sigma(\Gamma) \subset Y(\Gamma)$.

If for any game $\Gamma \in \mathcal{G}'$ the set $\sigma(\Gamma)$ consists of a single payoff vector, then the solution is called *single-valued* or a *value.*

Evidently, Definition 2 of efficient single-valued solutions agree with Definition 1 on the classes $\mathcal{G}_c$ and $\mathcal{G}_s$ except, possibly, for the condition 2) in Definition 1. This condition applied to the classes $\mathcal{G}_c$ and $\mathcal{G}_s$ would mean that the solution set for each game should be contained in its core that is in general not necessary for TU game solutions.

We shall call allocation methods whose ranges do not satisfy the second condition in Definition 1 *unconstrained allocation methods.*

Moreover, we restrict ourselves by symmetric, anonymous, and continuous allocation methods and cooperative game values respectively.

Definition 3 An allocation method F is *symmetric*, if $t = F(x;T)$ and $x_i = x_j$ imply $t_i = t_j$.

This definition may be also called by "equal treatments of equals": equal claims of agents lead to equal gains (losses).

For TU game values we need a stronger property:

Definition 4 A value Φ for the class $\mathcal{G}$ is *anonymous,* if $\Phi_{\pi(i)}(N, \pi v) = \Phi_i(N, v)$ for all games $\langle N, v\rangle$, all $i \in N$ and every permutation π of N. Here the game $\langle N, \pi v\rangle$ is defined by $(\pi v)(\pi S) := v(S)$ for all $S \subseteq N$;

Definition 5 A value Φ for the class $\mathcal{G}_{\mathcal{N}}$ is *symmetric* or satisfies the *equal treatment property*, if the equalities $v(S\cup\{i\}) = v(S\cup\{j\})$ for all $S \not\ni i, j$ and some $i, j \in N$ imply $\Phi_i(N, v) = \Phi_j(N, v)$.

It is obvious that if a value Φ is anonymous then it is symmetric, but not vice versa, and that Definitions 3 and 5 coincide on the classes $\mathcal{G}_c, \mathcal{G}_s$.

Consider now the consistency property both for allocation methods and for TU game solutions.

Definition 6 A solution F for allocation problems is *consistent* if for any $S \subset N$

$$t = F(x, T) \implies t_S = F(x_S, \sum_{i\in S} t_i). \tag{1}$$

Pairwise consistency requires (1) only for S, $|S| = 2$.

This property says that a solution vector remains to be a fair allocation when restricted to any subgroup of the agents. The same property is applied to the characterization of TU game solutions:

Definition 7 A solution σ is *consistent* in a class $\mathcal{G}' \subset \mathcal{G}_{\mathcal{N}}$ of games, if for any game $\langle N, v\rangle \in \mathcal{G}'$ from the class the *reduced game*, $\langle N \setminus R, v^x_{N\setminus R}\rangle$ being obtained when a coalition R leaves the game with the payoffs x_R prescribed them by the solution, also belongs to the class $\mathcal{G}'$ and $x = (x_{N\setminus R}, x_R) \in \sigma(N, v)$, implies $x_{N\setminus R} \in \sigma(N \setminus T, v^x_{N\setminus R})$.

Consistency of a solution, as in the allocation problems case, describes the property that for every solution vector, any of its parts remains the solution of a reduced game when the complementary players leave the game with payoffs prescribed them by the solution vector. However, Definition 7 does not answer what is the reduced game itself. In contrast with Definition 6, the characteristic function of the reduced game is not defined uniquely by the definition of the initial game and the payoff vector. Therefore, in order to give the formal definition of consistency first it is necessary to define reduced games themselves. Several definitions of the reduced games and the same number of the definitions of solution consistency are known. There is the detailed survey of almost all of them in Thomson (1996). We recall here only the most popular definitions of the reduced games.

The first one is due to Davis and Maschler (1965), who defined it for every payoff vector x as follows:

$$v^x_{N\setminus R}(S) = \begin{cases} v(N) - x_(R), & \text{if } S = N \setminus R, \\ \max_{Q\subset R}(v(S\cup Q) - x(Q)) & \text{otherwise.} \end{cases} \qquad (2)$$

The linearly reduced games (Ruiz et al. (1998)) are defined for $R = \{i\}$ by

$$v^x_{N\setminus i}(S) = \begin{cases} v(N) - x_i, & \text{if } S = N \setminus \{i\}, \\ w_{n,s}v(S) + (1 - w_{n,s})(v(S\cup i) - x_i) & \text{otherwise,} \end{cases} \qquad (3)$$

where $n = |N|, s = |S|, \quad w_{n,s} \in [0,1]-$ weighting coefficients. For arbitrary sets of players leaving the game the characteristic function of the linearly reduced game is obtained by the repeated application of (3).

The subgame is a particular case of (3) when $w_{n,s} = 1$ for all $n, s = 1,\dots,n-2$, and the complement reduced games — when $w_{n,s} = 0$.

Note that in all definitions of the reduced games the reduced value of the grand coalition $N \setminus R$ is equal to $v(N) - x(R)$. If $x \in \sigma(N,v)$ is a solution vector, then this definition preserves its efficiency in any reduced game. Therefore, in the sequel we also follow this definition and will define only the values of reduced characteristic functions for coalitions different from the grand ones.

It is worthwhile to compare Definition 6 with those (2) and (3) for the classes $\mathcal{G}_c$ and $\mathcal{G}_s$ with the help of the following diagram:

$$\begin{array}{ccc} \langle N, T; \{d_i\}_{i\in N}\rangle & \longrightarrow & \langle N, v\rangle \\ \downarrow & & \downarrow \\ \langle N\setminus R, T - \sum_{i\in R} x_i; \{d_j\}_{j\in N\setminus R}\rangle & \longrightarrow & \langle N\setminus R, v^x\rangle. \end{array} \qquad (4)$$

Here $\langle N, T; \{d_i\}_{i\in N}\rangle$ is an allocation problem, $\langle N, v\rangle$ the corresponding to this problem TU game from the class $\mathcal{G}_c$ or $\mathcal{G}_s$, $x \in \mathbb{R}^N_+$ is an arbitrary vector satisfying Definition 1. In the second line of the diagram the corresponding reduced allocation problem and game are placed. The question is: for what definitions of the reduced games and payoff vectors the diagram is commutative? It is easy to check that the answer is positive for the Davis–Maschler (2) definition of the reduced games and every efficient vector x. [1]

However, e.g. for the linear reduced games defined in (3) the diagram is commutative only for the trivial case $T = \sum_{i\in N} d_i$.

The definitions of the reduced game providing TU game consistency of the extended parametric methods and commutativity of the diagram are found in the last section of the paper.

3 Parametric allocation methods and TU games solutions

In this section we recall definitions and properties of the parametric allocation methods and define their TU game analogues.

3.1 Parametric allocation methods

Definition 8 An allocation method F (a value Φ) is *continuous*, if it is jointly continuous in all arguments (the function $\Phi(N, v)$ is jointly continuous in all variables $v(S), S \subseteq N$.)

