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THE DIFFERENCE BETWEEN MANIPULABILITY INDEXES IN IC AND IANC MODELS

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THE DIFFERENCE BETWEEN MANIPULABILITY INDEXES IN IC AND IANC MODELS[2]

We consider the problem of manipulability of social choice rules in the impartial anonymous and neutral culture model (IANC) and provide a new theoretical study of the IANC model, which allows us to analytically derive the difference between the Nitzan-Kelly index in the Impartial Culture (IC) and IANC models. We show in which cases this difference is almost zero, and in which the Nitzan-Kelly index for IANC is the same as for IC. However, in some cases this difference is large enough to cause changes in the relative manipulability of social choice rules. We provide an example of such cases.

JEL Classification: C6, D7.

Keywords: anonymity, neutrality, IC, IANC, manipulability.

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1 Introduction

Procedures aggregating individual preferences into a collective choice differ in their vulnerability to manipulation. We say that manipulation can occur if any voter can achieve a better voting result for himself by misrepresenting his preferences. Gibbard (1973) and Satterthwaite (1975) started detailed research into the problem of manipulability. They proved that any non-dictatorial social choice rule with at least three possible outcomes is manipulable. Satterthwaite provided the definition of a strategy-proof procedure, explaining that it is a voting scheme in which no manipulation can occur. These studies have given rise to a number of extensions and generalizations of the Gibbard-Satterthwaite theorem.

Following Gibbard and Satterthwaite, Barbera (1977) later studied the possibility of constructing a satisfactory social choice procedure, proving that a social choice rule that satisfies the unanimity condition and does not leave "too much" to chance must be either uniformly manipulable or dictatorial.

The problem of manipulability was widely investigated by J. Kelly. In Kelly (1977) it was proved that, without an assumption of single-valuedness, rules that satisfy both non-dictatorship and strategy-proofness could exist. However, Duggan and Schwartz (2000) proved that, assuming three or more alternatives, non-manipulability and non-dictatorship are inconsistent with citizens' sovereignty and residual resoluteness. Citizens' sovereignty means that the social choice rule could produce any alternative as a result. Residual resoluteness assumes that a rule does not produce ties if all preferences are the same (say, x above y) or if just one voter deviates (putting y above x).

The next problem in studying manipulability was to compare social choice procedures in their vulnerability to manipulation. The first approach is measuring the probability that in a randomly chosen preference profile manipulation is possible. Since it was introduced in Nitzan (1985) and Kelly (1988), we call this measure the Nitzan-Kelly's index. The latter also considers an approach that takes into account the number of profiles where manipulation is very unlikely to occur, although still possible. In Kelly (1993) the first method was developed and supported by computational results on the relative manipulability of social choice rules.

Aleskerov and Kurbanov (1999) and Aleskerov et al (2011) continued this line of research. The first paper contains the results of computational experiments that reveal the degree of manipulability of social choice rules. In addition, the authors introduced some new indexes for evaluating manipulability. In Aleskerov et al (2011), which is fundamental to this study, manipulability is studied in two ways. It extends the number of voters in the computational experiment and uses different methods of expanding preferences.

All the listed articles focus on individual manipulations under impartial culture assumption. The impartial culture model was introduced in Guilbaud (1952). This model assumes that a set of all preference profiles is used for generating voters' preferences. Another important probabilistic model is the impartial anonymous culture model (IAC), first described in Kuga and Nagatani (1974) and Gehrlein and Fishburn (1976). The question of manipulability of social choice rules in the IAC model was thoroughly investigated by Pritchard and Wilson (2007), Lepelley and Valognes (2002), Favardin and Lepelley (2006), and Slinko (2006). These four publications are devoted to the study of coalitional manipulations.

In this paper, we consider the impartial anonymous and neutral culture model (IANC), in which both names of voters and names of alternatives do not matter. In this model, some preference profiles are regarded as equivalent in terms of permutations of individuals and alternatives. Therefore, the set of all preference profiles splits up into equivalence classes. The first investigation of this model was started in Egecioglu (2005) and extended in Egecioglu and Giritligil (2009). They introduced a way of calculating the number of anonymous and neutral equivalence classes and an algorithm for their uniform random generation. However, this model has not been thoroughly analyzed yet. Particularly, a way of analyzing the difference of indexes in IC and IANC without conducting a computational experiment has not been investigated in the literature.

In the IC model, the Nitzan-Kelly's index is a proportion of manipulated profiles in the set of all preference profiles. In the IANC model we consider not profiles, but equivalence classes, and the Nitzan-Kelly's index in IANC is a proportion of manipulated equivalence classes. The reason why we consider such a difficult model as IANC is that every sensible social choice rule satisfies both anonymity and neutrality, it means that any two preference profiles that differ in permutation of voters and (or) names of alternatives will be both either manipulable or not with respect to those rules. We can regard an equivalence class as a type of group preference, so, considering only representatives of equivalence classes, we do not count preference profiles of the same type twice.

The difficulty is that computational experiments in the IANC model have rather high complexity, and the number of equivalence classes is still very large (see Tab. 1 below). The algorithm for generating representatives of equivalence classes was introduced in Egecioglu (2005). However, we should know whether the results of computational experiments in IANC would differ from those in the basic IC model. Then, if this difference is not zero, could it significantly influence the relative manipulability of social choice rules?

Using combinatorial methods and elements of group theory, we derive the difference of indexes in IANC and IC models. We study properties of equivalence classes with maximal and minimal number of elements and evaluate the maximal difference of probabilistic measures for the number of voters and alternatives up to 10.

Tab. 1. The number of anonymous and neutral equivalence classes (first row of each cell) and the number of preference profiles (second row)

		The number of voters					
		3	**4**	**5**	**6**	**7**	**8**
The number of alternatives	**3**	10	24	42	83	132	222
		216	1296	7776	46656	279936	1679616
	4	111	762	4095	19941	84825	329214
		13824	331776	7962624	191102976	$4.586*10^{9}$	$1.1*10^{11}$
	5	2467	76044	1876255	39096565	703593825	$1.117*10^{10}$
		1728000	$2.073*10^{8}$	$2.488*10^{10}$	$2.986*10^{12}$	$3.583*10^{14}$	$4.3*10^{16}$

We show that for an exact number of voters and alternatives this difference is almost zero and, consequently, any probabilistic measure in the IANC model will be equal to those in the IC model. At the same time, this difference could be large enough to cause changes in the relative manipulability of social choice rules. We provide an example of such a situation and compute the manipulability indexes for four social choice rules in IC and IANC for the case of three alternatives. We compare the relative manipulability of these rules and the difference of indexes for each rule in both models. After that, we explain it in terms of the anonymous and neutral culture model.

2 Definitions, notions, and theoretical basis

In this paper we use the notions for the impartial anonymous and neutral culture model introduced in Egecioglu (2005). First, there is a set of alternatives A, consisting of m elements, and a set of individuals (or voters) $N = \{1, 2, ..., n\}$ with n elements. A preference profile is defined as a matrix consisting of n vectors that represent voters' preferences by ordering m alternatives. The preference profile is $\vec{P} = \{P_1, P_2, ..., P_n\}$, and preference of the i-th individual is P_i, a linear order.

