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GAME-THEORETIC MODEL OF FINANCIAL MARKETS WITH TWO RISKY ASSETS*

Victor Domansky, Victoria Kreps†

Abstract

We consider multistage bidding models where two types of risky assets (shares) are traded between two agents that have different information on the liquidation prices of traded assets. These prices are random integer variables that are determined by the initial chance move according to a probability distribution $\mathbf{p}$ over the two-dimensional integer lattice that is known to both players. Player 1 is informed on the prices of both types of shares, but Player 2 is not. The bids may take any integer value.

The model of n-stage bidding is reduced to a zero-sum repeated game with lack of information on one side. We show that, if liquidation prices of shares have finite variances, then the sequence of values of n-step games is bounded. This makes it reasonable to consider the bidding of unlimited duration that is reduced to the infinite game $G_\infty(\mathbf{p})$. We offer the solutions for these games.

We begin with constructing solutions for these games with distributions $\mathbf{p}$ having two- and three-point supports. Next, we build the optimal strategies of Player 1 for bidding games $G_\infty(\mathbf{p})$ with arbitrary distributions $\mathbf{p}$ as convex combinations of his optimal strategies for such games with distributions having two- and three-point supports. To do this we construct the symmetric representation of probability distributions with fixed integer expectation vectors as a convex combination of distributions with not more than three-point supports and with the same expectation vectors.

Key words: financial market, random walk of prices, asymmetric information, repeated game, optimal strategy, extreme points of convex sets.

JEL classification: C72, C73, D44

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1 Introduction. Modeling financial markets by repeated games with asymmetric information

Random fluctuations in stock market prices are usually explained by the effect from multiple exogenous factors subjected to accidental variations. The work of De Meyer and Saley (2002) proposes a different strategic motivation for these phenomena. The authors assert that the Brownian component in the evolution of prices on the stock market may originate from the asymmetric information of stockbrokers on events determining market prices. "Insiders" are not interested in the immediate revelation of their private information. This forces them to randomize their actions and results in the appearance of an oscillatory component in price evolution.

De Meyer and Saley demonstrate this idea on a model of multistage bidding between two agents for one-type risky assets (shares). The liquidation price of a share depends on a random "state of nature" Before the bidding starts a chance move determines the "state of nature" and therefore the liquidation value of shares once and for all. Player 1 is informed on the "state of nature", but Player 2 is not. Both players know the probability of a chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, ..., n$ both players simultaneously propose their prices for one share. The maximal bid wins and one share is transacted at this price. If the bids are equal, no transaction occurs. Each player aims to maximize the value of his final portfolio (money plus the liquidation value of obtained shares).

In this model the uninformed Player 2 should use informed Player 1's history of moves to update his beliefs about the state of nature. In fact, at each step Player 2 may use the Bayes rule to re-estimate the posterior probabilities of a chance move outcome, or, at least, the posterior expectations of a liquidation price per share. Player 1 could control these posterior probabilities.

Thus Player 1 faces a problem of how to best use his private information without revealing it to Player 2. Using a myopic policy – in which a high bid is posted if the liquidation price is high and a low bid is posted if this price is low – is not optimal for Player 1, because it fully reveals the state of nature to Player 2. On the other hand, a strategy that does not depend on the state of nature, while revealing no information to Player 2, does not allow Player 1 to take advantage of his superior knowledge. Thus Player 1 must maintain a delicate balance between taking advantage of his private information and concealing it from Player 2.

De Meyer and Saley consider a model where a share's liquidation price takes only two values and players may make arbitrary bids. They reduce this model to a zero-sum repeated game with lack of information on one side, as introduced by Aumann and Maschler (1995), but with continual action sets. De Meyer and Saley show that these n-stage games have values (i.e. the guaranteed gains of Player 1 are equal to the guaranteed losses of Player 2) in finding these values and the optimal strategies of players. De Meyer and Saley demonstrate that as n tends to infinity, the values infinitely increase with rate $\sqrt{n}$. It is shown that Brownian Motion appears in the asymptotics of transaction prices generated by these strategies.

It is more natural to assume that players may assign only discrete bids proportional to a minimal currency unit. In our papers (Domansky, 2007), (Domansky and Kreps, 2007) we investigate a model with two possible values fpr liquidation price and discrete admissible bids. We show that, unlike De Meyer and Saley's (2002) model as n approches ∞, the sequence of guaranteed gains of the insider is bounded from above and converges. It is reasonable to consider bidding with an infinite number of steps. We construct the optimal strategies for corresponding infinite games. We write out explicitly the random process formed by the prices of transactions at sequential steps. The transaction prices perform a symmetric random walk over the admissible bids between two possible values of liquidation price with absorbing extreme points. The absorption of transaction prices reveals the true share's price by Player 2.

In our works (Domansky and Kreps, 2009), (Domansky and Kreps, 20011) we consider a model where any integer non-negative bids are admissible. The liquidation price of share $C_{\mathbf{p}}$ may take any nonnegative integer values $k = 0, 1, 2, \ldots$ according to a probability distribution $\mathbf{p} = (p_0, p_1, p_2, \ldots)$.This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information for Player 2 and with countable state and action spaces. For constructing the optimal strategy of Player 1 (the insider) with an arbitrary liquidation price per share that has finite variance, we use the symmetric representation of distributions over an integer lattice with fixed integer mean values that are convex combinations of distributions with two-point supports and that have the same mean values. The solutions for games with two-point distributions were obtained by Domansky (2007).

We show that if the random variable $C_{\mathbf{p}}$, determining the liquidation price of a share has a finite mathematical expectation $\mathbf{E}[C_{\mathbf{p}}]$, then the values $V_n(\mathbf{p})$ of n-stage games $G_n(\mathbf{p})$ exist (i.e. the guaranteed gain of Player 1 is equal to the guaranteed loss of Player 2). If the variance $\mathbf{D}[C_{\mathbf{p}}]$ is infinite, then, as n approaches ∞, the sequence $V_n(\mathbf{p})$ diverges.

On the contrary, if the variance $\mathbf{D}[C_{\mathbf{p}}]$ is finite, then, as n approaches ∞, the sequence of values $V_n(\mathbf{p})$ for the games $G_n(\mathbf{p})$ is bounded from above and converges. So it is reasonable to consider the games $G_\infty(\mathbf{p})$ with an infinite number of steps. We explicitly construct the optimal strategies for these games. It is shown that the insider optimal strategy generates a symmetric random walk of posterior mathematical expectations over the set of positive integer numbers with absorption.

In section 2 we introduce the model of multistage bidding where two types of risky assets (shares) are traded between two agents having different information on the liquidation prices of traded assets. These prices are integer random variables that are determined by the initial chance move for the whole period of bidding according to a probability distribution $\mathbf{p} \in \Delta(\mathbb{Z}^2)$ over two-dimensional integer lattice that is known to both players . Player 1 knows the prices of both types of shares. Player 2 does not have this information, but does know, that Player 1 is an insider.

At each step of bidding both players simultaneously make their integer bids, i.e. they post their prices for each type of shares. The player who posts the larger price for a share of a given type buys one share of this type from his opponent at this price. Any integer bids are admissible. Players aim to maximize the values of their final portfolios, calculated as money plus obtained shares evaluated by their liquidation prices.

The described model of n-stage bidding is reduced to the zero-sum repeated game $G_n(\mathbf{p})$ with lack of information on one side and with two-dimensional one-step actions with components corresponding to bids for each type of assets.

It is easy to show that if the expectations of share prices are finite, then the values $V_n(\mathbf{p})$ of n-stage bidding games $G_n(\mathbf{p})$ exist. The value of such a game does not exceed the sum of values of games modeling the bidding with one-type shares. This means that a simultaneous bidding of two types of risky assets is less profitable for the insider than separate bidding of one-type shares. This is explained by the fact that the simultaneous bidding leads to revealing more insider information, because the bids for shares of each type provide information on shares of the other type.

In section 3 we show that if both share prices have finite variances, then the values $V_n(\mathbf{p})$ of n-stage bidding games do not exceed the function $H(\mathbf{p})$ which is the smallest piecewise linear function equal to the one half of the sum of share price variances for distributions with integer expectations of both share prices.

To prove this we define the set of strategies $\tau^*(\mathbf{p})$ for Player 2 that ensure these upper bounds. The strategy $\tau^*(\mathbf{p})$ is a direct combination of Player 2' optimal strategies for the games with one-type risky asset. The initial bids are the integer parts of expectations for corresponding liquidation prices. At step $t > 1$, the bid for a given type of share depends on the result of bidding for this share type at the previous step. If the buyer was Player 1, then the next bid increases for one unit; if the buyer was Player 2, then the next bid decreases for one unit; if there was a tie, then the next bid remains the same.

This makes it reasonable to consider the bidding of unlimited duration without an artificial restriction given beforehand for number of steps. This bidding model is reduced to the infinite game $G_\infty(\mathbf{p})$. We show that this game terminates naturally when the posterior expectations of both liquidation prices come close enough to their real values. We further show that the value $V_\infty(\mathbf{p})$ coincides with $H(\mathbf{p})$. It is observed that $H(\mathbf{p})$ is the sum of values for infinite games with one-type assets studied in (Domansky and Kreps, 2009).

In section 4 we construct optimal strategies σ^* for Player 1 that ensure $H(\mathbf{p})$ for games $G_\infty(\mathbf{p})$ with two states. We base this on the results for games with one-type assets and with two states obtained by Domansky (2007).

The defined strategy σ^* of Player 1 generates an asymmetric random walk of posterior probabilities by adjacent points of the lattice formed with those probabilities where at least one of the price expectations is an integer value. The probabilities of jumps provide martingale characteristics of posterior probabilities and with absorption at extreme points.