Young (1987) has found the general form of symmetric, continuous, and pairwise consistent allocation methods. They are *parametric*:

Definition 9 An allocation method F is *parametric*, with *representation* $f : \mathbb{R}^1_{++} \times [a, b] \to \mathbb{R}^1$, $f(d, a) = 0, f(d, b) = x$, if

$$t = F(d, T) \iff \exists \lambda \in [a, b] \text{ s.t. } \quad \forall i \;\; t_i = f(d_i, \lambda)$$

and $\sum_i t_i = T$.

Theorem 1 (Young (1987)) . *A continuous allocation method F is symmetric and pairwise consistent if and only if it is representable by a continuous parametric function.*

[1] The diagram is also commutative for the reduced games due to Hart —Mas-Colell (1989) for a solution vector $x = \Phi(N, v)$.

In the proof of the Theorem 1 it is shown that the continuous parametric methods are *monotone*:

Definition 10 An allocation method F is *(strictly) monotonic*, if

$$T > T' \implies F(d;T) \geq F(d;T')$$
$$(T > T' \implies F(d;T) > F(d;T'))\,.$$

It is clear that if a parametric method F is strictly monotonic, then it is representable by a function $f(d,\lambda)$ which is strictly monotonic in the second variable and, hence, there exists the inverse function $f^{-1}(d,t) := e(d,t)$. In this notation x is the parametric solution to the allocation problem $\langle N, T; \{d_j\}_{j\in N}\rangle$ if x is a solution vector satisfying the equalities

$$e(d_i, x_i) = \lambda \quad \text{for all } i \in N. \tag{5}$$

In the same paper Young has shown that the solution of a parametric method minimize an additively separable objective function. Since in the rest of the paper we consider only strictly monotonic parametric methods, we cite the result only for such parametric methods:

Theorem 2 (Young (1987)) . *A symmetric, continuous, and strictly monotonic allocation method F is pairwise consistent if and only if its solutions satisfy the following equation:*

$$F(N,T:\{d_j\}_{j\in N}) = \arg \min_{\substack{x\in R^N_+ \\ 0\leq d_i \leq x_i}} \sum_{i\in N} \int_0^x f^{-1}(d,y)\,dy. \tag{6}$$

For surplus sharing problems one more condition on x is: $d_i \leq x_i$; and for cost allocation problems the condition is: $d_i \geq x_i$.

It is easy to check that the Lagrange conditions for the problem (6) lead to the equalities (5).

3.2 Excess solutions for TU games

There are solutions for TU games which also could be called "parametric" because they coincide with the parametric methods on the class of two-person games. For example,

Definition 11 A solution Φ for a class $\mathcal{G}' \subset \mathcal{G}_N$ is *translation covariant*, if $\Phi(N, v+a) = \Phi(N,v) + a$ for all $a \in \mathbb{R}^n$ and all games $\langle N, v\rangle \in \mathcal{G}'$. Here the game $\langle N, v+a\rangle \in \mathcal{G}'$ is defined by $(v+a)(S) := v(S) + \sum_{j\in S} a_j$ for all $S \subseteq N$;

Observe that translation covariant solutions possess the following property: let Φ be any translation covariant solution, $\langle N, v\rangle$, $\langle N, w\rangle$ be two arbitrary games. Then the equalities

$$v(S) - x(S) = w(S) - y(S) \text{ for all } S \subset N$$

imply

$$x \in \Phi(N, v) \iff y \in \Phi(N, w). \tag{7}$$

Recall that the difference $v(S) - x(S)$ is the excess value of a coalition S w.r.t. x. The relation (7) means that if for two TU games with the same player's sets for some payoff vectors x, y the excess vectors coincide, then both x, y belong to a translation covariant solution or do not belong simultaneously.

For any surplus allocation problem $\langle N, T; d\rangle$ considered as a game from the class $\mathcal{G}_s$ such a solution is defined by the equalities:

$$x = \Phi(N, T; d) \iff \Phi_i(N, T; d) - x_i = \Phi_j(N, T; d) - x_j, \ \forall \ i, j \in N, \ \sum_i \Phi_i(x, T) = T. \tag{8}$$

The equalities (8) imply that Φ is the *egalitarian* method (Moulin (2001)). This method is parametric with representation $f(d, \lambda) = d - \lambda = e^{-1}(d, \lambda)$.

For cost allocation problems the allocation method, corresponding to any translation covariant single-valued solution for the class $\mathcal{G}_c$, is the *uniform loss* method (Moulin (2001)) which maximizes the lexmin ordering applied to the vectors of losses $(x_i - d_i), i \in N$.

In the rest part of the paper we consider only the correspondence between allocation methods for surplus sharing problems and TU game solutions.

Another example is the *proportional* solutions defined for positive TU games with help of the *proportional* excess $e(v, x) = v/x$ as follows:

Definition 12 A solution Φ is *proportional* for the class of positive TU games $\mathcal{G}_+$ if for any $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}_+$ the equalities

$$\frac{v(S)}{x(S)} = \frac{w(S)}{y(S)} \text{ for all } S \subset N$$

imply (7).

The parametric representation of proportional solutions on the class of allocation problems is obtained with representation $f(d, \lambda) = d/\lambda$.

Definitions 11 and 12 may be extended to an arbitrary *excess function* $e : \mathbb{R}^2 \to \mathbb{R}^1$ associating with each TU game $\langle N, v\rangle$ every its payoff vector x and a coalition $S \subset N$ a negative utility $e(v(S), x(S))$ of the vector x for the coalition S.

Definition 13 A solution Φ for a class $\mathcal{G}' \subset \mathcal{G}_{\mathcal{N}}$ is called an *excess* solution, if there exists an excess function $e : \mathbb{R}^2 \to \mathbb{R}^1$ such that for any two games $\langle N, v \rangle, \langle N, w \rangle \in \mathcal{G}' \subset \mathcal{G}_{\mathcal{N}}$ the equalities

$$e(v(S), x(S)) = e(w(S), y(S)) \text{ for all } S \subset N \tag{9}$$

imply (7).

It is clear that each continuous excess function which is strictly monotonic in the second variable defines the parametric, possibly unconstrained, solution Φ_e for allocation problems with representation

$$f(x, \lambda) = e^{-1}(x, \lambda) \tag{10}$$

where the inverse function e^{-1} is taken w.r.t. the second variable.

In the sequel we consider only continuous and strictly monotonic in the second variable excess functions.

In contrast with parametric allocations methods, one excess function can define many excess TU game solutions. In fact, the equalities (9) imply that the excess function e defining an excess solution is determined up to any monotonous transformation $g : 2^N \times \mathbb{R} \to \mathbb{R}$ such that the equalities

$$g(S, e(v(S), x(S))) = g(S, e(w(S), y(S))) \text{ for all } S \subset N$$

are equivalent to those (9).