The total number of different preference profiles is $(m!)^n$. Impartial culture model assumes that each voter selects his or her preferences out of $m!$ possible linear orders and each of $(m!)^n$ preference profiles is equally likely. The set of all preference profiles with n voters and m alternatives is denoted by $\Omega(m, n)$.

As mentioned above, in the impartial anonymous and neutral culture model there is no difference between voters and between the names of alternatives. For example, in this model the following three profiles are considered as the same representation of preferences:

$$\vec{P} = \begin{array}{cc} \underline{P_1} & \underline{P_2} \\ y & y \\ x & z \\ z & x \end{array} \qquad \vec{P}' = \begin{array}{cc} \underline{P_1'} & \underline{P_2'} \\ x & x \\ y & z \\ z & y \end{array} \qquad \vec{P}'' = \begin{array}{cc} \underline{P_1''} & \underline{P_2''} \\ y & y \\ z & x \\ x & z \end{array}.$$

Therefore, we have a partition of $\Omega(m, n)$ into anonymous and neutral equivalence classes (ANECs). Any preference profile from a given ANEC can be taken as the representative profile (or root). In other words, ANEC is a set of preference profiles that can be generated from each other by permuting voters' preferences and renaming alternatives.

The permutation of voters (or columns) is denoted as σ, an element of a symmetric group S_n, and the permutation on the set of alternatives is $\tau \in S_m$. The pair of permutations σ and τ is denoted by $g = (\sigma, \tau)$. $G = S_n \times S_m$ is the group of all permutations $g = (\sigma, \tau)$, and it acts on the set of all preference profiles. There are $n!$ permutations of voters and $m!$ permutations of alternatives and, therefore, the number of permutations $g = (\sigma, \tau)$ is

$$|G| = n!m!.$$

A partition λ of n is defined as a weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_z)$, such that $\lambda_1 + \lambda_2 + ... + \lambda_\alpha = n$, where λ_i is called a part of λ. For example, (3,2,1,1) is a partition of 7 into 4 parts. The type of partition is denoted by $1^{\alpha_1} 2^{\alpha_2} ... n^{\alpha_n}$, which means that a partition λ has α_i parts of size *i* for each *i* from 1 to *n*.

Each permutation can be represented via cycle decomposition. Therefore, σ defines a partition λ of *n*, and τ defines a partition μ of *m* in such a way that parts of partitions λ and μ are the lengths of cycles in σ and τ, respectively. The sum $\alpha = \alpha_1 + \alpha_2 + ...$ is the total number of cycles in permutation σ. For any partition λ we define a number

$$z_\lambda = 1^{\alpha_1} 2^{\alpha_2} ... n^{\alpha_n} \alpha_1! \alpha_2! ... \alpha_n!.$$

The set of all permutations of a given cycle type $1^{\alpha_1} 2^{\alpha_2} ... n^{\alpha_n}$, is called a conjugacy class. Thus, the number of permutations in a conjugacy class is evaluated as

$$z_\lambda^{-1} n!.$$

The image of a profile $\vec{P}$ under the permutation $g = (\sigma, \tau)$ is denoted by $\vec{P}^g$. Anonymous and neutral equivalence class $\theta_{\vec{P}}$ can be defined as a subset of Ω: $\{\vec{P}^g \mid g \in G\}$. Profiles $\vec{P}_1, \vec{P}_2$ are regarded as equivalent if there exists a permutation $g \in G$ such that $\vec{P}_1^g = \vec{P}_2$.

If for a given permutation g there exists a profile $\vec{P}$, such that $\vec{P}^g = \vec{P}$ then $\vec{P}$ is called a fixed-point of g. The set of all fixed points for g is

$$F_g = \{P \in \Omega \mid P^g = P\}.$$

For a given profile $\vec{P}$ a set of all permutations that do not change $\vec{P}$ is a subgroup of G and is called a stabilizer of $\vec{P}$. This set is defined as

$$G_{\vec{P}} = \{g \in G \mid \vec{P}^g = \vec{P}\}.$$

The number of elements in the orbit of $\vec{P}$ (or its equivalence class) can be evaluated as a ratio

$$|\theta_{\vec{P}}| = |G| / |G_{\vec{P}}|.$$

An indicator function $\chi(S)$ of statement S is defined as follows

$$\chi(S) = \begin{cases} 1, & if \quad S \quad is \quad True, \\ 0, & if \quad S \quad is \quad False. \end{cases}$$

$GDC(\lambda)$ is the greatest common divisor of the parts of λ, $LCM(\lambda)$ is a least common multiple of the parts of λ. The number of anonymous and neutral equivalence classes for n voters and m alternatives was found in Egecioglu (2005). It is given by

$$R(m,n) = \sum_{\mu} z_{\mu}^{-1} \begin{pmatrix} \frac{n}{d} + \frac{m!}{d} - 1 \\ \frac{m!}{d} - 1 \end{pmatrix},$$

where $d = LCM(\mu)$ and binomial coefficient for an integer k, $0 \le k \le x$ is defined as

$$\binom{x}{k} = C_x^k = \begin{cases} \frac{x!}{k!(x-k)!}, x \in \mathbb{N}, \\ 0, x \notin \mathbb{N}. \end{cases}$$

For n and $m!$ being relatively prime the number of equivalence classes

$$R(m,n) = \frac{1}{m!} \binom{n + m! - 1}{m! - 1}.$$

In addition, we give some definitions on manipulability. The preference profile where all individuals express their true preferences except the i-th individual is denoted by $\vec{P}_{-i} = \{P_1, ..., P_{i-1}, P_i', P_{i+1}, ..., P_n\}$. P_i' is the deviation of the i-th individual from his true preferences P_i.

The social choice (or the outcome of aggregating procedure) with respect to the profile $\vec{P}$ is denoted by $C(\vec{P})$. As in Aleskerov et al. (2010), the case of multiple choice is considered. That means that the result of an aggregating procedure might consist of several elements.

Consider a preference profile $\vec{P}"$ from the example provided above. How can one decide, what is better for the first individual: the set $\{x, y, z\}$ or $\{z\}$? To answer this question, several methods of expanding preferences are proposed. In this study, Leximin and Leximax, two lexicographic methods introduced in Pattanaik (1978), are considered.

On the basis of a linear order representing voter's preferences on the set of alternatives, expanded preferences order all the subsets of A. In the *Leximin* method, the worst alternatives of two sets are compared, and the set where the better alternative is contained is considered as the better set. If they are the same, then the second-worst alternatives are compared and so on. In the *Leximax* method of expanding preferences, the best alternatives are compared, then the second-best alternatives and so on. EP_i denotes the expanded preferences of individual *i*.

For example, if voter i prefers alternative x to alternative y and alternative y to alternative *z*, then, according to the *Leximin* method,

$$\{x\}EP_i\{x,y\}EP_i\{y\}EP_i\{x,z\}EP_i\{x,y,z\}EP_i\{y,z\}EP_i\{z\}.$$

According to *Leximax*

$$\{x\}EP_i\{x,y\}EP_i\{x,y,z\}EP_i\{x,z\}EP_i\{y\}EP_i\{y,z\}EP_i\{z\}.$$

In the case of multiple choice manipulation is defined as follows: if for individual *i* $C(\vec{P}_{-i})EP_iC(\vec{P})$, then manipulation takes place.