In section 5 we construct optimal strategies σ^* of Player 1 that ensure $H(\mathbf{p})$ for games $G_\infty(\mathbf{p})$ with three states. The martingale of posterior mathematical expectations generated by the optimal strategy of Player 1 for the game with the three-point support distribution represents a symmetric random walk over points of integer lattice lying within the triangle spanned across the support points of distribution. The symmetry is broken the moment that the walk hits the triangle boundary. From this moment, the game turn into one of games with distributions having two-point supports.

Further we consider the games $G_\infty(\mathbf{p})$ with prices given by arbitrary probability distributions $\mathbf{p} \in \Delta(\mathbb{Z}^2)$. We get the solution for the games $G_\infty(\mathbf{p})$ as combinations of the solutions to games with two and three states obtained in sections 4 and 5. To realize the idea in sections 6 and 7 we construct symmetric representations of distributions over $\mathbb{R}^2$ with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values.

These representations are two-dimensional analogs of the following easily verified formula for distributions $\mathbf{p}$ over $\mathbb{R}^1$ with a mean value u:

$$\mathbf{p} = \int_{x=u^-}^{\infty} \mathbf{p}(dx) \int_{y=-\infty}^{u^+} \frac{x-y}{\int_{t=u}^{\infty}(t-u)\cdot \mathbf{p}(dt)} \cdot \mathbf{p}^u_{x,y} \cdot \mathbf{p}(dy),$$

where, for $y < u < x$, distributions $\mathbf{p}^u_{x,y} = ((x-u)\cdot\delta^y + (u-y)\cdot\delta^x)/(x-y)$, δ^x is the degenerate distribution with the single-point support x, and $\mathbf{p}^u_{x,u} = \mathbf{p}^u_{u,y} = \delta^u/2$. A draft of this construction can be seen in (Domansky, 2011).

In section 8 we construct Player 1' optimal strategy in a bidding game for two types of shares with an arbitrary distribution having an integer expectation vector (k, l), as a convex combination of his optimal strategies for such games with distributions having not more than three-point supports. If the state chosen by chance move is (k, l), then Player 1 stops the game. In this case he cannot receive any profit from his informational advantage.

If the state chosen by the chance move is $(x, y) \neq (k, l)$, then he chooses one or two complementary points by means of the lottery with the conditional probabilities of these complements.

He then plays his optimal strategy for the state (x, y) in a game with a distribution having either two- or three-point support that is the state (x, y) and the chosen complement.

We get the solutions for infinite games with arbitrary probability distributions over a two-dimensional integer lattice with finite component variances. Both players have optimal strategies. The optimal strategy for Player 2 is a direct combination of his optimal strategies for the games with one-type of risky asset. The value of such game is equal to the sum of values for corresponding games with one risky asset. Thus, the profit that Player 2 gets under simultaneous n-step bidding in comparison with separate bidding for each type of shares disappears in the game of unbounded duration.

As for the case with one-type of risky assets the appearance of a random walk of transaction prices is demonstrated. But the symmetry of this random walk is broken at the final stages of the game.

2 Repeated games with one-sided information modeling multistage bidding with two types of risky assets

We consider repeated games $G_n(\mathbf{p})$ with incomplete information on one side (Aumann and Maschler, 1995) modeling the bidding with two types of risky assets described in the introduction.

Two players with opposite interests have money and two types of shares. The liquidation prices of both share types may take any integer values x and y.

At stage 0 a chance move determines the "state of nature" s and therefore the liquidation prices of shares (s^1, s^2) for the whole period of bidding n according to the probability distribution $\mathbf{p}$ over $\mathbb{Z}^2$ known to both Players. Player 1 is informed about the result of chance move z, Player 2 is not. Player 2 knows that Player 1 is an insider.

At each subsequent stage $t = 1, \ldots, n$ both Players simultaneously propose their bids, meaning prices for one share of each type, $(i_t^1, i_t^2) \in \mathbb{Z}^2$ for Player 1 and $(j_t^1, j_t^2) \in \mathbb{Z}^2$ for Player 2. The bids are announced to both Players before proceeding to the next stage. The maximal bid wins and one share is transacted at this price. Therefore, if $i_t^e > j_t^e$, Player 1 gets one share of type $e = 1, 2$ from Player 2 and Player 2 receives the sum of money i_t^e from Player 1. If $i_t^e < j_t^e$, Player 2 gets one share of type e from Player 1 and Player 1 receives the sum j_t^e from Player 2. If $i_t^e = j_t^e$, then no transaction of shares of type e occurs. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

This n-stage model is described by a zero-sum repeated game $G_n(\mathbf{p})$ with incomplete information for Player 2 and with countable state space $S = \mathbb{Z}^2$ and with countable action spaces

$I = \mathbb{Z}^2$ and $J = \mathbb{Z}^2$. The one-step gain $a(s, i, j)$ of Player 1 corresponding to the state $s = (s^1, s^2)$ and the actions $i = (i^1, i^2)$ and $j = (j^1, j^2)$ is given with the sum $\sum_{e=1}^2 a^e(s^e, i^e, j^e)$, where

$$a^e(s^e, i^e, j^e) = \begin{cases} j^e - s^e, & \text{for } i^e < j^e; \\ 0, & \text{for } i^e = j^e; \\ -i^e + s^e, & \text{for } i^e > j^e. \end{cases}$$

At the end of the game Player 2 pays to Player 1 the sum

$$\sum_{t=1}^{n} a(s, i_t, j_t),$$

where s is the result of a chance move. This description is a common knowledge of both Players.

At the step t it is enough for both Players to take into account the sequence $(i_1, \ldots, i_{t-1})$ of Player 1's previous actions only. Thus, a mixed behavioral strategy σ for Player 1, who is informed on the state, is a sequence of moves

$$\sigma = (\sigma_1, \ldots, \sigma_t, \ldots),$$

where the move $\sigma_t = (\sigma_t(s))_{s \in S}$ and $\sigma_t(s) : I^{t-1} \to \Delta(I)$ is the probability distribution used by Player 1 to select his action at stage t, given the state s and previous observations. Here $\Delta(\cdot)$ is the set of probability distributions over $(\cdot)$.

A strategy τ for uninformed Player 2 is a sequence of moves

$$\tau = (\tau_1, \ldots, \tau_t, \ldots),$$

where $\tau_t : I^{t-1} \to \Delta(J)$.

Note that here we define infinite strategies fitting for games of arbitrary duration. A pair of strategies (σ, τ) creates a probability distribution $\Pi_{(\sigma,\tau)}$ over $(I \times J)^\infty$. The payoff function of the game $G_n(\mathbf{p})$ is

$$K_n(\mathbf{p}, \sigma, \tau) = \sum_{s \in S} \mathbf{p}(s) h_n^s(\sigma, \tau), \tag{2.1}$$

where

$$h_n^s(\sigma, \tau) = \mathbf{E}_{(\sigma,\tau)}[\sum_{t=1}^{n} a(s, i_t, j_t)] \tag{2.2}$$

is the s-component of the n-step vector payoff $h_n(\sigma, \tau)$ for the pair of strategies (σ, τ). Here the expectation is taken with respect to the probability distribution $\Pi_{(\sigma,\tau)}$. Thus we consider n-step games $G_n(\mathbf{p})$ with total (non-averaged) payoffs which differs from the classical model of Aumann and Maschler.

We also consider the infinite games $G_\infty(\mathbf{p})$. For certain pairs of strategies (σ, τ), the payoff function $K_\infty(\mathbf{p}, \sigma, \tau)$, given by the infinite series (2.1),(2.2) with $n = \infty$, may be indefinite. If we restrict the set of Player 1's admissible strategies to strategies with nonnegative one-step gains

$$\sum_{s\in S}\mathbf{p}(s)\mathbf{E}_{(\sigma_1(s),j)}a(s,i,j)$$

against any action j of Player 2, then the payoff function of the game $G_\infty^m(p)$ becomes completely definite (may be infinite). Player 1 has many strategies, ensuring him a nonnegative one-step gain against any action of Player 2. In fact, any reasonable strategy of Player 1 should possess this property.

For the initial probability $\mathbf{p}$, the strategy σ ensures the n-step payoff

$$w_n(\mathbf{p},\sigma)=\inf_\tau K_n(\mathbf{p},\sigma,\tau).$$

The strategy τ ensures the n-step vector payoff $\mathbf{h}_n(\tau)$ with components

$$h_n^s(\tau)=\sup_{\sigma(s)} h_n^s(\sigma(s),\tau).$$

Now we describe the recursive structure of $G_{n+1}(\mathbf{p})$. A strategy σ may be regarded as a pair $(\sigma_1,(\sigma(i))_{i\in I})$, where $\sigma_1(i|s)$ is the probability over I depending on s, and $\sigma(i)$ is a strategy depending on the first action $i_1=i$.

Analogously, a strategy τ may be regarded as a pair $(\tau_1,(\tau(i))_{i\in I})$, where τ_1 is the probability over J.

A pair $(\mathbf{p},\sigma_1)$ induces the probability distribution π over $S\times I$, $\pi(s,i)=\mathbf{p}(s)\sigma_1(i|s)$. Let

$$\mathbf{q}\in\Delta(I),\quad \mathbf{q}(i)=\sum_S\mathbf{p}(s)\sigma_1(i|s),$$

be the marginal distribution of π on I (total probabilities of actions), and let

$$\mathbf{p}(\cdot|i)\in\Delta(S),\quad \mathbf{p}(s|i)=\mathbf{p}(s)\sigma_1(i|s)/\mathbf{q}(i),$$

be the conditional probability on S given $i_1=i$ (a posterior probability).