For anonymous excess solutions the function g may depend only on the size of coalitions. An example of such transformations is the *per capita* excess function, defined by the classical excess $e(x, y) = x - y$ and the function $g(s, t) = \dfrac{t}{s}$: such that $g(s, v(S) - x(S)) = \dfrac{v(S) - x(S)}{s}$. Nevertheless, for technical simplicity we shall keep the definition of the excess functions as functions of two variables. It is of no difficulty to add their dependence on coalitions' sizes in what follows.

The excess solutions admit a natural definition of their consistency. It will be the subject of the next section.

4 Consistency for excess TU game solutions

The aim of this section is to describe all anonymous, continuous, and consistent TU game excess values under some conditions on the reduced games. The main problem is to define TU game consistency itself, because, as it had been already mentioned in section 2, it may be done by different definitions of reduced games.

4.1 Some properties of the reduced games

Since we deal only with the excess solutions, among which we are looking for consistent ones, it is natural to define the reduced values of the corresponding excess values. It turns out to be simpler than directly defining the characteristic function values for the reduced games. Such an approach for proportional excess functions had been applied by the author in Yanovskaya (2002).

Let an excess function e be given. Consider an arbitrary TU game $\langle N, v\rangle \in \mathcal{G}_{\mathcal{N}}$. Let $T \subset N$ be any coalition of players leaving the game, x be a payoff vector for $\langle N, v\rangle$.

We would like to define a reduced game $\langle N \setminus T, v_T^x\rangle$ on the player set $N \setminus T$. It is possible to define the characteristic function v_T^x by different ways. Therefore, we may impose some assumptions for this function. We shall formulate them as axioms which the reduced game should satisfy. Some axioms describe conditions on the characteristic function v_T^x, the other ones deal with the values of the excess function $e(v_T^x(S), x(S)), S \subset N \setminus T$.

1. *Independence of inessential players.* The values of excess function for the reduced game $e(v_T^x(S), x(S))$ depend only on the values $e(v(S), x(S))$ and $e(v(S \cup Q), x(S \cup Q))$, $Q \subset T$.

Therefore, we can denote the excess $e(v_T^x(S), x(S))$ as a function

$$e(v_{N\setminus T}^x(S), x(S)) = \varphi_{N,S,T}(\{e(v(S \cup Q)), x(S \cup Q))\}_{Q\subset T}). \tag{11}$$

In the statement of other axioms we suppose the property 1 to be fulfilled.

2. *Path Independence.* For any coalitions $T_1, T_2, \subset N$ such that $T_1 \cap T_2 = \emptyset$, $S \subset N \setminus (T_1 \cup T_2)$ and a payoff vector x,

$$v_{T_1T_2}^x(S) = v_{T_2T_1}^x(S),$$

where $v_{T_1T_2}^x(S)$ is defined from the equality

$$e(v_{T_1T_2}^x(S), x(S)) = \varphi_{N,S,T_1}(\{e(v_{T_2}^x(S \cup Q), x(S \cup Q))\}_{Q\subset T_1},).$$

Note that the properties 1 and 2 imply that it suffices only to consider the functions $\varphi_{N,S,i}(e(v(S), x(S)), e(v(S \cup i), x(S \cup i))$ for one-person coalitions $T = \{i\}$. In fact, the path independence property permits to define all functions $\varphi_{N,S,T}$ by the known functions $\varphi_{N,S',i}$, $S' \supset S, i \in N \setminus S'$.

The property 2 is also supposed to be fulfilled in the sequel. For simplicity of notation we omit the index N in $\varphi_{N,S,i}$ when it does not lead to a confusion. The next four properties are rather technical:

3. *Monotonicity.* The functions $\varphi_{N,S,i}$ are monotone increasing in each variable.

4. *Normalization condition.*

   $\varphi_{N,S,i}(a,a) = a$ for any $a \in \mathbb{R}^1$.

5. *Continuity* The functions $\varphi_{N,S,i}$ are continuous in both variables.

6. *Anonymity.*

$$\varphi_{N,S,i} = \varphi_{n,s} \tag{12}$$

   for some function $\varphi_{n,s} : \mathbb{R}^2 \to \mathbb{R}^1$, where $n = |N|$, $s = |S|$.

Observe that anonymity implies that for any coalition $T \subsetneqq N \setminus S$

$$\varphi_{N,S,T} = \varphi_{n,s,t} \tag{13}$$

for some functions $\varphi_{n,s,t} : \mathbb{R}^{t+1} \to \mathbb{R}^1$, where $t = |T|$.

We give a short discussion of the properties 1–6, which further will be considered as axioms defining the characteristic functions of the reduced games.

Axiom 1 states that in the reduced games the value of the excess for a coalition S w.r.t. to a payoff vector x in the reduced game depends only on the the excess values of the initial game for the coalition S and all coalitions consisting of S and the leaving players and w.r.t. the same payoff vector.

All known definitions of the reduced games satisfy this property for the excesses $e(x,y) = x - y$ and $e(x,y) = \dfrac{x}{y}$ for positive games.

The second independence axiom, axiom 2, states independence of the way of reducing: it doesn't matter in what order we reduce a game and whether the reducing on a player subset is made at once or by a consecutive elimination of players. This axiom is naturally fulfilled for the Davis–Maschler definition of the reduced games and for complement consistency due to Moulin (1985). As for linear dependence of reduced games on the initial ones ("linear consistency"), this property was used by Yanovskaya and Driessen (2001) for the definition of weighting coefficients.

Axiom 3 is a property of strict monotonicity of excesses of the reduced games in excesses of the initial one. The definition of Davis–Maschler of the reduced games (2) does not meet this property: the excesses of the reduced games are only non-decreasing in both variables. Moreover, this axiom also excludes from consideration the subgames as reduced games, and also the complement reduced games (Moulin (1985)). But on the other hand, strict monotonicity permits separability theorems be applied (see Gorman (1968)) for determining the general form of reduced games.

Axiom 4 states that the equality of both excess values $e(v(S), x(S))$ and $e(v(S \cup i), x(S \cup i)$, $i \notin S$ implies that their common value coincides with the corresponding excess value for the coalition S in the reduced game after leaving the player i.

It is clear that both axioms 2 and 3 imply that all excess values $e(v^x(S), x(S))$ in the reduced game are between those $e(v(S), x(S))$, $e(v(S \cup i), x(S \cup i))$ in the initial game , i.e. they are some "means" of these values.

Continuity of the functions $\varphi_{N,S,i}$ assumed by axiom 5 does not need any comment.

Anonymity is a standard property. It states that the definition of the reduced excess for some coalition depends only on its size. However, axiom 6 turns out to be unnecessary for consistency of anonymous solutions: there are anonymous solutions (e.g. the core) which are consistent w.r.t. definitions of the reduced games not satisfying axiom 6. In the sequel anonymity is only supposed for simplification of expressions which are rather lengthy even under the condition of anonymity.