The Nitzan-Kelly's index of manipulability is the ratio

$$NK = \frac{d_0}{(m!)^n},$$

where d_0 is the number of profiles in which manipulation is possible.

In the IANC model this index is calculated over the set of roots of equivalence classes

$$NK_{IANC} = \frac{r_0}{R(m,n)},$$

where r_0 is the number of roots in which manipulation is possible, and $R(m,n)$ is the total number of roots.

3 Anonymous and neutral equivalence classes

In this section, we reveal some properties of anonymous and neutral equivalence classes in order to evaluate the maximal difference of indexes in the IC and IANC models. We discuss the problem of difference in terms of manipulability, but all these results are applicable to the study of any other probability in the IC and IANC models. First, we consider what properties cause this difference. Let us consider a hypothetical example of a set Ω consisting of ten preference profiles. Assume that there are four ANECs: two classes of cardinality 2, one class of cardinality 5, and the last one has only one preference profile.

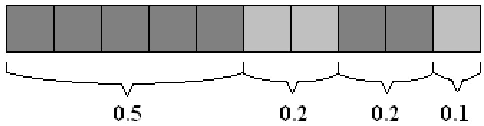

Fig. 1. A hypothetical example of four equivalence classes.

We can assume that only profiles from the biggest equivalence class are manipulable. Consequently, the manipulability index in the IC model is 0.5, while in the IANC model this index is equal to 0.25 because only 1 of 4 equivalence classes is manipulable. So, we can see that this difference results from an inequality of equivalence classes. In the IANC model all equivalence classes are equally likely.

Therefore, the manipulability index in IC exceeds the index in IANC if the average cardinality of equivalence classes that are manipulable exceeds the average cardinality of all equivalence classes. On the contrary, the manipulability index is less in IC than in IANC, if the average cardinality of equivalence classes that are manipulable is less than the average cardinality of all equivalence classes.

To start with, we consider equivalence classes that have the least and the greatest cardinality.

Theorem 1 (*Anonymous and neutral equivalence class with the minimal number of elements*) The minimal number of elements in an anonymous and neutral equivalence class is $m!$. This class is unique for the case of $n \geq 3$.

Next, we determine the cardinality of maximal equivalence class. For any preference profile belonging to this class, the number of stabilizing permutations is minimal. For some *m* and *n*, there exist preference profiles that have a stabilizer consisting only of an identity permutation.

Theorem 2 (*Anonymous and neutral equivalence class with the maximal number of elements*) If $m! > n$, then the maximal number of elements in an anonymous and neutral equivalence class is $m!n!$.

In contrast to minimal equivalence classes, there could be more than one maximal equivalence class. Evaluating the exact number of such classes is a rather difficult computational and combinatorial problem and requires some specific calculations in most cases. However, we can estimate this number by an interval, which is, surprisingly, not only rather small, but also tends to zero as *m* and *n* grow.

First, let us denote the set of preference profiles consisting of different columns by $\tilde{\Omega}$. The number of equivalence classes on this set is denoted by $\tilde{R}(m,n)$. Finally, $\tilde{F}_g$ is a set of fixed points $\vec{P}$ from $\tilde{\Omega}$,

$$\tilde{F}_g = \{\vec{P} \in \tilde{\Omega} \mid \vec{P}^g = \vec{P}\}.$$

Lemma 3 The number of fixed-points from $\tilde{\Omega}$ for some permutation $g = (\sigma, \tau)$ is equal to

$$\left|\tilde{F}_g\right| = \begin{cases} \prod_{j=0}^{\alpha} (m! - j \cdot t), & if \quad \lambda_1 = \lambda_2 = \ldots = \lambda_\alpha = t, \\ 0, & otherwise. \end{cases}$$

where $t = LCM(\mu)$.

Since the preference profiles in a maximal equivalence class in the case $m! > n$ always consist of different columns, our next step is to evaluate the number of ANECs on $\tilde{\Omega}$.

Theorem 4 For any *m* and *n* such that $m! > n$, the number of equivalence classes on $\tilde{\Omega}$ is equal to

$$\tilde{R}(m,n) = \sum_{\lambda} \sum_{\mu} z_\lambda^{-1} z_\mu^{-1} \chi(S(\lambda,\mu)) \prod_{j=0}^{\alpha-1} (m! - j \cdot t),$$

where $S(\lambda,\mu) = (\lambda_1 = \lambda_2 = \ldots = \lambda_\alpha = t)$.

Theorem 4 allows us to make an important corollary concerning maximal equivalence classes.

Corollary 5 For any *m* and *n* such that $m! > n$ a), the number of maximal ANEC satisfies the following inequality

$$\frac{2(m!-1)!}{(m!-n)!n!} - \tilde{R}(m,n) \le R_{\max}(m,n) \le \tilde{R}(m,n).$$

b) If m and n are such that $n > m$ and n is a prime number, then the number of maximal ANEC is equal to $\tilde{R}(m,n)$.

4 Evaluating the difference of the Nitzan-Kelly's index

In this section we apply theoretical results to the problem of evaluating the maximal difference of manipulability indexes. As mentioned earlier, the inequality of ANECs' cardinality causes this difference. We illustrate this in the following diagram. On the x-axis are 24 ANECs for a case with 3 alternatives and 4 voters. The value on the y-axis corresponds to the number of elements in the ANEC. The line on this figure displays the average number of elements in the classes.

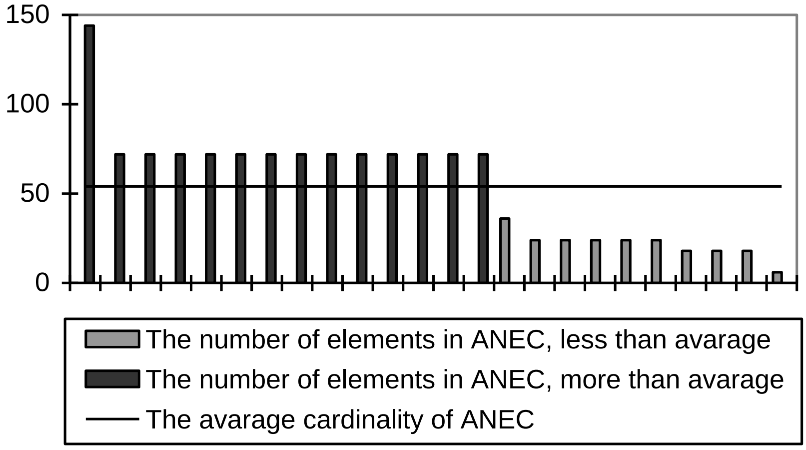

Fig. 2. The set of ANEC for 3 alternatives and 4 voters.

We have already concluded that the manipulability index in IC exceeds the index in IANC when the average cardinality of equivalence classes that are manipulable exceeds the average cardinality of all equivalence classes. On the other hand, the difference between the index in the IC and IANC models is negative when the average cardinality of equivalence classes that are manipulable is less than the average cardinality of all equivalence classes. Consequently, the absolute value of difference is maximal when all the classes θ, such that $|\theta|>|\theta_{av}|$, and only they are either manipulable or not manipulable.