Conversely, any set of total probabilities of actions $\mathbf{q}\in\Delta(I)$ and posterior probabilities $(\mathbf{p}(\cdot|i)\in\Delta(S))_{i\in I}$, satisfying the equality

$$\sum_{i\in I}\mathbf{q}(i)\mathbf{p}(\cdot|i)=\mathbf{p},$$

define a certain random move of Player 1 for the current probability $\mathbf{p}$. The posterior probabilities contain all information about the previous history of the game, that is essential for Player 1. Thus, to define a strategy for Player 1, it is sufficient to define the random move of Player 1 for any current posterior probability.

The following recursive representation for the payoff function corresponds to the recursive representation of strategies:

$$K_{n+1}(\mathbf{p},\sigma,\tau) = K_1(\mathbf{p},\sigma_1,\tau_1) + \sum_{i\in I} \mathbf{q}(i) K_n(\mathbf{p}(\cdot|i),\sigma(i),\tau(i)).$$

Let, for all $i \in I$, the strategy $\sigma(i)$ ensure the payoff $w_n(\mathbf{p}(\cdot|i),\sigma(i))$ in the game $G_n(\mathbf{p}(\cdot|i))$. Then the strategy $\sigma = (\sigma_1,(\sigma(i))_{i\in I})$ ensures the payoff

$$w_{n+1}(\mathbf{p},\sigma) = \min_{j\in J} \sum_{i\in I} [\sum_{s\in S} \mathbf{p}(s)\sigma_1(i|s)a(s,i,j) + \mathbf{q}(i) w_n(\mathbf{p}(\cdot|i),\sigma(i))]. \tag{2.3}$$

Let, for all $i \in I$, the strategy $\tau(i)$ ensure the vector payoff $\mathbf{h}_n(\tau(i))$. Then the strategy $\tau = (\tau_1,(\tau^n(i))_{i\in I})$ ensures the vector payoff $\mathbf{h}_{n+1}(\tau)$ with the components

$$h^s_{n+1}(\tau) = \max_{i\in I} \sum_{j\in J} \tau_1(j)(a(s,i,j) + h^s_n(\tau(i)) \quad \forall s \in S. \tag{2.4}$$

The game $G_n(\mathbf{p})$, where $n \in \mathbb{N} \cup \{\infty\}$, has a value $V_n(\mathbf{p})$ if

$$\inf_\tau \sup_\sigma K_n(\mathbf{p},\sigma,\tau) = \sup_\sigma \inf_\tau K_n(\mathbf{p},\sigma,\tau) = V_n(\mathbf{p}).$$

Players have optimal strategies σ^* and τ^* if

$$V_n(\mathbf{p}) = \inf_\tau K_n(\mathbf{p},\sigma^*,\tau) = \sup_\sigma K_n(\mathbf{p},\sigma,\tau^*),$$

or, as in the notation introduced above,

$$V_n(\mathbf{p}) = w_n(\mathbf{p},\sigma^*) = \sum_{s\in S} \mathbf{p}(s) h^s_n(\tau^*).$$

For $n \in \mathbb{N}$ the values $V_n(\mathbf{p})$ should satisfy Bellman optimality equations:

$$V_{n+1}(\mathbf{p}) = \inf_{\tau_1} \sup_{\sigma_1} [K_1(\mathbf{p},\sigma_1,\tau_1) + \mathbf{q}(i) V_n(\mathbf{p}(\cdot|i))]. \tag{2.5}$$

The value $V_\infty(\mathbf{p})$ should satisfy Bellman optimality equation:

$$V_\infty(\mathbf{p}) = \inf_{\tau_1} \sup_{\sigma_1} [K_1(\mathbf{p},\sigma_1,\tau_1) + \mathbf{q}(i) V_\infty(\mathbf{p}(\cdot|i))]. \tag{2.6}$$

For probability distributions $\mathbf{p}$ with finite supports, the games $G_n(\mathbf{p})$, being games with finite state and action spaces, have values $V_n(\mathbf{p})$. The functions V_n are continuous and concave in $\mathbf{p}$. Both players have optimal strategies $\sigma^*_n(\mathbf{p})$ and $\tau^*_n(\mathbf{p})$. The value of such a game does not exceed the sum

$$V_n(\mathbf{p}^1) + V_n(\mathbf{p}^2)$$

of values for games modeling the bidding with one-type shares, where $\mathbf{p^1}$ and $\mathbf{p^2}$ are the marginal distributions of the distribution $\mathbf{p}$. This follows from the fact that Player 2 can guarantee himself the loss that does not exceed this sum exploiting the direct combination of optimal strategies $\tau_n^*(\mathbf{p}^1)$ and $\tau_n^*(\mathbf{p}^2)$ for the single asset games $G_n(\mathbf{p}^1)$ and $G_n(\mathbf{p}^2)$ as a strategy for the two asset game $G_n(\mathbf{p})$.

Consider the set M^1 of probability distributions $\mathbf{p}$ with finite first moments

$$m_1^1[\mathbf{p}] = \sum_{s\in\mathbb{Z}^2} s^1 \cdot \mathbf{p}(s^1, s^2) < \infty; \quad m_1^2[\mathbf{p}] = \sum_{s\in\mathbb{Z}^2} s^2 \cdot \mathbf{p}(s^1, s^2) < \infty.$$

For $\mathbf{p} \in M^1$, the liquidation prices of both shares have finite expectations $\mathbf{E_p}[s^1] = m_1^1[\mathbf{p}]$, $\mathbf{E_p}[s^2] = m_1^2[\mathbf{p}]$. The set M^1 is a convex subset of the Banach space $L^1(\mathbb{Z}^2, \{|s^1| + |s^2|\})$ of mappings $\mathbf{l} : \mathbb{Z}^2 \to R^1$ with the norm

$$||\mathbf{l}||^1_{\{|s^1|+|s^2|\}} = \sum_{s\in\mathbb{Z}^2} \mathbf{l}(s^1, s^2) \cdot |s^1| + |s^2|.$$

Let $\mathbf{p}_1, \mathbf{p}_2 \in M^1$. Then, for "reasonable" strategies σ and τ,

$$|K_n(\mathbf{p}_1, \sigma, \tau) - K_n(\mathbf{p}_2, \sigma, \tau)| < n||\mathbf{p}_1 - \mathbf{p}_2||^1_{\{|s^1|+|s^2|\}}.$$

Therefore, the payoff of the game $G_n(\mathbf{p})$ with $\mathbf{p} \in M^1$ can be approximated using the payoffs of games $G_n(\mathbf{p}_k)$ with probability distributions $\mathbf{p}_k$ having finite support. The next theorem follows immediately from this fact.

Theorem 2.1. *If $\mathbf{p} \in M^1$, then the games $G_n(\mathbf{p})$ have values $V_n(\mathbf{p})$. The values $V_n(\mathbf{p})$ are positive and do not decrease, as the number of steps n increases.*

Remark 2.2. If the random variable $C_\mathbf{p}$ does not belong to L^2, then, as n approaches ∞, the sequence $V_n(\mathbf{p})$ diverges.

3 Upper bounds for values $V_n(\mathbf{p})$

Here we consider the set $M^2(\mathbb{Z}^2)$ of probability distributions $\mathbf{p} = (p(u,v))$ over the two-dimension integer lattice $\mathbb{Z}^2$ with finite second moments

$$m_u^2[\mathbf{p}] = \sum_{u,v=-\infty}^{\infty} u^2 \cdot p(u,v) < \infty, \quad m_v^2[\mathbf{p}] = \sum_{u,v=-\infty}^{\infty} v^2 \cdot p(u,v) < \infty.$$

The set M^2 is a closed convex subset of Banach space $L^1(\mathbb{Z}^2, \{u^2+v^2\})$ of mappings $\mathbf{l} : \mathbb{Z}^2 \to R^1$ with the norm

$$||\mathbf{l}|| = \sum_{u,v=-\infty}^{\infty} |l(u,v)|(u^2 + v^2).$$

For $\mathbf{p} \in M^2(\mathbb{Z}^2)$, the random variables u and v, determining the prices of shares, belong to L^2 and have finite variances

$$\mathbf{D_p}[u] = m_u^2[\mathbf{p}] - (m_u^1[\mathbf{p}])^2, \quad \mathbf{D_p}[v] = m_v^2[\mathbf{p}] - (m_v^1[\mathbf{p}])^2.$$

The main result of this section is that, for $\mathbf{p} \in M^2(\mathbb{Z}^2)$, the sequence $V_n(\mathbf{p})$ of values remains bounded as $n \to \infty$.

To prove this we define the set of infinite strategies $\tau^{(k,l)}$ of Player 2, suitable for the games $G_n(\mathbf{p})$ with arbitrary n.

Definition 3.1. *The first move $\tau_1^{(k,l)}$ is the action (k,l). For $t > 1$, the e-th component of the move $\tau_t^{(k,l)}$, $e = 1, 2$, depends on the last observed pair of e-th components of actions (i_{t-1}^e, j_{t-1}^e) for both players:*

$$j_t^e = \begin{cases} j_{t-1}^e - 1, & \text{if } i_{t-1}^e < j_{t-1}^e \ ; \\ j_{t-1}^e, & \text{if } i_{t-1}^e = j_{t-1}^e; \\ j_{t-1}^e + 1, & \text{if } i_{t-1}^e > j_{t-1}^e. \end{cases}$$

Proposition 3.2. *For the state $s = (u, v) \in \mathbb{Z}^2$ the strategy $\tau^{(k,l)}$ ensures the payoff*

$$\max_{\sigma} K_n^{a,b}(\sigma, \tau^{(k,l)} | (u, v)) \leq (u - k)(u - k - 1)/2 + (v - l)(v - l - 1)/2. \tag{2.1}$$

Proof. The strategy $\tau^{(k,l)}$ prescribes that Player 2 will operate separately with each of the assets. Hence Player 1 can do the same. Therefore the assertion follows from Proposition 1 of Domansky and Kreps (2009). This proves Proposition 3.2.