4.2 Characterization of the reduced games

In this subsection we describe all reduced games satisfying axioms 1–6. The proof of the following theorem is based on some lemmas on strict separability of social welfare functions which can be found in (Yanovskaya 2002)). In order to make the paper self-contained, they are placed in Appendix as well.

Theorem 3 *Given a TU game $\langle N, v\rangle \in \mathcal{G}_\mathcal{N}$ and a payoff vector x, then the reduced game $\langle N \setminus S, v^{xs}\rangle$ satisfies axioms 1–6 if and only if its characteristic function is defined by repeated application the following definition of the reduced games*
$\langle N \setminus i, v^x\rangle$, when only one player leaves the game:

$$\psi_{n-1,s}(e(v^x(S), x(S)) = \psi_{n,s}(e(v(S), x(S)) + \psi_{n,s+1}(e(v(S \cup i), x(S \cup i)), \tag{14}$$

where the collection of continuous monotonically increasing functions $\psi_{n,s} : \mathbb{R}^1 \to \mathbb{R}^1$, for $n = 3, \ldots, s \leq n-2$, satisfies the equalities

$$\psi_{n-1,s} = \psi_{n,s} + \psi_{n,s+1} \quad \textit{for all } n = 3, \ldots, s < n-2. \tag{15}$$

Proof. Let a collection of functions $\varphi_{n,s} : \mathbb{R}^2 \to \mathbb{R}^1$, $n = 1, \ldots,\ s < n-2$ satisfy axioms 2–6. Then Lemma 3 implies (14), and Lemma 4 implies (15).

Now we prove the 'if' part of the Theorem.

Let the reduced game $\langle N \setminus i, v^x\rangle$ be defined by equalities (14) and (15). Then the functions $\varphi_{n,s}$ defined in (11) and (12) satisfy axioms 3 – 6. We have only to check Path Independence, i.e. the equality

$$\varphi_{n,s,2}\Big(e(v(S),x(S)),\ e(v(S\cup i),x(S\cup i)),\ e(v(S\cup j),x(S\cup j)),\ e(v(S\cup\{i,j\}),x(S\cup\{i,j\}))\Big) =$$
$$\varphi_{n,s,2}\Big(e(v(S),x(S)),\ e(v(S\cup j),x(S\cup j)),\ e(v(S\cup i),x(S\cup i)),\ e(v(S\cup\{i,j\}),x(S\cup\{i,j\}))\Big). \tag{16}$$

for all $S \subsetneqq N\setminus\{i,j\}$ and all values $v(S), v(S\cup i), v(S\cup j), v(S\cup\{i,j\}),\ x(S), x_i, x_j$.

By definition of the functions $\varphi_{n,s}$ and using both equalities (15),(14), the left part of (16) equals

$$\begin{aligned}
&\varphi_{n-1,s}\Big(\varphi_{n,s}(e(v(S),x(S)), && e(v(S\cup i)),x(S\cup i)),\\
&\varphi_{n,s+1}(e(v(S\cup j),x(S\cup j)), && e(v(S\cup\{i,j\}),x(S\cup\{i,j\}))\Big) =\\
&\varphi_{n-1,s}\Big((\psi_{n,s}+\psi_{n,s+1})^{-1} && (\psi_{n,s}(e(v(S),x(S))+\psi_{n,s+1}(e(v(S\cup i),x(S\cup i)),\\
&(\psi_{n,s+1}+\psi_{n,s+2})^{-1} && (\psi_{n,s+1}(e(v(S\cup j),x(S\cup j))+\psi_{n,s+2}(e(v(S\cup\{i,j\}),x(S\cup\{i,j\}))\Big) =\\
&(\psi_{n-1,s}+\psi_{n-1,s+1})^{-1} && \Big(\psi_{n,s}(e(v(S),x(S))+\psi_{n,s+1}(e(v(S\cup i),x(S\cup i))+\\
& && \psi_{n,s+1}(e(v(S\cup j),x(S\cup j)))+\psi_{n,s+2}(e(v(S\cup\{i,j\}),x(S\cup\{i,j\}))\Big).
\end{aligned} \tag{17}$$

The right hand of the equalities (17) is invariant under the permutation of i and j. Therefore, we would obtain the same value if we begin with the right-hand side in (16), i.e. equality (16) has been proved.

□

Theorem 3 defines reduced games through collections of functions, satisfying equalities (15). Observe that the functions $\psi_{2,1}$ in (14) may be chosen arbitrary.

4.3 Consistent excess values for TU games

The following Theorem describes the general form of efficient, anonymous, and consistent excess values.

Theorem 4 *For each continuous and strictly monotonic in the second variable excess function e, any corresponding excess value Φ for the class $\mathcal{G}_+$ is efficient, anonymous, and consistent in the definition (14) of the reduced games if and only if*

$$\Phi(N,v) = \arg\min_{x\in X_+(N,v)} \sum_{S\subset N} \tau_{n,s}(v(S),x(S)), \tag{18}$$

where $\tau_{n,s}:\mathbb{R}^2_{++}\to\mathbb{R}^1_{++}$ is a collection of strictly convex and differentiable in the second variable functions, $n=2,\dots,\ s\le n-1$, satisfying

$$\frac{\partial \tau_{n,s}(x,y)}{\partial y} = \psi_{n,s}\left(e(x,y)\right), \tag{19}$$

and the functions $\psi_{n,s}$ satisfy the condition of Theorem (3).

Proof. The 'if' part is checked directly: Let Φ be defined by (18) and $\Phi(N,v)=x$. Note that the minimum in (18) is attained in a unique strictly positive point because of strict convexity of the functions $\tau_{n,s}$.

Then the Lagrange conditions for Φ are the following:

$$\sum_{S\ni j}\psi_{n,s}\left(e(v(S),x(S))\right)=\sum_{S\ni k}\psi_{n,s}\left(e(v(S),x(S))\right)\quad\text{for all } j,k\in N. \tag{20}$$

Substituting (14) into (20) we obtain

$$\sum_{\substack{S\ni j\\ S\subset N\setminus\{i\}}}\psi_{n-1,s}\left(e(v^x_{N\setminus\{i\}}(S),x(S))\right)=\sum_{\substack{S\ni k\\ S\subset N\setminus\{i\}}}\psi_{n-1,s}\left(e(v^x_{N\setminus\{i\}}(S),x(S))\right) \tag{21}$$
$$\text{for all } j,k\in N\setminus\{i\},\ s<n-1.$$

The equalities (21) are just the Lagrange conditions for x^i, i.e. the vector x without $i-th$ component x_i, to be the solution of the problem

$$\min_{\substack{y\in\mathbb{R}^{N\setminus\{i\}}_{++}\\ y(N\setminus\{i\})=v(N)-x_i}}\sum_{j\in N\setminus\{i\}}\tau_{n-1,s}(e(v(S),y(S))). \tag{22}$$

Therefore, $x^i=\Phi(N\setminus i,v^{x,i})$.