In general, the maximal difference between manipulability indexes in the IC and IANC models is calculated as follows

$$\max \Delta_{IANC} = \left| \frac{d^*}{(m!)^n} - \frac{r^*}{R(m,n)} \right|,$$

where d^* is the number of profiles in all equivalences classes θ, such that $|\theta|>|\theta_{av}|$ (or $|\theta|<|\theta_{av}|$), and r^* is the number of such classes.

We continue to consider the case of $m!>n$ and suggest evaluating this difference by calculating the number of classes with cardinality that exceeds the average. This approach is justified by the fact that for small n the only classes such that $|\theta|>|\theta_{av}|$ are the classes with a maximal number of elements. Thus, the difference is calculated as

$$\max \Delta_{IANC} = \left| \frac{R_{\max}(m,n)}{R(m,n)} - \frac{R_{\max}(m,n)\cdot m!n!}{(m!)^n} \right|.$$

However, for a certain value of n the second maximal cardinality of ANECs also begins to exceed the average. To evaluate this value, we can use the following approximation. For simplicity, the number of equivalence classes is calculated as in the case of n and $m!$ being relatively prime

$$R(m,n) = \frac{1}{m!}\binom{n+m!-1}{m!-1}.$$

Then the average number of elements in ANEC is

$$\frac{(m!)^n}{\frac{1}{m!}\binom{n+m!-1}{m!-1}}.$$

The number of elements in ANEC, which is k times less than the maximal class, is

$$\frac{m!n!}{k}.$$

Such n that satisfies the inequality

$$\frac{(m!)^n}{\frac{1}{m!}\binom{n+m!-1}{m!-1}} < \frac{m!n!}{k},$$

which is simplified to

$$k\cdot(m!)^n < \frac{(n+m!-1)!}{(m!-1)!},$$

is denoted by n_k. In other words, n_k is the number of voters for which the cardinality of ANEC that is k times less than the maximal class begins to exceed the average cardinality of equivalence classes.

For example, if $m=3$, then $n_2=4$; if $m=4$, then $n_2=7$; if $m=5$, then $n_2=14$; and if $m=6$, $n_2=33$. Thus, when $n<n_2$, it is enough to know the cardinality and the number of maximal ANEC to evaluate the maximal difference of manipulability indexes in IC and IANC.

Using Corollary 5, we get the difference in the case of $m! > n$ estimated by the interval

$$\left(\frac{2(m!-1)!}{(m!-n)!n!} - R_d(m,n)\right) \cdot \left|\frac{1}{R(m,n)} - \frac{n!}{(m!)^{n-1}}\right| \le \max \Delta_{IANC} \le R_d(m,n) \cdot \left|\frac{1}{R(m,n)} - \frac{n!}{(m!)^{n-1}}\right|.$$

And an exact value of the maximal difference for m and n such that $n > m$ and n is a prime number

$$\max \Delta_{IANC} = R_{\max}(m,n) \cdot \left|\frac{1}{R(m,n)} - \frac{n!}{(m!)^{n-1}}\right|.$$

The following figures illustrate the behavior of the difference for the number of alternatives from 3 to 10 and the same number of voters. For the case of three and four alternatives, this difference is large enough to cause changes in the relative manipulability of social choice rules. At the same time, for the case of six or more alternatives, it becomes so small and insignificant that we can take indexes in the IC model equal to those in the IANC model. However, we should also take into account that this difference increases up to a certain value, which is to be calculated in future research.

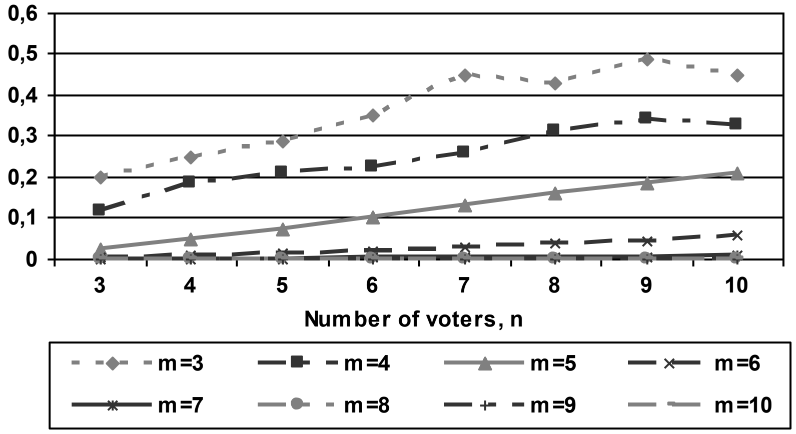

Fig. 3. Maximal difference of indexes in the IC and IANC models.

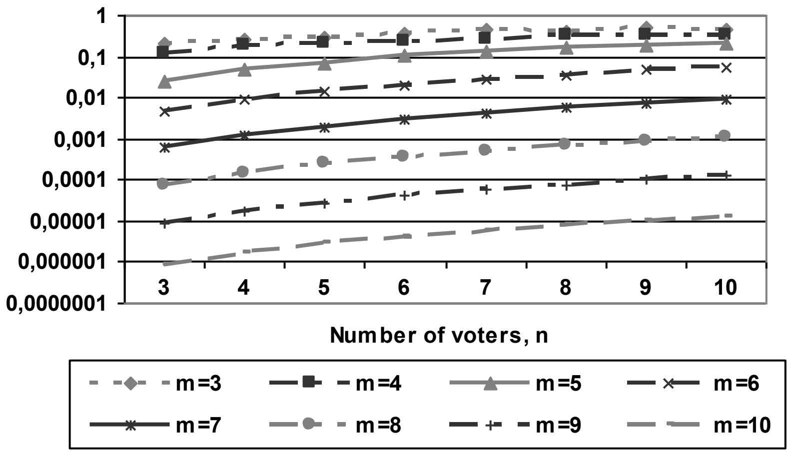

Fig. 4. Maximal difference of indexes in the IC and IANC models, log scale.

5 Manipulability of social choice rules in IC and IANC models

Using the results of the theoretical study from the previous section, we calculate the maximal difference of the Nitzan-Kelly's indexes in the IC and IANC models and compare it with the actual difference of this index for four social choice rules. First, we give a formal definition of these rules.

1. Plurality Rule. This rule chooses the best alternative for the maximal number of voters.

$$a \in C(\vec{P}) \Leftrightarrow [\forall x \in A \quad n^{+}(a,\vec{P}) \geq n^{+}(x,\vec{P})]$$

where $n^{+}(a,\vec{P}) = card\{i \in N \mid \forall y \in A \quad aP_i y\}$.

2. Approval Voting. Social choice is an alternative at the place of q or higher in the preferences of the maximal number of voters.

$$a \in C(\vec{P}) \Leftrightarrow [\forall x \in A \quad n^{+}(a,\vec{P},q) \geq n^{+}(x,\vec{P},q)]$$

where $n^{+}(a,\vec{P}) = card\{i \in N \mid \forall y \in A \quad aP_i y\}$.