□

Set

$$H(\mathbf{p}) = 1/2 \cdot (\mathbf{D_p}[u] + \mathbf{D_p}[v] - \alpha(\mathbf{p})(1 - \alpha(\mathbf{p})) - \beta(\mathbf{p})(1 - \beta(\mathbf{p}))) \tag{3.2}$$

where $\alpha(\mathbf{p}) = \mathbf{E_p}[u] - ent[\mathbf{E_p}[u]]$, $\beta(\mathbf{p}) = \mathbf{E_p}[v] - ent[\mathbf{E_p}[v]]$ and $ent[x]$, $x \in R^1$ is the integer part of x.

$H(\mathbf{p})$ is a continuous, concave, and piecewise linear function over $M^2(\mathbb{Z}^2)$. The domains of linearity of function $H(\mathbf{p})$ are

$$L(k,l) = \{\mathbf{p} : \mathbf{E_p}[u] \in [k, k+1], \mathbf{E_p}[v] \in [l, l+1]\}, \quad (k,l) \in \mathbb{Z}^2.$$

Its peak points are

$$\Theta(k,l) = \{\mathbf{p} : \mathbf{E_p}[u] = k, \mathbf{E_p}[v] = l\}.$$

Theorem 3.3. *For $\mathbf{p} \in M^2(\mathbb{Z}^2)$, the values $V_n(\mathbf{p})$ are bounded from above by the function $H(\mathbf{p})$.*

For $\mathbf{p} \in L(k,l)$ the upper bound H is ensured with the strategy $\tau^{(k,l)}$. For $\mathbf{p} \in \Theta(k,l)$ the upper bound H is ensured with the strategies $\tau^{(k,l)}$, $\tau^{(k-1,l)}$, $\tau^{(k,l-1)}$, and $\tau^{(k-1,l-1)}$.

Proof. It follows from Proposition 3.2 that the following upper bound for $V_n(\mathbf{p})$ does not depend on n:

$$V_n(\mathbf{p}) \leq \min_{(k,l)} \frac{1}{2} \sum_{u,v=-\infty}^{\infty} ((u-k)(u-k-1) + (v-l)(v-l-1)) \cdot p(u,v) \tag{3.3}$$

Observe that, if $\mathbf{E}_{\mathbf{p}}[u] - k = \alpha$, $\mathbf{E}_{\mathbf{p}}[v] - l = \beta$, then

$$\frac{1}{2} \sum_{u,v=-\infty}^{\infty} ((u-k)(u-k-1) + (v-l)(v-l-1)) \cdot p(u,v)$$

$$= \frac{1}{2}(\mathbf{D}_{\mathbf{p}}[u] + \mathbf{D}_{\mathbf{p}}[v] - \alpha(\mathbf{p})(1-\alpha(\mathbf{p})) - \beta(\mathbf{p})(1-\beta(\mathbf{p}))).$$

Consequently, for $\mathbf{p} \in L(k,l)$ the minimum in formula (3.3) is attained on (k,l), and the equality (2.2) holds. In particular, for $\mathbf{p} \in \Theta(k,l)$, this minimum is attained on (k,l), $(k-1,l)$, $(k,l-1)$, and $(k-1,l-1)$.

□

Corollary 3.4. *The strategies $\tau^m, m = 0, 1, \dots$ guarantee the same upper bound $H(\mathbf{p})$ for the upper value of the infinite game $G_\infty(\mathbf{p})$.*

4 Solutions for games $G_\infty(\mathbf{p})$ with two states

In this section we show that, for games $G_\infty(\mathbf{p})$ with the support of distribution $\mathbf{p}$ containing two states $z_1, z_2 \in \mathbb{Z}^2$, the value $V_\infty(\mathbf{p})$ is equal to $H(\mathbf{p})$.

A distribution with the support $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ is uniquely determined with expectations fpr coordinates. For any point $w = (u,v) = z_1, z_2$, $p_i \in [0,1]$, $p_1 + p_2 = 1$, the distribution $\mathbf{p}^w_{z_1,z_2}$ such that $\mathbf{E}_{\mathbf{p}^w_{z_1,z_2}}[x] = u$, $\mathbf{E}_{\mathbf{p}^w_{z_1,z_2}}[y] = v$, is given with probabilities $\mathbf{p}^w_{z_1,z_2}(z_i) = p_i$.

Without loss of generality we assume that one of these points is $(0,0)$. Thus there are two states $0 = (0,0)$ and $z = (x,y)$, where x and y are integers and $x > 0$. The distribution $\mathbf{p}^{pz}_{z,0}$ can be depicted with a scalar parameter $p \in [0,1]$ – the probability of state z. For definiteness set $y > 0$.

Observe that the function $H(p)$ is equal to the sum of values

$$H(p) = V^x_\infty(p) + V^y_\infty(p) \tag{4.1}$$

of one asset games $G^x_\infty(p)$ and $G^y_\infty(p)$ considered in Domansky (2007).

The function $V_\infty^m(p)$ is a piecewise linear continuous concave function of $p \in [0,1]$. The set of its break points is the regular lattice $\{k/m, k = 0,\dots,m\}$ with values $V_\infty^m(k/m) = k(m-k)/2$. Therefore, for $p \in [k/m,(k+1)/m]$,

$$V_\infty^m(p) = (pm-k)(k+1)(m-k-1)/2 + (1-pm+k)k(m-k)/2$$
$$= k(m-k)/2 + (pm-k)(m-2k-1)/2. \tag{4.2}$$

For $p \in [(k-1)/m, k/m]$,

$$V_\infty^m(p) = k(m-k)/2 - (k-pm)(m-2k+1)/2. \tag{4.3}$$

Thus the function $H(p)$ is a piecewise linear continuous concave function of $p \in [0,1]$. The set of its break points is the irregular lattice $D(x,y) \subset [0,1]$:

$$D(x,y) = \{k/x, k = 0,\dots,x\} \cup \{l/y, l = 0,\dots,y\}.$$

Further we enumerate the points of the lattice $D(x,y)$ in ascending order $D(x,y) = \{p_i\}$, $i = 0,1,\dots,I$, $p_0 = 0$, $p_I = 1$, $p_i < p_{i+1}$.

According to Corollary 3.4 the optimal strategy τ^* guarantees to Player 2 the loss not exceeding the function $H(p)$. Therefore it is sufficient to show that there is an optimal strategy σ^* of Player 1 that guarantees him this gain at the break points of function $H(p)$, i.e. for the initial probability p belonging to the lattice $D(x,y)$.

Now we present a definition of first moves for the strategy σ^* for $p_i \in D(x,y)$.

Definition 4.1. *For any initial probability p_i the first move of the strategy σ^* makes use of two actions a_i^- and a_i^+.*
For $p_i = k/x \neq l/y$, these actions are $a_i^- = (k-1,l)$ and $a_i^+ = (k,l)$.
For $p_i = l/y \neq k/x$, these actions are $a_i^- = (k,l-1)$ and $a_i^+ = (k,l)$.
For $p_i = k/x = l/y$, these actions are $a_i^- = (k-1,l-1)$ and $a_i^+ = (k,l)$.
The posterior probabilities $p(z|a_i^-)$ and $p(z|a_i^+)$ are the left and the right adjacent points p_{i-1} and p_{i+1} of the lattice $D(x,y)$ respectively.
Consequently the total probabilities of actions are

$$q(a_i^-) = \frac{p_{i+1}-p_i}{p_{i+1}-p_{i-1}}, \quad q(a_i^+) = \frac{p_i-p_{i-1}}{p_{i+1}-p_{i-1}}.$$

This first move is realized with the following conditional probabilities of action a_i^+:

$$f^*(a_i^+|z) = \frac{(p_i-p_{i-1})p_{i+1}}{(p_{i+1}-p_{i-1})p_i}, \quad f^*(a_i^+|0) = \frac{(p_i-p_{i-1})(1-p_{i+1})}{(p_{i+1}-p_{i-1})(1-p_i)}.$$

As the posterior probabilities also belong to the lattice $D(x,y)$ this set of moves defines the infinite strategy σ^*. The defined strategy σ^* of Player 1 generates the asymmetric random walk of posterior probabilities of state z by adjacent points of the irregular lattice $D(x,y)$ with the probabilities of jumps that provide the martingale characteristics for posterior probabilities and with absorption at the extreme points $p_0 = 0$ and $p_I = 1$.

Theorem 4.2. *The value $V_\infty(p)$ of the game $G_\infty(p)$ with two states 0 and $z = (x,y)$, and with the probability p of the state z is equal to the function $H(p)$. Both players have optimal strategies.*
For the initial probability $p_i \in D(x,y)$, one of optimal strategies of Player 1 is the strategy σ^* of Definition 4.1.
For the initial probability $p \in (k/x,(k+1)/x) \cap (l/y,(l+1)/y)$ a unique optimal strategy of Player 2 is the strategy $\tau^* = \tau^{k,l}$, defined in Definition 3.1. Any optimal strategy for adjacent intervals is also optimal for points of the lattice $D(x,y)$.
Proof. At first we show that the one-step gain of Player 1 corresponding to the first move σ_1^* combined with the optimal gain H at the points of posterior probabilities generated by this move and weighted by total probabilities of actions satisfy Bellman optimality equations.