Let us prove the 'only if' part. Let Φ be an excess value which is efficient, anonymous, and consistent in the definition (18) value and $\langle N,v\rangle$ be an arbitrary positive game, $\Phi(N,v)=x$. For arbitrary $i,j\in N$ consider the reduced game on the set $\{i,j\}$ w.r.t. the vector x. Then by iterating the definition (14) and putting $\psi_{2,1}(t)\equiv t$, we obtain

$$e(v^x_{i,j}(i),x_i)=\left(\sum_{q=0}^{n-2}\binom{n-2}{q}\psi_{n,1+q}\right)^{-1}\left(\sum_{Q\subset N\setminus\{i,j,\}}\psi_{n,1+q}(e(v(\{i\}\cup Q),x(\{i\}\cup Q)))\right). \tag{23}$$

Since the value Φ is anonymous and consistent, for the reduced game $\langle\{i,j\},v^x_{i,j}\rangle$ we should have

$$e(v^x_{i,j}(i),x_i)=e(v^x_{i,j}(j),x_j). \tag{24}$$

Substituting (24) in (23) we obtain

$$\sum_{Q\subset N\setminus\{i,j,\}}\psi_{n,1+q}\left(e(v(\{i\}\cup Q),x(\{i\}\cup Q))\right)==\sum_{Q\subset N\setminus\{i,j\}}\psi_{n,1+q}\left(e(v(\{j\}\cup Q),x(\{j\}\cup Q))\right),$$

The last equality holds for any $i,j\in N$. Thus, it can be rewritten as

$$\sum_{\substack{S\ni i\\ S\subset N}} \psi_{n,s}\left(e(v(S),x(S))\right) = \sum_{\substack{S\ni j\\ S\subset N}} \psi_{n,s}\left(e(v(S),x(S))\right) \quad \text{for any } i,j\in N. \tag{25}$$

The equalities (25) coincide with (20). Therefore, the vector x minimizes the sum $\sum_{S\subset N}\tau_{n,s}(e(v(S),x$

□

Compare Theorem 4 with Theorem 2. Both objective functions in (6) and (18) are additively separable. The components in (6) and those in (18), corresponding to one element coalitions, coincide up to monotonic functions $\psi_{n,s}$ in (18) and (19). Just because of the functions $\psi_{n,s}$, a parametric solution, in general, differs from the corresponding consistent TU game solution on the class $\mathcal{G}_s$. (18)

5 Double consistent surplus sharing methods

In this section we describe all continuous and strictly monotonic parametric methods for surplus sharing problems which are consistent in the sense of TU game excess consistency (14), (15) for the class $\mathcal{G}_s$. As it had been already noted in subsection 3.1, excess functions for strictly monotonic parametric methods are defined up to monotone transformations. Thus, it suffices only to define their level lines.

Lemma 1 *If an unconstrained strictly monotonic parametric method Φ with representation f for surplus sharing problems is consistent in Definition 7 to the class $\mathcal{G}_s$ of TU games, where the reduced games are defined by the equalities (14), then the level lines of the excess function $e=f^{-1}$ are straight lines.*

Proof. Let $\langle N=\{1,2,3\},v\rangle$ be a three-person TU game from the class $\mathcal{G}_s$. Consider all its reduced games on two-player sets, in particular on the player set $\{i,j\}$. Then for some functions $\psi_{3,1},\psi_{3,2}$ the following equalities hold for all $x\in X(N,v)$:

$$\begin{aligned} e(v^x_{i,j}(i),x_i)) &= \psi_{3,1}(e(v(i),x_i)+\psi_{3,2}(e(v(i)+v(k),x_i+x_k) \\ e(v^x_{i,j}(j),x_j)) &= \psi_{3,1}(e(v(j),x_j)+\psi_{3,2}(e(v(j)+v(k),x_j+x_k). \end{aligned} \tag{26}$$

Let $x=\Phi(N,v)$ and at the same time x be a parametric solution of the three-person allocation problem $\langle\{v(i)\}_{i=1,2,3},v(N)\rangle$ defined by the excess function e. Then, by Theorem 1 and (10) we should have

$$e(v(i),x_j)=e(v(j),x_j). \tag{27}$$

From (26), (27), and arbitrariness of the choice $i,j\in\{1,2,3\}$ it follows that

$$e(v(i)+v(k),x_i+x_k)=e(v(j)+v(k),x_j+x_k)=e(v(i)+v(j),x_i+x_j). \tag{28}$$

We could take $v(i) = v(j)$ for any pair of players i, j. Then by symmetry of the solution x for the allocation problem we should have $x_i = x_j$ and equalities (28) imply

$$e(v(i), x_i) = e(v(j), x_j) \iff e(2v(i), 2x_i) = e(2v(j), 2x_j). \tag{29}$$

Therefore from (28) and (29) it follows

$$e(v(i), x_i) = e(v(j), x_j) \implies = e\left(\frac{x_i + x_j}{2}, \frac{v(i) + v(j)}{2}\right). \tag{30}$$

The last relation means that the level lines of the function e are straight lines. Therefore,

$$e(v, x) = c \implies v = kx + b, \tag{31}$$

where the parameters k and b, in general, may depend on c. □

As the level lines of a function cannot intersect, then for the general case, when both the values of the characteristic functions and the players' payoffs may be arbitrary numbers, we should have $k(c) \equiv k$, i.e. the level lines are parallel. In this case we have $e(v, x) = \varphi(x - \lambda v)$ for some increasing function φ.

However, for surplus sharing problems the domain of the excess functions is the positive orthant $\mathbb{R}^2_{++}$, and the level lines may intersect outside of it. Because of the implication (26),(27) $\implies$ (28) this may happen only in zero point. If all they intersect in it, then $e(v, x) = \dfrac{x}{v}$ for all $x, v \in \mathbb{R}^1_{++}$. Suppose that at least one level line has a form $x = av + b$, $b \neq 0$, and let $k > 0, b > 0$. Then either all the level lines are parallel straight lines, or there are some level lines intersecting in zero point. By the property (27) $\implies$ (28) the line $y = ax$ is also a level line. If $x = Av$ is another level line of e, then all the lines $x = qv, a \leq q \leq A$ are also the level lines of e. Let A be the maximal number possessing this property (A may be equal to infinity). It means that other level lines have the form $x = Av + b$. Note that there exist no level lines $x = kv$ with $k < a$.

Thus, the general form of the excess functions, satisfying the conditions of lemma 1 is the following:

$$e(v, x) = const \iff \begin{cases} \frac{v}{x} = c, & \text{if } a \leq \frac{x}{v} \leq A, \\ x = av + b_1(c), & \text{if } \frac{x}{v} < a, \\ x = Av + b_2(c), & \text{if } \frac{x}{v} > A. \end{cases} \tag{32}$$

for some $0 \leq a \leq A \leq \infty$.