3. Borda's Rule. For each alternative in the i-th preference,s the number $r_i(x,\vec{P})$ is counted calculated as follows

$$r_i(x,\vec{P}) = card\{b \in A : xP_i b\}.$$

The sum of $r_i(x,\vec{P})$ over all $i \in N$ is called a Borda's count.

$$r(a,\vec{P}) = \sum_{i=1}^{n} r_i(a,\vec{P}).$$

Borda's rule chooses an alternative with the maximal Borda's count.

$$a \in C(\vec{P}) \Leftrightarrow [\forall b \in A, \quad r(a,\vec{P}) \geq r(b,\vec{P})].$$

4. Black's procedure. Chooses a Condorset winner, if it exists, and, if it does not exist, the winner of Borda's rule.

We compute the Nitzan-Kelly's indexes both in impartial culture and impartial anonymous and neutral culture model using the Leximin and Leximax method of expanding preferences in 3-alternatives voting. After that, we calculate the difference of these indexes

$$\Delta K_{IANC} = \frac{d_0}{(m!)^n} - \frac{r_0}{R(m,n)},$$

Fig. 5. represents the differences calculated for the Leximin method (for graphs representing the Nitzan-Kelly's index for the Leximax method, see Appendix B). The maximal difference is represented by the lowest and the highest border on figure 5. As can be seen from this graph, the difference is negative only for approval voting rule. This fact can be explained as follows: preference profiles in which manipulation is possible often belong to equivalence classes with a small number of elements.

The plurality rule has the highest level of difference for $3 \le n \le 10$. These two facts cause the changes in the relative manipulability of social choice rules. Figures 6 and 7 illustrate the behavior of the Nitzan-Kelly's index in IC and IANC. The approval rule turns out to be the most manipulable in the IANC model, while under the IC assumption it is the second least manipulable rule. The relative position of the plurality rule changed in the opposite direction. However, Black's procedure is the least manipulable in both cultures.

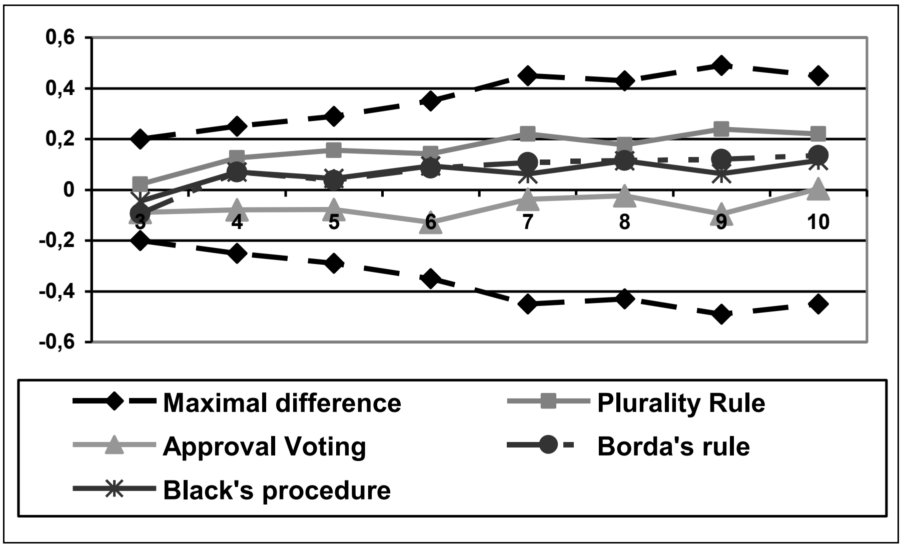

Fig. 5. The difference of the Nitzan-Kelly's index in IC and IANC, Leximin

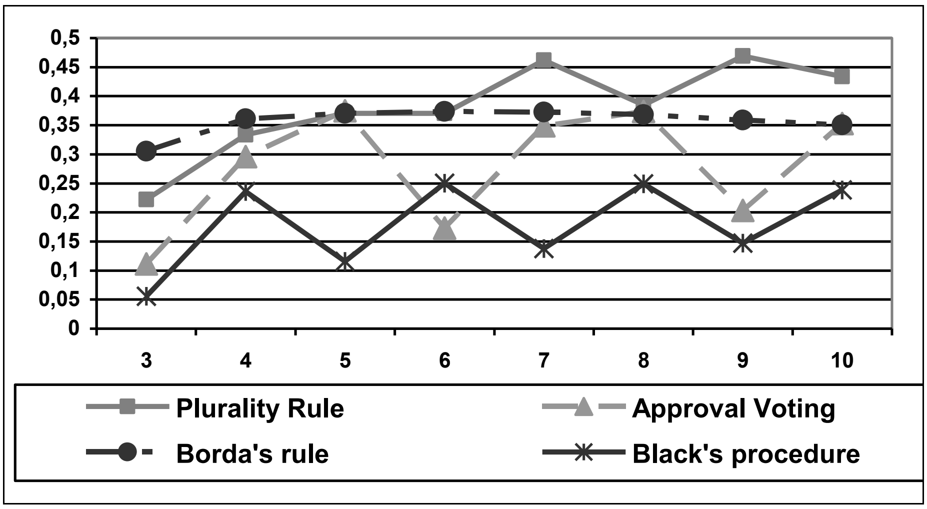

Fig. 6. The Nitzan-Kelly's index for the Leximin method in the IC model

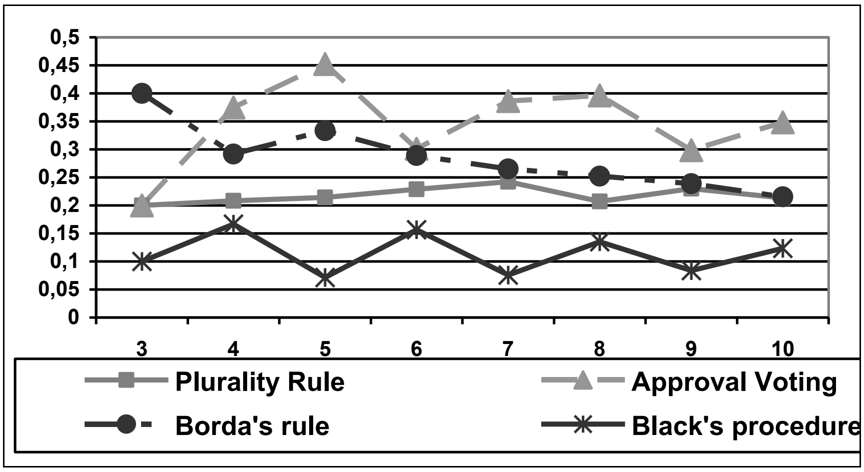

Fig. 7. The Nitzan-Kelly's index for the Leximin method in the IANC model

6 Concluding remarks

Anonymity and neutrality are the basic axioms in social choice theory. The IANC model, based on these axioms, assumes that both names of voters and names of alternatives are immaterial. We introduce combinatorial instruments that allow us to study the properties of social choice rules under IANC assumption. Since computational experiments in this model have a rather high complexity, we present an alternative way of analyzing properties of anonymous and neutral social choice rules.