For $p_i = k/x \neq l/y$, the one-step gain of Player 1 corresponding to the first move σ_1^* in the game $G_\infty(0,z,p)$ is equal to his gain in the one-asset game $G_\infty^x(p)$

$$\min_{(k',l')} K_1(\sigma_1^*,(k',l')|0,z,p_i) = \min_{k'} K_1^{m_1}(\sigma_1^*,k'|p_i)$$
$$= \frac{x(p_{i+1}-p_i)(p_i-p_{i-1})}{p_{i+1}-p_{i-1}}. \tag{4.4}$$

Here the minimum in the left part is attained at $(k',l') = (k-1,l)$ and (k,l), and the minimum in the right part is attained at $k' = k-1$ k.

For this move, taking into account (4.2), (4.3), (4.4), we get

$$\min_{k'} K_1^x(\sigma_1^{x,y},k'|p_i) + q(k-1)V_\infty^x(p_{i-1}) + q(k)V_\infty^x(p_{i+1})$$
$$= \frac{x(p_{i+1}-p_i)(p_i-p_{i-1})}{p_{i+1}-p_{i-1}}$$
$$+\frac{p_{i+1}-p_i}{p_{i+1}-p_{i-1}}(k(x-k)/2 - x(p_i-p_{i-1})(x-2k+1)/2)$$
$$+\frac{p_i-p_{i-1}}{p_{i+1}-p_{i-1}}(k(x-k)/2 + x(p_{i+1}-p_i)(x-2k-1)/2)$$
$$= k(x-k)/2 = V_\infty^x(p_i). \tag{4.5}$$

Thus the Bellman optimality equation is fulfilled for a one-asset game. On the other hand, three points p_{i-1}, p_i and p_{i+1} are situated on the same linearity interval of function $V_\infty^y(p)$, i.e.

$$q(k-1)V_\infty^y(p_{i-1}) + q(k)V_\infty^y(p_{i+1}) = V_\infty^y(p_i). \tag{4.6}$$

Summing (4.5) and (4.6), and also taking into account (4.1) we obtain

$$\min_{(k',l')} K_1(\sigma_1^*, (k',l')|0, z, p_i) + q(k-1,l)H(p_{i-1}) + q(k,l)H(p_{i+1}) = H(p_i), \tag{4.7}$$

i.e., for $p_i = k/x \neq l/y$ and for the move σ_1^* in the game $G_\infty(0, z, p)$, function H satisfies the Bellman optimality equation.

For $p_i = l/y \neq k/x$, the proof of this fact is analogous with replacement of x and y.

For $p_i = k/x = l/y$, the Bellman optimality equations (4.5) are fulfilled for both one-asset games $G_\infty^x(p)$ and $G_\infty^y(p)$. Summing these optimality equations we obtain the optimality equation for the two-asset game $G_\infty(p)$.

Thus function H satisfies the Bellman optimality equation for all initial probabilities $p_i \in D(x, y)$. Iterating this optimality equation and taking into account the fact that a random walk of posterior probabilities generated by the strategy σ^* terminates in a finite mean number of steps, we see that, for the initial probability $p_i \in D(x, y)$, the strategy σ^* guarantees Player 1 the gain of $H(p_i)$.

□

5 Solutions for games $G_\infty(\mathbf{p})$ with three states

In this section we show that, for games $G_\infty(\mathbf{p})$ with the support of distribution $\mathbf{p}$ containing three states $z_1, z_2, z_3 \in \mathbb{Z}^2$, the value $V_\infty(\mathbf{p})$ coincides with $H(\mathbf{p})$.

We assume that three points

$$z_1 = (x_1, y_1), \quad z_2 = (x_2, y_2), \quad z_3 = (x_3, y_3), \quad z_1, z_2, z_3 \in \mathbb{Z}^2$$

are enumerated counterclockwise. It follows that, for $w \in \triangle(z_1, z_2, z_3)$, $\det[z_i - w, z_{i+1} - w] \geq 0$, where $\det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$. Notice that arithmetical operations with subscripts are fulfilled in modulo 3.

A distribution with the support z_1, z_2, z_3 is uniquely determined with expectations of coordinates. For any point $w = (u, v) \in \triangle(z_1, z_2, z_3)$ the distribution $\mathbf{p}^w_{z_1,z_2,z_3}$ such that

$$\mathbf{E}_{\mathbf{p}^w_{z_1,z_2,z_3}}[x] = u, \quad \mathbf{E}_{\mathbf{p}^w_{z_1,z_2,z_3}}[y] = v,$$

is given with probabilities

$$\mathbf{p}^w_{z_1,z_2,z_3}(z_i) = \frac{\det[z_{i+1} - w, z_{i+2} - w]}{\sum_{j=1}^3 \det[z_j - w, z_{j+1} - w]}. \tag{5.1}$$

Observe that $\sum_{j=1}^3 \det[z_j - w, z_{j+1} - w] = \det[z_1 - z_3, z_2 - z_3]$ does not depend on w.

According to Corollary 3.4 the optimal strategy τ^* guarantees Player 2 the loss not exceeding $H(\mathbf{p})$. It follows from Theorem 3.2 that, for $\mathbf{p}^w_{z_1,z_2,z_3}$ with $w=(u,v)$ belonging to the boundary of the triangle $\triangle(z_1,z_2,z_3)$, the equality $V_\infty(\mathbf{p}^w_{z_1,z_2,z_3}) = H(\mathbf{p}^w_{z_1,z_2,z_3})$ holds. For other points $w=(u,v)\in\triangle(z_1,z_2,z_3)$, the function $H(\mathbf{p}^w_{z_1,z_2,z_3})$ is the least concave majorant of its values at the points $\mathbf{p}^w_{z_1,z_2,z_3}$ with $w=(u,v)\in\mathbb{Z}^2$ and at the boundary of $\triangle(z_1,z_2,z_3)$. Therefore this is sufficient to show that there is a strategy σ^* for Player 1 that guarantees him $H(\mathbf{p}^w_{z_1,z_2,z_3})$, for $w=(u,v)\in\mathbb{Z}^2$.

For the point $w=(u,v)\in\mathbb{Z}^2$ that belongs to the triangle $\triangle(z_1,z_2,z_3)$

$$H(\mathbf{p}^w_{z_1,z_2,z_3}) = \frac{1}{2}\Big(\sum_{i=1}^{3}(x_i^2+y_i^2)\mathbf{p}^w_{z_1,z_2,z_3}(z_i) - (u^2+v^2)\Big). \tag{5.2}$$

For $\mathbf{p}^w_{z_1,z_2,z_3}$ with $w=(u,v)\in\mathbb{Z}^2$, the first step of strategy σ^* may efficiently use the actions $(u-1,v-1)$, $(u,v-1)$, $(u-1,v)$ and (u,v). With the help of these actions Player 1 can perform moves such that the modulus of difference between the posterior expectations of each coordinate and its initial expectation is not more than one.

There are several types of optimal first moves for Player 1. In particular, the first moves σ_1^{NE-SW} (north-east – south-west), σ_1^{NW-SE}, and their probabilistic mixtures. Denote $e=(1,1), \bar{e}=(1,-1)$. The first move σ_1^{NE-SW} exploits only two actions $w-e$ and w with posterior expectations $w-b\cdot e$ and $w+a\cdot e$. The first move σ_1^{NW-SE} makes use of actions $(u-1,v)$ and $(u,v-1)$ with posterior expectations $w-b\bar{e}$ and $w+a\bar{e}$.

Further we define the first move σ_1^{NE-SW} both in terms of posterior expectations and in terms of the conditional probabilities of actions. We assume w.l.o.g. that $w=0\in\triangle(z_1,z_2,z_3)$. The span of this move is defined with a mutual disposition of the points $-e$, e and the triangle $\triangle(z_1,z_2,z_3)$. If $z_i=k\cdot e$ for some $i=1,2,3,\ k>0$, then put $a=1$. If $z_i=k\cdot -e$ for some $i=1,2,3,\ k>0$, then put $b=1$.

If $z_i\neq k\cdot e,\ i=1,2,3,\ k>0$, then there is a unique $i=i^+$ such that the half-line starting at 0 and passing through e crosses the side z_{i^+}, z_{i^++1} of the triangle $\triangle(z_1,z_2,z_3)$. If $z_i\neq k\cdot -e$, $i=1,2,3,\ k>0$, then there is a unique $i=i^-\neq i^+$ such that the half-line starting at 0 and passing through $-e$ crosses the side z_{i^-}, z_{i^-+1}. Put

$$a=\min\Big(\frac{\det[z_{i^+}, z_{i^++1}]}{\det[e, z_{i^++1}-z_{i^+}]},1\Big), \quad b=\min\Big(\frac{\det[z_{i^-}, z_{i^-+1}]}{\det[-e, z_{i^-+1}-z_{i^-}]},1\Big).$$

Definition 5.1. *The first move σ_1^{NE-SW} for the game $G_\infty(\mathbf{p}^0_{z_1,z_2,z_3})$ makes use of actions $-e$ and 0. The posterior expectations are*

$$\mathbf{E}_p[z|-e] = -b\cdot e, \quad \mathbf{E}_p[z|0] = a\cdot e.$$

The total probabilities of actions are

$$q(e) = a/(b+a), \quad q(0) = b/(b+a).$$

This move is realized with the conditional probabilities of actions:

$$f^*(-e|z_i) = \frac{a\det[z_{i+1} + b\cdot e, z_{i+2} + b\cdot e]}{(b+a)\det[z_{i+1}, z_{i+2}]}, \quad i = 1, 2, 3;$$

$$f^*(0|z_i) = \frac{b\det[z_{i+1} - a\cdot e, z_{i+2} - a\cdot e]}{(b+a)\det[z_{i+1}, z_{i+2}]}, \quad i = 1, 2, 3.$$

Remark. The martingale of posterior expectations generated by the optimal strategy of Player 1 is a symmetric random walk over the adjacent points of the lattice $\mathbb{Z}^2$ disposed inside the triangle $\triangle(z_1, z_2, z_3)$. The symmetry of this random walk is broken at the moment when it hits the triangle boundary. Beginning from this moment the game degenerates into one of two-point games with the distribution support being either z_{i^+}, z_{i^++1}, or z_{i^-}, z_{i^-+1}.