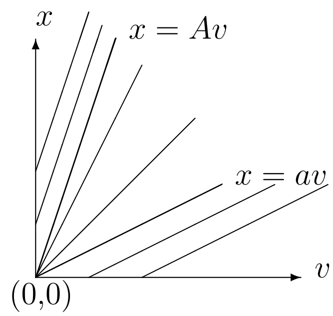

Figure 1: Level lines of the excess function (32)

□

Reveal among these excess functions those for which the corresponding parametric solutions for surplus sharing problems satisfy the core property.

Corollary 1 *Under the assumptions of Lemma 1 a solution* Φ *for surplus sharing problems satisfies the core property* $\Phi_i(N, v) \geq v(i) \, \forall i \in N$ *if and only if the corresponding excess function has the following form:*

$$e(x, y) = const \iff \begin{cases} \dfrac{x}{y} = c, & \textit{if } \dfrac{y}{x} \leq A, \\ y = Ax + b(c), & \textit{if } \dfrac{y}{x} \geq A \end{cases} \tag{33}$$

for some $A \geq 1$.

Proof. The 'if' part.
Let an excess function (33) be given. Then, taking into account (10), we can define the corresponding parametric allocation method Φ^A for surplus sharing problems as follows:

$$\Phi_i^A(x, T) = \begin{cases} \dfrac{x_i}{\sum_{j=1}^n x_j} T, & \text{if } T \leq A \sum_{j=1}^n x_j, \\ A x_i + \dfrac{1}{n}\left(T - A \sum\limits_{j=1}^n x_j\right), & \text{if } T \geq A \sum_{j=1}^n x_j. \end{cases} \tag{34}$$

The inequality $\sum_{i \in N} x_i \leq T$, implies the core property $\Phi_i^A(x, T) \geq x_i$ for all $i \in N$.

The 'only if' part.
Let an excess function e be defined in (32). Then the corresponding parametric solutions for surplus sharing problems are described by the following two-dimensional family $\Phi^{a,A}$:

$$\Phi_i^{a,A}(x, T) = \begin{cases} a x_i + \dfrac{1}{n}\left(T - a \sum\limits_{j=1}^n x_j\right), & \text{if } T \leq a \sum_{j=1}^n x_j, \\ \dfrac{x_i}{\sum_{j=1}^n x_j} T, & \text{if } a \sum_{i \in N} x_i \leq T \leq A \sum_{j=1}^n x_j, \\ A x_i + \dfrac{1}{n}\left(T - A \sum\limits_{j=1}^n x_j\right), & \text{if } T \geq A \sum_{j=1}^n x_j. \end{cases} \tag{35}$$

Let the solutions $\Phi^{a,A}$ satisfy the core property $\Phi_i^{a,A}(x,T) \geq x_i$ for all $i \in N$. The inequality $T \geq \sum_{i \in N} x_i$ implies that both numbers a, A are greater or equal to 1. Let us show that the solutions $\Phi^{a,A}$ can not be defined by the first line in (35). In fact, for all T satisfying the inequality

$$T \leq a \sum_{j=1}^{n} x_j$$

it should hold

$$a x_i + \frac{1}{n}\left(T - a \sum_{j=1}^{n} x_j\right) \geq x_i,$$

that means

$$(a-1)x_i \geq \frac{1}{n}\left(a \sum_{j=1}^{n} x_j - T\right) \quad \forall i = 1, \ldots, n. \tag{36}$$

The last inequalities must hold for any $x \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n x_i < T$. Put $x = (1, \frac{\varepsilon}{n-1}, \ldots, \frac{\varepsilon}{n-1})$, where $\varepsilon > 0$ is a sufficiently small number. Then $\sum_{i=1}^n x_i = 1 + \varepsilon$, and if T is enough close to $\sum_{i=1}^n x_i : T = 1 + \varepsilon + \alpha$, then for this case inequalities (36) for $i \neq 1$ have the form:

$$(a-1)\varepsilon \geq \frac{n-1}{n}\left((a-1)(1+\varepsilon) - \alpha\right). \tag{37}$$

It is easy to notice that the inequality (34) does not hold for a fixed $a > 1$ and sufficiently small ε and α.

Thus, we obtain the one-dimensional family of sharing solutions (37). Observe that this family coincides with that in Moulin (1987), Theorem 1, describing all consistent and decentralizable surplus sharing solutions.

Lemma 1 and Corollary 1 imply the main result of the section:

Theorem 5 *A strictly monotonic parametric method Φ for surplus sharing problems satisfies the core property and is consistent in Definition 7 to the class $\mathcal{G}_s$ of TU games, where the reduced games are defined by equalities (14) if and only if the level lines of the corresponding excess function satisfy (34)*

Proof. We should prove only the 'if' part of the Theorem.

Let Φ be a strictly monotonic parametric method for surplus sharing problems with excess function e. By Corollary 1 it has the form (34). Consider an arbitrary three-person game $\langle N = \{1,2,3\}, v\rangle \in \mathcal{G}_s$ and let $x = \Phi(N, v(N); \{v(i)\}_{i \in N})$. Then equalitites (27) hold. The equalities (34) imply the relation (29) and, hence, (28). Now equalities (27) and (28) imply the equalities $e(v_{ij}^x(i), x_i) = e(v_{ij}^x(j), x_j)$ for all $i, j = 1, 2, 3$ that means consistency of Φ when reducing a three-person players' set on two-person ones.

By induction in the number of players it is not difficult to check the TU-game consistency of F on the whole class $\mathcal{G}_s$.

□

In spite of double consistency of the values satisfying the conditions the Theorem diagram (4) may be not commutative for the corresponding definition of the reduced games. In fact, for usual excess function $e(v, x) = v - x$ equality (14) has the following form for the surplus sharing problems, where $v(S) = \sum_{i \in S} d_i$:

$$\psi_{n-1,s}\left(e(v^x(S), x(S))\right) = \psi_{n,s}\left(e(\sum_{i \in S} d_i - x(S))\right) + \psi_{n,s+1}\left(e(\sum_{j \in S \cup \{i\}} d_j, x(S \cup \{i\})\right), \quad (38)$$

and taking into account that the solution vector x for the parametric allocation method, corresponding to the excess into consideration satisfies the equalities

$$x_i - d_i = x_j - d_j = \lambda \ \forall i, j \in N,$$

we should have for commutativity of the diagram the equalities

$$\psi_{n-1,s}(s\lambda) = \psi_{n,s}(s\lambda) + \psi_{n,s+1}((s+1)\lambda), \quad (39)$$

which are inconsistent with condition (15) in Theorem 3.

Thus, the last result reveals among the excess functions satisfying the conditions of Theorem 5 those for which diagram (4) is commutative.