In the IC model, the Nitzan-Kelly's index is the probability that any preference profile independently drawn from the set of all preference profiles will be manipulable. In the IANC model, it is the same probability on the set of anonymous and neutral equivalence classes. How do these probabilities differ from each other? Using methods of combinatorics and group theory, we evaluate the number and cardinality of anonymous and neutral equivalence classes with a maximal and minimal number of elements. We evaluate maximal difference of any probability (such as the Nitzan-Kelly's index) in the IC and IANC models. We show that for the case of three and four alternatives, the maximal possible difference is high enough to cause changes in the relative manipulability of social choice rules. For the case of six or more alternatives, this difference is almost zero, and all probabilities in the IC model are equal to those in the IANC model. This means that studying some properties of choice rules in the IANC model may not require complicated computational experiments.

To illustrate how a transition from IC to IANC can change the situation, we analyze the actual difference of manipulability indexes of four social choice rules in the IC and IANC models with Leximin and Leximax extension methods.

Appendix A

Theorem 1 *The minimal number of elements in anonymous and neutral equivalence class is* $m!$. *This class is unique for cases where* $n \geq 3$.

Proof. Firstly, we show that the maximal number of stabilizing permutations is $n!$. If profile $\vec{P}$ consists of equal columns, then the only stabilizing permutations for $\vec{P}$ are $g = (\sigma, \tau)$, such that σ is any permutation of voters, and τ is an identity permutation. Then, let us consider a profile $\vec{P}$ in which at least two voters have different preferences. If σ is a permutation of voters, τ_1 and τ_2 are different permutations of alternatives, then both $g_1 = (\sigma, \tau_1)$ and $g_2 = (\sigma, \tau_2)$ cannot be stabilizing permutations for the same profile. Therefore, the cardinality of stabilizer cannot exceed $n!$ and minimal cardinality of the equivalence class is $m!n!/n! = m!$.

Let us consider a preference profile with two columns. The first column is $(a_1, a_2, ..., a_m)$, and the second is $(a_{\tau(1)}, a_{\tau(2)}, ..., a_{\tau(m)})$, such that permutation τ has an order of 2. For this profile the cardinality of the stabilizer group also equals 2 and the number of equivalence classes with a minimal number of elements is greater than 1 in this case.

Now assume that $n \geq 3$. Suppose that there exists a profile $\vec{P}$ in which at least two voters have different preferences and the stabilizer for $\vec{P}$ has exactly $n!$ elements. For each permutation of voters σ, there must exist a permutation of alternatives τ, such that $g = (\sigma, \tau)$ is a stabilizing permutation for $\vec{P}$. Let us try to build a permutation τ for such permutations σ, that have a cycle (i, j) and a cycle (k) (σ fixes k-th column), where $i, j \neq k$, $k = 1..n$. This means that corresponding permutations τ should not permute alternatives and only profile with all identical columns could be a fixed point for such $g = (\sigma, \tau)$.

Since all profiles consisting of similar preferences belong to the same equivalence class, we can conclude that it is unique. An equivalence class with a minimal number of elements is denoted by $\theta_{\min}$. The cardinality of such a class is

$$\left|\theta_{\min}\right| = \frac{m!n!}{n!} = m! \blacksquare$$

Theorem 2 *The maximal number of elements in anonymous and neutral equivalence class is* $m!n!$, *if and only if* $m! > n$.

Proof. First, we show that $|\theta_{\max}| = m!n!$ implies $m! > n$. Suppose, that $|\theta_{\max}| = m!n!$, but $m! < n$. This means that the number of columns in preference profile $\vec{P}$ is greater than the number of column types and at least two columns in $\vec{P}$ are the same. Consequently, the number of stabilizing permutations is more than one and $|\theta_{\max}| < m!n!$. On the other hand, if $m! = n$, then the preference profile consists of all $m!$ different columns. Consider any permutation of order m, for example, (12 … m), which splits up a set of $m!$ columns into $m!-1$ non-intersecting orbits. In this case, the preference profile has more than one stabilizing permutation.

Secondly, we should prove that if $m! > n$ then $|\theta_{\max}| = m!n!$. In the case $m! > n$ for a preference profile consisting of different columns there cannot be a stabilizing permutation $g = (\sigma, \tau)$, such that σ permutes voters and τ is an identity permutation. If a permutation $g = (\sigma, \tau)$ renames alternatives, then in each row of its fixed point $\vec{P}$ there is the same number of permuted alternatives. If alternatives are permuted in the same cycle, then they are repeated in each row the same number of times. We can always build a profile consisting of different columns that does not satisfy this property the following way. Take any profile consisting of different columns that has at least two stabilizing permutations. Then reverse the order of any two alternatives such that columns are still not repeated and the mentioned property is not satisfied. Thus, the only stabilizing permutation for a described profile $\vec{P}$ with $m! > n$ is an identity permutation. The cardinality of equivalence class $\theta_{\max}$, consisting of such profiles, is

$$|\theta_{\max}| = \frac{m!n!}{1} = m!n! \quad \blacksquare$$

Lemma 3 The number of fixed points from the set of preference profiles consisting of different columns for some permutation $g = (\sigma, \tau)$ is equal to

$$\left|\tilde{F}_g\right| = \begin{cases} \prod_{j=0}^{\alpha} (m! - j \cdot t), & if \quad \lambda_1 = \lambda_2 = ... = \lambda_\alpha = t, \\ 0, & otherwise. \end{cases}$$

where $t = LCM(\mu)$.

Proof. Suppose, for some $\vec{P} \in \tilde{\Omega}$ holds $\vec{P}^g = \vec{P}$. Let us consider the permutation of alternatives τ. The order of this permutation t is equal to the least common multiple of cycle lengths, $t = LCM(\mu)$. If a column is $(a_1, a_2, ..., a_m)$, then after the action of permutation τ it becomes $(a_{\tau(1)}, a_{\tau(2)}, ..., a_{\tau(m)})$, after the second action of τ $(a_{\tau^2(1)}, a_{\tau^2(2)}, ..., a_{\tau^2(m)})$, and so on. Finally, $(a_{\tau^{t-1}(1)}, a_{\tau^{t-1}(2)}, ..., a_{\tau^{t-1}(m)})$ becomes again $(a_1, a_2, ..., a_m)$.

Next, we take any cycle of permutation of voters σ, $(i_1\ \ i_2 ...\ \ i_k)$. Let the i_1-th column be $(a_1, a_2, ..., a_m)$. Then, after the action of permutation σ it becomes the column number i_2. At the same time, τ permutes alternatives and the column number i_2 should be equal to $(a_{\tau(1)}, a_{\tau(2)}, ..., a_{\tau(m)})$, because the preference profile after permuting both voters and alternatives should be the same. Thus, if the i_2-th column $(a_{\tau(1)}, a_{\tau(2)}, ..., a_{\tau(m)})$ after permuting columns becomes the i_3-th column and $(a_{\tau(1)}, a_{\tau(2)}, ..., a_{\tau(m)})$ is mapped to $(a_{\tau^2(1)}, a_{\tau^2(2)}, ..., a_{\tau^2(m)})$ by τ, then the i_3-th column should be $(a_{\tau^2(1)}, a_{\tau^2(2)}, ..., a_{\tau^2(m)})$, etc. Finally, the i_k-th column becomes i_1-th and we conclude that should be equal to $(a_{\tau^{t-1}(1)}, a_{\tau^{t-1}(2)}, ..., a_{\tau^{t-1}(m)})$ in order to be mapped to $(a_1, a_2, ..., a_m)$ by permutation τ. Since repeated columns are not permitted, the length of the cycle $(i_1\ \ i_2 ...\ \ i_k)$, k, should be equal to $t = LCM(\mu)$. The same result holds for any other cycle of permutation σ, in other words, the length of all cycles in σ is the same, $\lambda_1 = \lambda_2 = ... = \lambda_\alpha$, or $GCD(\lambda) = LCM(\lambda)$, and equals $LCM(\mu)$.