If $a < 1$, then after observing the action 0 the next game is $G_\infty(\mathbf{p}^{ae}_{z_{i^+}, z_{i^++1}})$ with the probabilities of states

$$p(z_{i^+}) = \frac{\det[e, z_{i^++1}]}{\det[e, z_{i^++1} - z_{i^+}]}, \quad p(z_{i^++1}) = \frac{\det[z_{i^+}, e]}{\det[e, z_{i^++1} - z_{i^+}]}.$$

If $b < 1$, then after observing the action $-e$ the next game is $G_\infty(\mathbf{p}^{-be}_{z_{i^-}, z_{i^-+1}})$ with the probabilities of states

$$p(z_{i^-}) = \frac{\det[e, z_{i^-+1}]}{\det[e, z_{i^-+1} - z_{i^-}]}, \quad p(z_{i^-+1}) = \frac{\det[z_{i^-}, e]}{\det[e, z_{i^-+1} - z_{i^-}]}.$$

Theorem 5.2. *The value $V_\infty(\mathbf{p}^0_{z_1,z_2,z_3})$ of the game $G_\infty(\mathbf{p}^0_{z_1,z_2,z_3})$ is equal to the function $H(\mathbf{p})$ given by (3.2). Both players have optimal strategies.*
The optimal strategy for Player 2 is given by Definition 3.1.
For $w = (u, v) \in \mathbb{Z}^2$, one of optimal strategies of Player 1 is the strategy σ^* of Definition 4.1.
Proof. Taking into account Corollary 3.4 and Theorem 4.2 this is sufficient to show that the one-step gain corresponding to the first move σ_1^{NE-SW} of the optimal strategy for Player 1 combined with the gain $H(\mathbf{p})$ at the points of posterior probabilities generated by this move and weighted by total probabilities of actions satisfies Bellman optimality equations.

The best replies of Player 2 to the first move σ_1^{NE-SW} are actions 0, $-e$, $(-1, 0)$, and $(0, -1)$. The corresponding one-step gain of Player 1 is equal to $2ab/(b + a)$. In fact,

$$K_1(\sigma_1^{NE-SW}, 0|\mathbf{p}^0_{z_1,z_2,z_3}) = -q(-e)\mathbf{E}_p[x + y| - e] = 2ab/(b + a);$$

$$K_1(\sigma_1^{NE-SW}, -e|\mathbf{p}^0_{z_1,z_2,z_3}) = q(0)\mathbf{E}_p[x + y|0] = 2ab/(b + a).$$

For actions $(0, -1)$ and $(-1, 0)$ of Player 2 the proof is analogous.

It follows from (5.1) and (5.2) that

$$H(\mathbf{p}^0_{z_1,z_2,z_3}) = \frac{\sum_{i=1}^3 (x_i^2 + y_i^2)\det[z_{i+1}, z_{i+2}]}{2\det[z_1 - z_3, z_2 - z_3]};$$

$$H(\mathbf{p}^{ae}_{z_1,z_2,z_3}) = H(\mathbf{p}^{0}_{z_1,z_2,z_3}) - a\frac{\sum_{i=1}^{3}(x_i^2+y_i^2)\det[z_{i+1}-z_{i+2},e]}{2\det[z_1-z_3,z_2-z_3]} - a;$$

$$H(\mathbf{p}^{-be}_{z_1,z_2,z_3}) = H(\mathbf{p}^{0}_{z_1,z_2,z_3}) + b\frac{\sum_{i=1}^{3}(x_i^2+y_i^2)\det[z_{i+1}-z_{i+2},e]}{2\det[z_1-z_3,z_2-z_3]} - b.$$

We get

$$2ab/(b+a) + q(-e)H(\mathbf{p}^{-be}_{z_1,z_2,z_3}) + q(0)H(\mathbf{p}^{ae}_{z_1,z_2,z_3}) = H(\mathbf{p}^{0}_{z_1,z_2,z_3}),$$

i.e., for $\mathbf{p}^{0}_{z_1,z_2,z_3}$ and for the move σ_1^{NE-SW} in the game $G_\infty(\mathbf{p}^{0}_{z_1,z_2,z_3})$, function H satisfies Bellman optimality equation.

□

6 Decompositions of univariate distributions as patterns for bivariate distributions

We now consider the games $G_\infty(\mathbf{p})$ with prices given by arbitrary probability distributions $\mathbf{p} \in \Delta(\mathbb{Z}^2)$. We get the solution for the games $G_\infty(\mathbf{p})$ as combinations of the solutions of games with two and three states that were obtained in sections 4 and 5. To study this idea, in sections 6 and 7 we construct symmetric representations of distributions over $\mathbb{R}^2$ with the given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values.

We investigate the set $\mathbf{P}(\mathbb{R}^2)$ of probability distributions $\mathbf{p}$ over the plane $\mathbb{R}^2 = \{z = (x, y)\}$ with finite first absolute moments

$$\int_{\mathbb{R}^2} |x| \cdot \mathbf{p}(dz) < \infty, \quad \int_{\mathbb{R}^2} |y| \cdot \mathbf{p}(dz) < \infty.$$

We denote mean values of the distribution $\mathbf{p}$ by $\mathbf{E}_\mathbf{p}[x]$ and $\mathbf{E}_\mathbf{p}[y]$:

$$\mathbf{E}_\mathbf{p}[x] = \int_{\mathbb{R}^2} x \cdot \mathbf{p}(dz) < \infty, \quad \mathbf{E}_\mathbf{p}[y] = \int_{\mathbb{R}^2} y \cdot \mathbf{p}(dz) < \infty.$$

We construct symmetric representations of convex sets of distributions with given mean values

$$\Theta(u, v) = \{\mathbf{p} \in \mathbf{P}(\mathbb{R}^2) : \mathbf{E}_\mathbf{p}[x] = u, \mathbf{E}_\mathbf{p}[y] = v\},$$

as convex hulls of their extreme points, which are distributions with supports containing not more than three points and with the same mean values. For extreme points of convex sets of distributions with the given moments see Winkler (1988).

As a pattern we take the symmetric representation of one-dimensional probability distributions over the integer lattice that was used in Domansky and Kreps (2009) for an analysis of bidding models with single-type asset. Let $\mathbf{p}$ be a probability distribution over the set of integers $\mathbb{Z}^1$ with zero mean value. Then

$$\mathbf{p} = \mathbf{p}(0)\cdot\delta^0 + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{k+l}{\sum_{t=1}^{\infty} t\cdot\mathbf{p}(t)}\mathbf{p}(-l)\mathbf{p}(k)\cdot\mathbf{p}^0_{k,-l}, \tag{6.1}$$

where $\mathbf{p}^0_{k,-l}$ is the probability distribution with the support $\{-l,k\}$ and with zero mean value. The representation (1) allows to reduce solving models with prices of assets given by arbitrary probability distributions over $\mathbb{Z}^1$ to solving such models with two-point distributions.

Formula (6.1) becomes more transparent if we take into account the equality

$$\sum_{t=1}^{\infty} t\cdot\mathbf{p}(t) = \frac{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}(s+t)\mathbf{p}(-t)\mathbf{p}(s)}{1-\mathbf{p}(0)} \tag{6.2}$$

Coefficients of decomposition (6.1) take a "symmetric" form

$$\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{k,-l}) = (1-\mathbf{p}(0))\frac{(k+l)\mathbf{p}(k)\mathbf{p}(-l)}{\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}(s+t)\mathbf{p}(-t)\mathbf{p}(s)}.$$

We mean just this form of coefficients saying that the representation (6.1) is symmetric. We aim for constructing an analogous representation of bivariate probability distributions.

We treat coefficients $\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{k,-l})$ of decomposition (6.1) as probabilities of extreme distributions with two-point supports. The probability of such distribution is proportional to the span of its support and to the probabilities of both support points. Choosing a point in accordance with a distribution $\mathbf{p}$ can be understood by means of two-step lottery: the first step chooses an extreme distribution and the second step chooses a point in its support. This treatment allows us to calculate the conditional probabilities of complementary points given one point $-l$ or k in a support of extreme distribution. We get

$$\mathbf{P}_{\mathbf{p}}(k|-l) = \frac{k\cdot\mathbf{p}(k)}{\sum_{t=1}^{\infty} t\cdot\mathbf{p}(t)}, \quad \mathbf{P}_{\mathbf{p}}(-l|k) = \frac{l\cdot\mathbf{p}(-l)}{\sum_{t=1}^{\infty} t\cdot\mathbf{p}(t)}. \tag{6.3}$$

Thus the conditional probability is the same for all given points on a half-line. It is proportional to the point probability and to its distance from the origin. This property is characteristic for this decomposition. In fact, if $\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{k,-l})$ is a probability distribution such that (6.3) is fulfilled, then

$$\mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{k,-l}\cap\{k\}) = \mathbf{P}_{\mathbf{p}}(k|-l)\cdot\mathbf{p}(-l) = \mathbf{P}_{\mathbf{p}}(\mathbf{p}^0_{k,-l})\frac{k}{k+l},$$

and therefore we get (6.1). Formula (6.1) can be written as

$$\mathbf{p} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{k+l}{\sum_{t=1}^{\infty} t\cdot\mathbf{p}(t)}\mathbf{p}(-l)\mathbf{p}(k)\cdot\mathbf{p}^0_{k,-l},$$

if we put $\mathbf{p}^0_{k,0} = \mathbf{p}^0_{0,-l} = \delta^0/2$.