Corollary 2 *A strictly monotonic double consistent allocation method defined in (34) provides commutativity of diagram (4) for the definition (14) of the reduced games if and only is it is proportional.*

Proof. it is clear that for surplus sharing problems $\langle N, T; d \rangle$, their proportional solution vectors x satisfy the equalities

$$\frac{v(S)}{x(S)} = \frac{v(\{i\})}{x(\{i\})} \ \forall i \in N, S \subset N,$$

where $\langle N, v \rangle \in \mathcal{G}_s$ is the TU game, correwponding to the surplus sharing problem, i.e. $v(S) = \sum_{j \in S} d_j$. Therefore, equalities (15) and (14) imply $v^x(S) = \sum_{j \in S} d_j$ and diagram (4) is commutative.

Let now Φ be any surplus sharing method satisfying (34). Then it is either proportional, or is defined by the second line in (34). The second case does not provide commutativity of the diagram. The proof of this fact is analogous to the proven before Corollary 2 non-commutativity of the diagram for the egalitarian method (see (38),(39)).

6 Concluding remarks

In the main theorems of the paper — Theorems 3 and 5 — two assumptions about strict monotonicity of characteristic functions of the reduced games and of parametric allocation methods have been made. However, perhaps these restrictions are not essential and the results will hold without them as well. In fact, the definition of the most popular Davis–Maschler reduced game can be obtained as the limit in $p \to \infty$ of the reduced games determined by the power functions $\psi_{n,s}(t) = w_{n,s} \cdot t^p$ (for the positive values of the excess functions) (Yanovskaya 2002b).

On the other hand, strict monotonicity of parametric methods seems to be natural within hydraulic interpretation because of practical impossibility of vessels with zero volume's parts as it happens in hydraulic metaphora of non strict monotonic parametric methods.

Moreover, some non-strictly monotonic parametric methods (such as e.g. the contested garment one) may be approximated by strictly monotonic ones. Whether such approximations are always possible is an open problem.

Acknowledgements

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Appendix

In the Appendix some properties of the functions $\varphi_{n,s}$ (12) and $\varphi_{n,s,t}$ (13) defining the characteristic function of the reduced games are established.

Lemma 2 . *Let $F : \mathbb{R}^4 \to \mathbb{R}^1$ be a continuous function, strictly increasing in each variable and satisfying the conditions:*
$(i) F(a,b,c,d) = \bar{F}(\xi_1(a,b), \xi_2(c,d))$, where $\bar{F}$ is an increasing in each variable function, and
$(ii) F(a,b,c,d) = F(a,c,b,d)$ for all a,b,c,d.
Then $F(a,b,c,d) = U(\xi_1(a) + \xi_2(b) + \xi_3(c) + \xi_4(d))$ for some continuous strictly increasing functions $U, \xi_i,\ i = 1,2,3,4$.

Proof. The first condition (i) means that the subsets of variables $(a,b),(c,d)$ of the function F are strictly separable from their complements (c,d), (a,b) respectively. The condition (ii) implies that the subsets (a,c), (b,d) are strictly separable from (b,d), (a,c) respectively. It is clear that any point $x \in \{a,b,c,d\}$ can be represented by an intersection or difference of the subsets $(a,b),(c,d),(a,c),(b,d)$. Other subsets are union of the points. Therefore, any subset of the set $\{a, b, c, d\}$ is strictly separable from its complement that means the complete strict separability of the function F. By Gorman's theorem (Gorman (1968)) this property together with continuity and monotonicity of the function F implies the additive form stated in the Lemma.

□

In the next Lemma we find the form of the functions $\varphi_{n,s}$ (12).

Lemma 3 *If the reduced game satisfies the axioms 1–6, then*

$$\varphi_{n,s}(a,b) = (\psi_{n,s} + \psi_{n,s+1})^{-1}\Big(\psi_{n,s}(a) + \psi_{n,s+1}(b)\Big)$$

for some continuous strictly increasing functions $\psi_{n,s}$, $s \leq n-2$ *such that* $\psi_{n,s}(0) = 0$.

Proof. From (17) and Lemma 2 it follows

$$\begin{aligned}\varphi_{n,s,2}(a,b,c,d) &= \varphi_{n-1,s}(\varphi_{n,s}(a,b), \varphi_{n,s+1}(c,d)) = \\ &\bar{U}(\xi^1_{n,s}(a) + \xi^2_{n,s}(b) + \xi^3_{n,s}(c) + \xi^4_{n,s}(d)).\end{aligned} \tag{40}$$

for some continuous increasing functions $\xi^i_{n,s}, i = 1,2,3,4,\ s < n-2$, where we denoted the values of excesses $e(v(S), x(S)), e(v(S\cup i), x(S\cup i), e(v(S\cup j), x(S\cup j), e(v(S\cup\{i,j\}),\ x(S\cup\{i,j\}))$ by a, b, c, d respectively.

Axiom 4 implies

$$\bar{U} = (\xi^1_{n,s} + \xi^2_{n,s} + \xi^3_{n,s} + \xi^4_{n,s})^{-1}.$$

Without loss of generality we may suppose that $\xi^i_{n,s}(0) = 0$ for all $i = 1,2,3,4, s \leq n-2$. Then by substituting in (40) consecutively $c, d = 0\ \ a, b = 0$, we obtain

$$\varphi_{n,s}(a,b) = \varphi_{n-1,s}(\cdot,0)^{-1}\Big((\xi^1_{n,s} + \xi^2_{n,s} + \xi^3_{n,s} + \xi^4_{n,s})^{-1}(\xi^1_{n,s}(a) + \xi^2_{n,s}(b))\Big);$$

$$\varphi_{n,s+1}(c,d) = \varphi_{n-1,s}(0,\cdot)^{-1}\Big((\xi^1_{n,s} + \xi^2_{n,s} + \xi^3_{n,s} + \xi^4_{n,s})^{-1}(\xi^4_{n,s}(c) + \xi^4_{n,s}(d)\Big).$$

Path independence axiom implies that $\xi^2_{n,s} = \xi^3_{n,s}$, and axiom 4 implies that the previous equalities have the form

$$\varphi_{n,s}(a,b) = (\psi_{n,s} + \psi_{n,s+1})^{-1}(\psi_{n,s}(a) + \psi_{n,s+1}(b)). \tag{41}$$

$$\varphi_{n,s+1}(c,d) = (\psi_{n,s+1} + \psi_{n,s+2})^{-1}(\psi_{n,s+1}(c) + \psi_{n,s+2}(d)), \tag{42}$$

where we denoted $\xi^1_{n,s} = \psi_{n,s}$, $\xi^2_{n,s} = \xi^3_{n,s} = \psi_{n,s+1}$, $\xi^4_{n,s} = \psi_{n,s+2}$.

□

Our next step is to reveal a link between the functions $\psi_{n,s}$ for different $n, s < n-2$.

Lemma 4 *In the conditions of Lemma 3 the functions $\psi_{n,s}$ satisfy the equalities*

$$\psi_{n-1,s} = \psi_{n,s} + \psi_{n,s+1}, \quad s \le n-2, n = 3, \ldots,$$

up to positive multipliers B_n.

Proof.