Suppose, a permutation $g = (\sigma, \tau)$ is such that $GCD(\lambda) = LCM(\lambda) = LCM(\mu)$ and σ consists of $\alpha = n/t$ cycles. The first column of the first cycle in σ we can choose from $m!$ different columns. The rest of the columns in this cycle are determined by permutation τ. Then, the first column of the second cycle in σ can be represented by any of $m! - t$ variants, since t columns are already used in the first cycle. The first column of the third cycle can be defined $m! - 2t$ different ways and so on. Inside any cycle there cannot appear any column that is already used, because a permutation τ splits up the set of all different columns into non-intersecting orbits of columns that can be produced one from another by this permutation. Columns in every cycle form one of such orbits.

Finally, we get an exact formula for the number of fixed-points from $\tilde{\Omega}$ for some permutation $g = (\sigma, \tau)$

$$\left|\tilde{F}_g\right| = \begin{cases} \prod_{j=0}^{\alpha} (m! - j \cdot t), & if \quad \lambda_1 = \lambda_2 = ... = \lambda_\alpha = t, \\ 0, & otherwise. \end{cases} \quad \blacksquare$$

Theorem 4 For any *m* and *n* such that $m! > n$, the number of equivalence classes on $\tilde{\Omega}$ is equal to

$$\tilde{R}(m,n) = \sum_\lambda \sum_\mu z_\lambda^{-1} z_\mu^{-1} \chi(S(\lambda,\mu)) \prod_{j=0}^{\alpha-1} (m! - j \cdot t),$$

where $S(\lambda,\mu) = (GCD(\lambda) = LCM(\lambda) = LCM(\mu))$.

Proof. Let $\tilde{R}$ be the number of ANECs on $\tilde{\Omega}$, and $\vec{P}^1, \vec{P}^2, ..., \vec{P}^{\tilde{R}}$ be the representatives of these classes. For any preference profile $\vec{P}$ holds $\left|\theta_{\vec{P}}\right| \cdot \left|G_{\vec{P}}\right| = \left|G\right|$.

Then, take the sum of these equalities over representatives

$$\sum_{i=1}^{\tilde{R}} \left|\theta_{\vec{P}^i}\right| \cdot \left|G_{\vec{P}^i}\right| = \tilde{R} \cdot \left|G\right|,$$

$$\tilde{R} = \frac{1}{\left|G\right|} \sum_{i=1}^{\tilde{R}} \left|\theta_{\vec{P}^i}\right| \cdot \left|G_{\vec{P}^i}\right|.$$

Since $\left|G_{\vec{P}^k}\right| = \left|G_{\vec{P}^l}\right|$ if $\vec{P}^k$ and $\vec{P}^l$ belong to the same ANEC, we can rewrite

$$\left|\theta_{\vec{P}^i}\right| \cdot \left|G_{\vec{P}^i}\right| = \sum_{\vec{P} \in \theta_{\vec{P}^i}} \left|G_{\vec{P}^i}\right|.$$

$$\tilde{R} = \frac{1}{\left|G\right|} \sum_{i=1}^{\tilde{R}} \left|\theta_{\vec{P}^i}\right| \cdot \left|G_{\vec{P}^i}\right| = \frac{1}{\left|G\right|} \sum_{i=1}^{\tilde{R}} \sum_{\vec{P} \in \theta_{\vec{P}^i}} \left|G_{\vec{P}^i}\right| = \frac{1}{\left|G\right|} \sum_{\vec{P} \in \tilde{\Omega}} \left|G_{\vec{P}}\right|$$

The sum of stabilizing permutations over all preference profiles from $\tilde{\Omega}$ is equal to the sum of all fixed-points from $\tilde{\Omega}$ for all permutations

$$\sum_{\vec{P} \in \tilde{\Omega}} \left|G_{\vec{P}}\right| = \sum_{\vec{P} \in \tilde{\Omega}} \sum_{g \in G_{\vec{P}}} 1 = \sum_{g \in G} \sum_{\vec{P} \in \tilde{F}_g} 1 = \sum_{g \in G} \left|\tilde{F}_g\right|.$$

$$\tilde{R} = \frac{1}{\left|G\right|} \sum_{g \in G} \left|\tilde{F}_g\right|.$$

Using Lemma 3.1 and denoting $S(\lambda, \mu) = (GCD(\lambda) = LCM(\lambda) = LCM(\mu))$ we get

$$\tilde{R}(m,n) = \frac{1}{\left|G\right|} \sum_{g \in G} \chi(S(\lambda, \mu)) \prod_{k=0}^{\alpha} (m! - k \cdot LCM(\mu)).$$

Since $\left|\tilde{F}_g\right|$ depends only on the cycle type of permutation $g = (\sigma, \tau)$, we can take the sum over all partitions λ and μ and multiply every component by the number of permutations of *n* with partition λ and the number of permutations of *m* with partition μ, $z_\lambda^{-1} n!$ and $z_\mu^{-1} m!$, respectively.

$$\tilde{R}(m,n) = \frac{1}{m!n!} \sum_{\lambda} \sum_{\mu} z_\lambda^{-1} n! \cdot z_\mu^{-1} m! \cdot \chi(S(\lambda, \mu)) \prod_{j=0}^{\alpha-1} (m! - j \cdot t) =$$

$$= \sum_{\lambda} \sum_{\mu} z_\lambda^{-1} z_\mu^{-1} \chi(S(\lambda, \mu)) \prod_{j=0}^{\alpha-1} (m! - j \cdot t). \blacksquare$$

Corollary 5 For any *m* and *n* such that $m! > n$:

a) the number of maximal ANEC satisfies the following inequality

$$\frac{2(m!-1)!}{(m!-n)!n!} - \tilde{R}(m,n) \le R_{\max}(m,n) \le \tilde{R}(m,n).$$

b) If m and n are such that $n > m$ and n is a prime number, then the number of maximal ANEC is equal to $\tilde{R}(m,n)$.