This formula can be easily generalized for probability distributions over the set of real numbers $\mathbb{R}^1$ with the zero mean value. Namely

$$\mathbf{p} = \int_{x=0^-}^{\infty} \mathbf{p}(dx) \int_{y=0^-}^{\infty} \frac{x+y}{\int_{t=0}^{\infty} t \cdot \mathbf{p}(dt)} \cdot \mathbf{p}^0_{x,-y} \cdot \mathbf{p}(-dy), \tag{6.4}$$

where, for $x, y > 0$, the distributions $\mathbf{p}^0_{x,-y} = (x \cdot \delta^{-y} + y \cdot \delta^x)/(x+y)$, and $\mathbf{p}^0_{x,0} = \mathbf{p}^0_{0,-y} = \delta^0/2$.

Formula (6.4) occurs in the offspring of Skorokhod representation (Skorokhod,1961) for a sequence of sums of independent centered random variables by means of a Brownian motion stopped at random times (see the survey of Obloj, 2004).

One of the steps of Skorohod's proof consists of a demonstration that any centered probability distribution on the real line can be disintegrated into centered distributions supported at two points each. Skorokhod employs another decomposition formula that works only for distributions with continuous distribution functions. In fact, the formula (6.4) can be employed as well.

Any centered probability distribution $\mathbf{p}$ on the real line can be represented as the distribution of the random variable $w(\tau)$, where $w(t), t \geq 0, w(0) = 0$ is a Brownian motion, the stopping time τ is the minimal root of the equation $(w(t) - \chi)(w(t) + \psi) = 0$, and the random vector $(\chi, \psi) \in \mathbb{R}^2_+$ is distributed with probabilities

$$\mathbf{P}\{(\chi, \psi) \in dx \times dy\} = \frac{x+y}{\int_{t=0}^{\infty} t \cdot \mathbf{p}(dt)} \cdot \mathbf{p}(dx)\mathbf{p}(-dy).$$

Obloj indicates that for the first time formula (6.4) was used in this context in the works of Hall (1968, 1969). Kallenberg attributes employing the same formula to Chung (see Kallenberg (1997), the proof of Lemma 12.4 in Chapter 12).

7 Decompositions of bivariate centered distributions

Here we construct symmetric representations of convex sets of distributions with given mean values

$$\Theta(u, v) = \{\mathbf{p} \in \mathbf{P}(\mathbb{R}^2) : \mathbf{E}_{\mathbf{p}}[x] = u, \mathbf{E}_{\mathbf{p}}[y] = v\},$$

as convex hulls of their extreme points. This is sufficient to provide Decomposition for the set $\Theta(0,0)$ of centered distributions. Extreme points of the set $\Theta(0,0)$ are the degenerate distribution δ^0 with the single-point support $0 = (0,0)$, distributions $\mathbf{p}^0_{z_1,z_2} \in \Theta(0,0)$ with two-point supports (z_1, z_2), and distributions $\mathbf{p}^0_{z_1,z_2,z_3} \in \Theta(0,0)$ with three-point supports (z_1, z_2, z_3).

The distribution $\mathbf{p}^0_{z_1,z_2} \in \Theta(0,0)$ with the two-point support $\{z_1, z_2\}$ such that $(0,0)$ belongs to the interval (z_1, z_2), i.e. $z_1 = ae_\psi$, $z_2 = -be_\psi$ where e_ψ is a unit vector with $\arg e_\psi = \psi$, $a, b \in \mathbb{R}^1_+$, is given by

$$\mathbf{p}^0_{ae_\psi, -be_\psi} = \frac{b \cdot \delta^{ae_\psi} + a \cdot \delta^{-be_\psi}}{a+b},$$

The distribution $\mathbf{p}^0_{z_1,z_2,z_3} \in \Theta(0,0)$ with the support $\{z_1, z_2, z_3\}$ such that $(0,0)$ belongs to the interior of the triangle $\triangle(z_1, z_2, z_3)$ is given by

$$\mathbf{p}^0_{z_1,z_2,z_3} = \frac{\sum_{i=1}^3 \det[z_{i+1}, z_{i+2}] \cdot \delta^{z_i}}{\sum_{j=1}^3 \det[z_j, z_{j+1}]},$$

where $\det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$. All arithmetical operations with subscripts are fulfilled modulo 3. Using polar coordinates $z_i = (r_i, \varphi_i)$, $z_{i+1} = (r_{i+1}, \varphi_{i+1})$ we get $\det[z_i, z_{i+1}] = r_i \cdot r_{i+1} \sin(\varphi_{i+1} - \varphi_i)$. Thus $|\det[z_i, z_{i+1}]|$ is equal to the area of the parallelogram spanned over the vectors z_i and z_{i+1}. If the points z_i, z_{i+1} are indexed counterclockwise, then $\det[z_i, z_{i+1}] \geq 0$.

Consider the set Δ^0 of non-ordered triples (z_1, z_2, z_3) that form triangles containing the point $(0,0)$:

$$\Delta^0 = \{(z_1, z_2, z_3), z_i \neq (0,0) : (0,0) \in \triangle(z_1, z_2, z_3)\}.$$

The set Δ^0 is manifold with a boundary. Its interior $\mathrm{Int}\Delta^0$ is the set of triples $(z_1, z_2, z_3) \in \Delta^0$ such that $(0,0)$ belongs to the interior of the $\triangle(z_1, z_2, z_3)$. Its boundary $\partial\Delta^0$ is the set of triples $(z_1, z_2, z_3) \in \Delta^0$ such that $(0,0)$ belongs to the boundary of the $\triangle(z_1, z_2, z_3)$.

If $(z_1, z_2, z_3) \in \partial\Delta^0$, then there is an index i such that $\det[z_i, z_{i+1}] = 0$. In this case $\arg z_{i+1} = \arg z_i + \pi (\mathrm{mod}\ 2\pi)$, the point $(0,0) \in [z_i, z_{i+1}]$ and the distribution $\mathbf{p}^0_{z_1,z_2,z_3}$ degenerates into the distribution $\mathbf{p}^0_{z_i,z_{i+1}}$ with the support $\{z_i, z_{i+1}\}$.

For $\psi \in [0, 2\pi)$, let R_ψ be the half-line $R_\psi = \{z : \arg z = \psi (\mathrm{mod}\ 2\pi)\}$. With each value $\psi \in [0, 2\pi)$ we associate the set $\Delta^0(\psi)$ of non-ordered couples

$$\Delta^0(\psi) = \{(z_1, z_2), z_i \neq (0,0) : \forall z \in R_\psi \quad (0,0) \in \triangle(z_1, z_2, z)\}.$$

Let $\mathrm{Int}\Delta^0(\psi)$ and $\partial\Delta^0(\psi)$ be the sets of non-ordered couples (z_1, z_2) such that, for $z \in R_\psi$, the triple (z_1, z_2, z) belongs to $\mathrm{Int}\Delta^0$ and to $\partial\Delta^0$ respectively. We take, that points (z_1, z_2) are indexed counterclockwise. This implies $\det[z_1, z_2] \geq 0$.

Using polar coordinates $z_1 = (r_1, \varphi_1)$, $z_2 = (r_2, \varphi_2)$ we get

$$\mathrm{Int}\Delta^0(\psi) = \{((r_1, \varphi_1), (r_2, \varphi_2)) : \psi < \varphi_1 < \pi + \psi, \pi + \psi < \varphi_2 < \pi + \varphi_1 (\mathrm{mod}\ 2\pi)\}. \tag{7.1}$$

The set $\partial\Delta^0(\psi)$ can be naturally represented as a conjunction of three non-intersecting sets:

$$\partial_1\Delta^0(\psi) = \{((r_1, \varphi_1), (r_2, \varphi_2)) : \psi < \varphi_1 < \pi + \psi, \varphi_2 = \pi + \psi (\mathrm{mod}\ 2\pi)\}.$$

In other words, the set of couples such that $(0,0)$ belongs to the side (z_2, z),

$$\partial_2\Delta^0(\psi) = \{((r_1, \varphi_1), (r_2, \varphi_2)) : \varphi_1 = \pi + \psi, \pi + \psi < \varphi_2 < 2\pi + \psi (\mathrm{mod}\ 2\pi)\},$$

i.e. the set of such couples that $(0,0)$ belongs to the side (z_1, z), and

$$\partial_3\Delta^0(\psi) = \{((r_1, \varphi_1), (r_2, \varphi_2)) : \psi < \varphi_1 < \pi + \psi, \varphi_2 = \pi + \varphi_1 (\mathrm{mod}\ 2\pi)\},$$

i.e. the set of couples such that $(0,0)$ belongs to the side (z_1,z_2) of the $\triangle(z_1,z_2,z)$.