Substituting (41) and (42) in (40) we obtain

$$\varphi_{n-1,s}\Big((\psi_{n,s} + \psi_{n,s+1})^{-1}(\psi_{n,s}(a) + \psi_{n,s+1}(b)), (\psi_{n,s+1} + \psi_{n,s+2})^{-1}(\psi_{n,s+1}(c) + \psi_{n,s+2}(d))\Big) = (\psi_{n,s} + \psi_{n,s+1} + \psi_{n,s+1} + \psi_{n,s+2})^{-1}(\psi_{n,s}(a) + \psi_{n,s+1}(b) + \psi_{n,s+1}(c) + \psi_{n,s+2}(d)). \tag{43}$$

Lemma 3 holds for any $s \le n-2$, $n = 3, \ldots$. Therefore, applying it to the equality (17) we obtain

$$\begin{aligned}
&\varphi_{n,s,2}(v(S), v(S \cup i), v(S \cup j), v(S \cup \{i,j\}) = \\
&(\psi_{n-1,s} + \psi_{n-1,s+1})^{-1}\Big(\psi_{n-1,s}\Big((\psi^1_{n,s} + \psi_{n,s+1})^{-1}(\psi(v(S)) + \psi_{n,s+1}(v(S \cup i))\Big) + \\
&\psi_{n-1,s+1}\Big((\psi_{n,s+1} + \psi_{n,s+2})^{-1}\Big((\psi^1_{n,s+1}(v(S \cup j)) + \psi_{n,s+2}(v(S \cup \{i,j\}))\Big)\Big) = \\
&(\psi_{n,s} + 2\psi_{n,s+1} + \psi_{n,s+2})^{-1}\Big(\psi_{n,s}(v(S)) + \psi_{n,s+1}(v(S \cup i)) + \\
&\psi_{n,s+1}(v(S \cup j)) + \psi_{n,s+2}(v(S \cup \{i,j\}))\Big).
\end{aligned} \tag{44}$$

The equalities (44) and axiom 2 imply that the expression

$$\begin{aligned}
&\psi_{n-1,s}\Big((\psi_{n,s} + \psi_{n,s+1})^{-1}(\psi_{n,s}(a) + \psi_{n,s+1}(b)\Big) + \\
&\psi_{n-1,s+1}\Big(\psi_{n,s+1} + \psi_{n,s+2})^{-1}(\psi_{n,s+1}(c) + \psi_{n,s+2}(d))\Big)
\end{aligned} \tag{45}$$

does not depend on the permutations of b and c. In particular, we may equalize the value of (45) for $(a,b,c,d) = (a,b,0,0)$ and $(a,0,b,0)$ respectively, and remembering that $\psi_{n,s}(0) = 0$ for all $n, s \le n-2$, we obtain

$$\begin{aligned}
&\psi_{n-1,s}\Big((\psi_{n,s} + \psi_{n,s+1})^{-1}(\psi_{n,s}(a) + \psi_{n,s+1}(b)\Big) = \\
&\psi_{n-1,s}(\psi_{n,s} + \psi_{n,s+1})^{-1}\psi_{n,s}(a) + \psi_{n-1,s+1}\Big((\psi_{n,s+1} + \psi_{n,s+2})^{-1}(\psi_{n,s+1}(b))\Big).
\end{aligned} \tag{46}$$

The equality (46) holds for any a, b. Taking $a = 0$ we obtain

$$\psi_{n-1,s}(\psi_{n,s} + \psi_{n,s+1})^{-1} = \psi_{n-1,s+1}(\psi_{n,s+1} + \psi_{n,s+2})^{-1} = f_{n,s}, \tag{47}$$

where by $f_{n,s}$ we denote both parts of the equality (47). Hence, (46) and (47) imply that the function $f_{n,s}$ is additive. Since the functions $\psi_{n,s}, \psi_{n-1,s}, \psi_{n,s+1}, \psi_{n,s+2}$ are continuous (axiom

5), the functions $f_{n,s}$ are linear, and the equalities $\psi_{n,s}(0) = 0$ for all $n, s \leq n-2$ imply that $f_{n,s}(t) = C_{n,s}t$.

Substituting $a = b$ in (46) we obtain

$$\psi_{n-1,s} = \psi_{n-1,s}(\psi_{n,s} + \psi_{n,s+1})^{-1}\psi_{n,s} + \psi_{n-1,s+1}(\psi_{n,s+1} + \psi_{n,s+2})^{-1}\psi_{n,s+1}. \tag{48}$$

The last equality together with (46) implies

$$\psi_{n-1,s} = B_{n,s}(\psi_{n,s} + \psi_{n,s+1})$$

for some constants $B_{n,s}$, and

$$\psi_{n-1,s+1} = B_{n,s}(\psi_{n,s+1} + \psi_{n,s+2}), \tag{49}$$

i.e. $B_{n,s} = B_n$ do not depend on s.

As the functions $\varphi_{n,s}$ in (41) and (42) (where n is fixed!) do not depend on multiplying $\psi_{n,s}$ by any positive constants B_n, we may put $B_n = 1$. This completes the proof. □

References

Davis M., Maschler M., 1965. The kernel of a cooperative game. Naval Research Logistic Quarterly 12, 223-259.

Gorman W.M., 1968. The structure of utility functions. Review of Economic Studies 35, 369–390

Hart, S., Mas–Colell, A., 1989. Potential, value and consistency. Econometrica 53, 589–614.

Kaminski M., 2000. 'Hydraulic' rationing. Mathematical Social Sciences 40, 131-155.

Moulin H., 1985. The separability axiom and equal sharing methods. Journal of Economic Theory 36, 120–148.

Moulin H., 1987. Equal or proportional division of a surplus, and other methods. International Journal of Game Theory 3, 161–186.

Moulin H., 1988. Axioms of cooperative decision making. In: Econometric Society Monographs. Vol.15, Cambridge University Press, Cambridge.

Moulin H., 2001. Axiomatic cost and surplus sharing methods In: Handbook on Social Choice and Welfare.

Ruis L.M., Valenciano F., Zarzuelo J.M., 1998. The family of least square values for transferable utilities games. Games and Economic Behaviour 4, 109–130.

Thomson W., 1996. Consistent allocation rules. Economics Department, University of Rochester, Rochester.

Yanovskaya E., Driessen T., 2001. On linear consistency of anonymous values for TU-games. International Journal of Game Theory 30, 601-609.

Yanovskaya E., 2002a. Consistency for proportional solutions. International Game Theory Review 4, 343-356.

Yanovskaya E., 2002b. A family of the least power values for cooperative games with transferable utilities, In: Gruber J., Tangian A., (Eds) Constructing and Applying Objective Functions. Proceedings of the Fourth International Conference on Econometric Decision Models, August 31-September 2, 2000. Springer-Verlag, Berlin, pp. 473-494.

Young H.P., 1987. On dividing an amount according to individual claims or liabilities. Mathematics of Operations Research 12, 397–414.

Young H.P., 1994. Equity in Theory and Practice. Princeton University Press, Princeton.

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