Proof. a) The second inequality is obvious, because preference profiles from maximal equivalence class always consist of different columns for $m! > n$. The total number of preference profiles consisting of different columns is

$$m! \cdot (m!-1) \cdot \ldots \cdot (m!+1-n) = \frac{(m!)!}{(m!-n)!}.$$

At the same time, it is the number of fixed points from $\tilde{\Omega}$ for the identity permutation. Since for this permutation $S(\lambda,\mu)$ is true, $(m!)!/(m!-n)!$ is included in the sum $m!n!\tilde{R}(m,n)$. The rest of the components of the sum, i.e.

$$m!n!\sum_{\lambda}\sum_{\mu} z_{\lambda}^{-1} z_{\mu}^{-1} \chi(S(\lambda,\mu)) \prod_{j=0}^{\alpha-1} (m!-j \cdot t) - \frac{(m!)!}{(m!-n)!}$$

form the sum of fixed points from $\tilde{\Omega}$ for all permutations except the identity permutation. The problem is that the sets $\tilde{F}_g$ intersect and we cannot find an exact number of preference profiles that have more than one stabilizing permutation. However, we can be sure that this number is not more than this sum. Consequently, the number of profiles having only one stabilizing permutation is not less than

$$\frac{(m!)!}{(m!-n)!} - \left(m!n!\sum_{\lambda}\sum_{\mu} z_{\lambda}^{-1} z_{\mu}^{-1} \chi(S(\lambda,\mu)) \prod_{j=0}^{\alpha-1} (m!-j \cdot t) - \frac{(m!)!}{(m!-n)!} \right) =$$

$$= \frac{2(m!)!}{(m!-n)!} - m!n!\sum_{\lambda}\sum_{\mu} z_{\lambda}^{-1} z_{\mu}^{-1} \chi(S(\lambda,\mu)) \prod_{j=0}^{\alpha-1} (m!-j \cdot t).$$

Dividing by the cardinality of maximal ANEC $m!n!$, we get

$$R_{\max}(m,n) \geq \frac{2(m!-1)!}{(m!-n)!n!} - \tilde{R}(m,n).$$

b) If n is a prime number, then σ either contains only one cycle of length n, or n cycles of length one. The first case means that the least common divisor of the τ cycle lengths should also be n, but we assumed $n > m$. Thus, this is impossible. The second case means that we only have an identity permutation in a stabilizer group of any profile from $\tilde{\Omega}$, i.e. $\tilde{\Omega}$ consists of preference profiles from maximal equivalence classes, and we have only one component in the sum $\tilde{R}(m,n)$

$$\tilde{R}(m,n) = R_{\max}(m,n) = \frac{(m!-1)!}{(m!-n)!n!}. \blacksquare$$

Appendix B

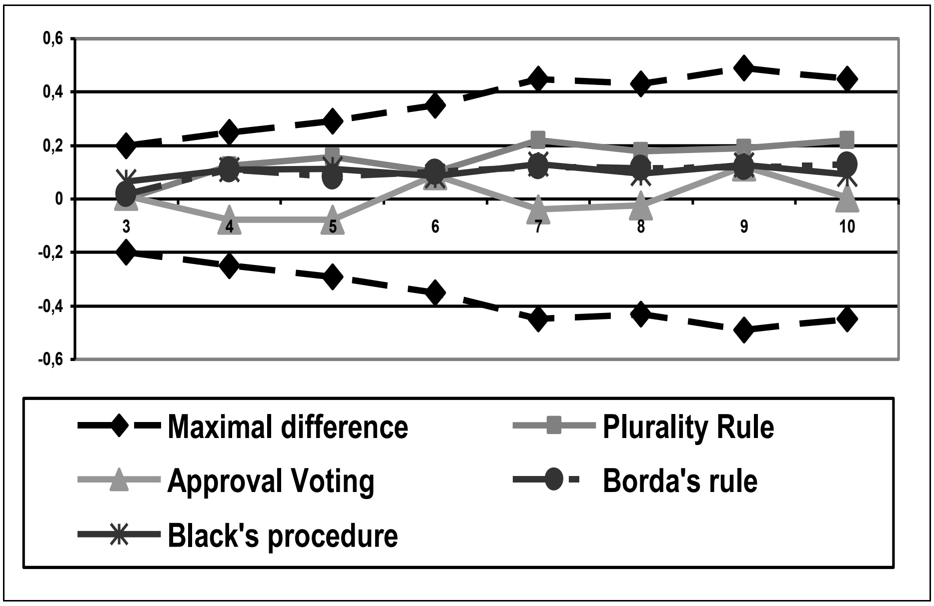

Fig. 8. The difference of the Nitzan-Kelly's index in IC and IANC, Leximin

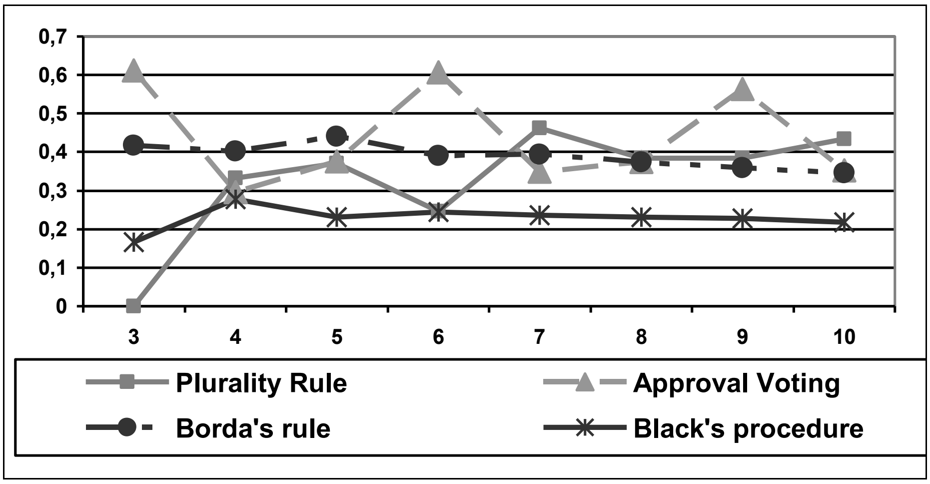

Fig. 9. The Nitzan-Kelly's index for the Leximax method in the IC model

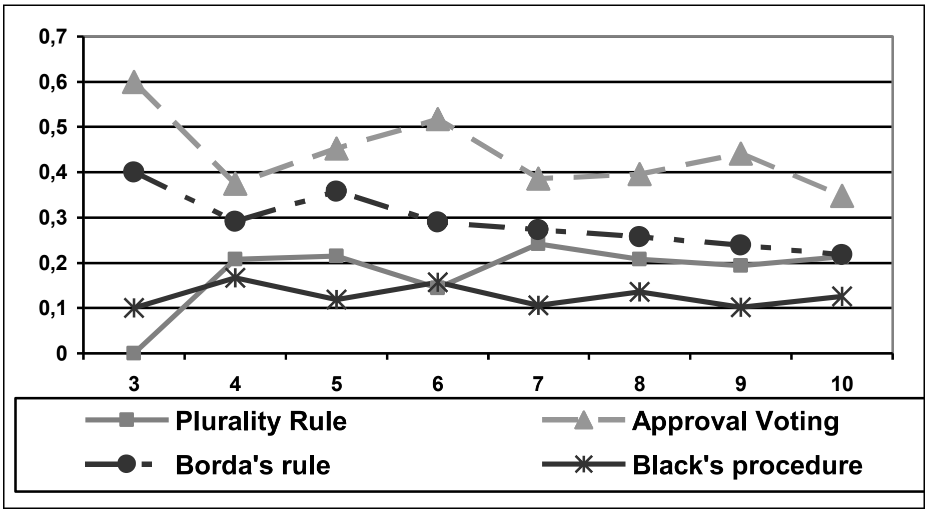

Fig. 10. The Nitzan-Kelly's index for the Leximax method in the IANC model

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