Now we introduce the value that plays the role of $\int_{t=0}^{\infty} t\cdot\mathbf{p}(dt)$, for symmetric representations of distributions over $\mathbb{R}^2$. Set

$$\Phi(\mathbf{p},\psi)=\int_{\mathrm{Int}\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)+1/2\int_{\partial\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2). \tag{7.2}$$

Using polar coordinates $z_1=(r_1,\varphi_1)$, $z_2=(r_2,\varphi_2)$ and taking into account (7.1) we get

$$\mathrm{Int}\Phi(\mathbf{p},\psi)=\int_{\mathrm{Int}\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)$$

$$=\int_{\varphi_1=\psi^+}^{\pi+\psi^-}\int_{r_1=0^+}^{\infty}\mathbf{p}(dr_1d\varphi_1)\int_{\varphi_2=\pi+\psi^+}^{\pi+\varphi_1^-}\int_{r_2=0^+}^{\infty}r_1\cdot r_2\cdot\sin(\varphi_2-\varphi_1)\mathbf{p}(dr_2d\varphi_2).$$

Consider the term

$$\partial\Phi(\mathbf{p},\psi)=\frac{1}{2}\int_{\partial\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)=\frac{1}{2}\sum_{i=1}^{3}\int_{\partial_i\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2).$$

For the set $\partial_3\Delta^0(\psi)$ the integrand is equal to zero. The integrals over the sets $\partial_1\Delta^0(\psi)$ and $\partial_2\Delta^0(\psi)$ differ from zero only if the measure $\mathbf{p}(R_{\psi+\pi})$ is more than zero. In this case

$$\int_{\partial_1\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)=\int_{R_{\psi+\pi}}r_2\mathbf{p}(dr_2)\cdot\int_{Hp_\psi}\det[e_\psi,z_1]\mathbf{p}(dz_1),$$

$$\int_{\partial_2\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)=\int_{R_{\psi+\pi}}r_1\mathbf{p}(dr_1)\cdot\int_{Hp_{\psi+\pi}}\det[z_2,e_\psi]\mathbf{p}(dz_2),$$

where e_ψ is a unit vector with $\arg e_\psi=\psi$, and Hp_φ is the half-plane

$$Hp_\varphi=\{z:\arg z\in(\varphi,\varphi+\pi)(\mathrm{mod}\ 2\pi)\}.$$

As $\mathbf{p}\in\Theta(0,0)$

$$\int_{Hp_\psi}\det[e_\psi,z_1]\mathbf{p}(dz_1)=\int_{Hp_{\psi+\pi}}\det[z_2,e_\psi]\mathbf{p}(dz_2),$$

and

$$\int_{\partial_1\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)=\int_{\partial_2\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2).$$

Thus

$$\partial\Phi(\mathbf{p},\psi)=\int_{\partial_2\Delta^0(\psi)}\det[z_1,z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)$$

$$=\int_{R_{\psi+\pi}}r_1\mathbf{p}(dr_1)\cdot\int_{\varphi_2=\pi+\psi^+}^{\pi+\varphi_1^-}\int_{r_2=0^+}^{\infty}r_2\cdot\sin(\varphi_2-\psi-\pi)\mathbf{p}(dr_2d\varphi_2).$$

Taking this and (7.1) into account we get

$$\Phi(\mathbf{p},\psi)=\int_{\varphi_1=\psi^+}^{\pi+\psi^+}\int_{r_1=0^+}^{\infty}\mathbf{p}(dr_1d\varphi_1)\int_{\varphi_2=\pi+\psi^+}^{\pi+\varphi_1^-}\int_{r_2=0^+}^{\infty}r_1\cdot r_2\cdot\sin(\varphi_2-\varphi_1)\mathbf{p}(dr_2d\varphi_2).$$

The next fact produces the base for constructing symmetric representations of distributions over $\mathbb{R}^2$.

Theorem 7.1. *For any distribution $\mathbf{p} \in \Theta(0,0)$ the quantity $\Phi(\mathbf{p},\psi)$ does not depend on ψ, i.e. this is an invariant $\Phi(\mathbf{p})$ of the distribution $\mathbf{p} \in \Theta(0,0)$.*

Remark 7.2. This theorem is a two-dimensional analog of the equality

$$\int_{t=0}^{\infty} t \cdot \mathbf{p}(dt) = \int_{t=0}^{\infty} t \cdot \mathbf{p}(-dt)$$

that holds for $\mathbf{p} \in \Theta(0) \subset \mathbf{P}(\mathbb{R}^1)$.

Corollary. *For any distribution $\mathbf{p} \in \Theta(0,0)$ the quantity $\Phi(\mathbf{p})$ has the following invariant representation that is a bivariate analog of formula (6.2):*

$$\Phi(\mathbf{p}) = \frac{(\int_{Int\Delta^0} + 1/2 \int_{\partial\Delta^0}) \sum_{j=1}^{3} \det[z_j, z_{j+1}] \mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3)}{1 - \mathbf{p}(0,0)}.$$

Now we will formulate the preliminary variant of the decomposition theorem for bivariate distributions.

Proposition 7.3. *Any distribution $\mathbf{p} \in \Theta(0,0)$ has the following symmetric decomposition into a convex combination of distributions with no more than three-point supports:*

$$\mathbf{p} = \mathbf{p}(0,0) \cdot \delta^0 + \int_{Int\Delta^0} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3} \mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3)$$

$$+1/2 \int_{\partial\Delta^0} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3} \mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3), \tag{7.3}$$

where $\Phi(\mathbf{p})$ is given by (7.2).

The last term of decomposition (7.3) contains all distributions $\mathbf{p}^0_{z_i,z_{i+1}}$ with two-point supports (z_i, z_{i+1}), where $z_i \in R_\psi$ and $z_{i+1} \in R_{\psi+\pi}$. In order for such combination of points to appear with nonzero probability, it is necessary that the measure $\mathbf{p}(R_\psi)$ and the measure $\mathbf{p}(R_{\psi+\pi})$ be greater than zero. This is possible for no more than a countable set $\Psi(\mathbf{p})$ of values ψ.

These considerations make possible the final formulation of the principal Theorem:

Theorem 7.4. *Any probability distribution $\mathbf{p} \in \Theta(0,0)$ has the following symmetric representation as a convex combination of distributions with one-, two-, and three-point supports:*

$$\mathbf{p} = \mathbf{p}(0,0) \cdot \delta^0 + \int_{Int\Delta^0} \frac{\sum_{j=1}^{3} \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3} \mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3)$$

$$+ \sum_{\Psi(\mathbf{p})} \frac{\partial\Phi(\mathbf{p},\psi)}{\Phi(\mathbf{p})} \int_{R_\psi} \int_{R_{\psi+\pi}} \frac{r_1 + r_2}{\int_{R_{\psi+\pi}} t\mathbf{p}(dt)} \mathbf{p}^0_{(r_1,\psi),(r_2,\psi+\pi)} \mathbf{p}(dr_2)\mathbf{p}(dr_1). \tag{7.4}$$

Note that the extension of this methodology to higher dimensions is absolutely straightforward for centered distributions over $\mathbb{R}^n$ that certainly do not include distributions with less than $(n+1)$-point supports in their decomposition. These are distributions without linear subspaces of non-zero measure, or, if there is such subspace, then this subspace has a half-subspace of zero measure.

Any centered distribution over $\mathbb{R}^n$ can be reduced to this form by subtracting a distribution that include only distributions with less than $(n+1)$-point supports in its decomposition.

8 Constructing optimal strategies for Player 1

In this section we construct optimal strategies for Player 1 making use of the decomposition for the initial distribution $\mathbf{p}$ developed above.

The coefficients of decomposition may be treated as probabilities of corresponding extreme distributions with not more than three-point supports. The choice of a point on the two-dimensional integer lattice in accordance with the distribution $\mathbf{p}$ can be realized by means of the two-step lottery: the first step chooses an extreme distribution and the second step chooses a point in its support. This treatment allows us to calculate the conditional probabilities of extreme distributions (i.e. one or two complementary points) given one point $(x, y) \neq (k, l)$ in the support of extreme distribution. These conditional probabilities turn to be the same for all points of any ray starting at (k, l).

Consequently, the following algorithm gives the optimal strategy for Player 1:

1. If the state chosen by chance move is $(0, 0)$, then Player 1 stops the game.

2. Let the state chosen by chance move be $z \neq (0, 0)$, and let $z = k \cdot w$, where $k \in \mathbb{N}$ and $w = (u, v)$ with (u, v) being a relatively prime pair of integers. Then Player 1 realize the Bernoulli trial with probabilities

$$\frac{\partial\Phi(\mathbf{p}, w)}{\Phi(\mathbf{p})}, \quad 1 - \frac{\partial\Phi(\mathbf{p}, w)}{\Phi(\mathbf{p})} = \frac{\mathrm{Int}\Phi(\mathbf{p}, w)}{\Phi(\mathbf{p})},$$

to choose between two-point and three-point distributions.

c) If two-point distributions are chosen, then Player 1 chooses a point $z_2 = -lw$ by means of lottery with probabilities

$$\frac{l \cdot \mathbf{p}(-lw)}{\sum_{t=1}^{\infty} t \cdot \mathbf{p}(-tw)}$$

and plays the optimal strategy $\sigma^*(\cdot|z)$ for the state $z = kw$ in the two-point game $G(\mathbf{p}^0_{kw,-lw})$.

d) If three-point distributions are chosen, then Player 1 chooses a pair of points z_2, z_3 by means of lottery with probabilities

$$\frac{\det[z_2, z_3]\mathbf{p}(z_2)\mathbf{p}(z_3)}{\mathrm{Int}\Phi(\mathbf{p}, w)}.$$

and plays the optimal strategy $\sigma^*(\cdot|z)$ for the state $z = z_1$ in the three-point game $G(\mathbf{p}^0_{z_1,z_2,z_3})$.

As the optimal strategies σ^* ensure Player 1 the gains equal to one half of the sum of component variances $\mathbf{D}_\mathbf{p}[u] + \mathbf{D}_\mathbf{p}[v]$ in the two and three-point games with $\mathbf{p} \in \Theta(k, l)$, and as the sum of component variances is a linear function over $\Theta(k, l) \cap M^2$, where M^2 is the class of distributions with finite second moment, we obtain the following result:

Theorem 8.1. *For any distribution $\mathbf{p} \in \Theta(k, l) \cap M^2$ the compound strategy depicted above ensures that Player 1 will gain $1/2 \cdot (\mathbf{D}_\mathbf{p}[u] + \mathbf{D}_\mathbf{p}[v])$ in the game $G(\mathbf{p})$.*

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