HIGHER SCHOOL OF ECONOMICS
N A T I O N A L R E S E A R C H U N I V E R S I T Y

*Iskakov Mikhail, Iskakov Alexey*

# EQUILIBRIUM IN SECURE STRATEGIES — INTUITIVE FORMULATION

Working Paper WP7/2012/06
Series WP7
Mathematical methods
for decision making in economics,
business and politics

Moscow
2012

**Iskakov, M., Iskakov, A.** Equilibrium in Secure Strategies — intuitive formulation : Working paper WP7/2012/06 [Text] / M. Iskakov, A. Iskakov ; National Research University "Higher School of Economics". – Moscow : Publishing House of the Higher School of Economics, 2012.

A new concept of equilibrium in secure strategies (EinSS) in non-cooperative games is presented. The EinSS coincides with the Nash Equilibrium when Nash Equilibrium exists and postulates the incentive of players to maximize their profit under the condition of security against actions of other players. The new concept is illustrated by a number of matrix game examples and compared with other closely related theoretical models. We prove the existence of equilibrium in secure strategies in two classic games that fail to have Nash equilibria. On an infinite line we obtain the solution in secure strategies of the classic Hotelling's price game (1929) with a restricted reservation price and linear transportation costs. New type of monopolistic equilibria in secure strategies are discovered in the Tullock Contest (1967, 1980) of two players.

*Key words*: Equilibrium in Secure Strategies, Hotelling, Tullock contest.

*JEL classification*: C72, D03, D43, D72, L12, L13.

*Mikhail Iskakov,* Trapeznikov Institute of Control Sciences RAS, Moscow, Russia

*Alexey Iskakov,* Trapeznikov Institute of Control Sciences RAS, Moscow, Russia

**Препринты Национального исследовательского университета «Высшая школа экономики» размещаются по адресу: http://www.hse.ru/org/hse/wp**

## Content


1. Introduction 4

2. Equilibrium in Secure Strategies 7

3. Solution in Objections and Counter Objections 12

4. Best Secure Response 16

5. Game with Uncertain Insider 19

6. Games with more than two players 23

7. EinSS in the Hotelling's Model 28

8. EinSS in the Tullock Contest 34

References 48

## 1. Introduction

Applications of Generalized Nash equilibria concepts is increasing steadily in the past 50 years. In general the objects of research are real-world applications or games with some additional mathematical structure (Facchinei and Kanzow, 2007). In this paper we propose new generalization of Nash equilibrium which introduce an additional criteria of security. We define the notion of threat and the notion of secure strategy which are the basis of the model of cautious behavior. In many practical situations the security considerations are indeed no less important than increasing profit. For example rational players do not break the rules if the expected penalty exceeds the profit from breaking the rules. In a similar way we assume that cautious player refuses from increasing his profit if it creates threat to lose more. He would rather prefer the greatest possible *secure* profit at a given strategies of other players. Taking into account this logic of behavior one can discover equilibrium positions which sometimes can not be revealed by standard logic of best responses. These equilibria positions we call *Equilibria in Secure Strategies (EinSS).* The first formulation of EinSS was published in Iskakov (2005). In this paper we present a reformulation of the concept and discuss all its aspects in detail. Generally speaking, the EinSS is realized when all players maximize their profits under the condition to avoid all threats from other players. We prove that any Nash-Cournot equilibrium is an Equilibrium in Secure Strategies. However, the EinSS can exist in games that fail to have Nash-Cournot equilibria as will be demonstrated by examples in this paper.

The concept of the EinSS creates natural associations with the concept of objections and counter objections developed by Aumann and Maschler (1961) for the cooperative games. Therefore we first of all compare our concept with *the solution in objections and counter objections* which is formulated as an imitation of the logic of Aumann and Maschler for the non-cooperative games. This logic implies that any threat existing in the profile shall be effectively contained by the counter threat of other players. This concept is obviously different from the EinSS where no threats in the profile allowed at all. In fact the EinSS describes more cautious behavior of players when they want to insure themselves against all threats whereas solution in objections and counter objections is a more risky approach. This difference can be clearly seen in the classic Hotelling's price game (1929) with the linear transport costs. Eaton and Lipsey (1978) assumed that undercutting through the whole market in this game shall be always contained by some counter threat which corresponds to the solution in objections and counter objections. In the EinSS approach players do assume that they can be driven out of the market and therefore keep their prices sufficiently low to secure themselves against such undercutting. As shown in M.Iskakov and A.Iskakov (2012) the EinSS approach results in the equilibrium solution of the Hotelling's price game with lower prices as compared with the solution based on the assumption of Eaton and Lipsey.

Our following step is to investigate the concept of *the best secure response (BSR)*. The Nash equilibrium is the profile in which the strategy of each player is the best response. In a similar way the strategy of each player in the EinSS turns out to be the best secure response. However, the set of profiles of best secure responses (or BSR-profiles) may be larger than the set of EinSS. An additional condition defined as *stability* makes the two coincide. Thereby we prove that a BSR-profile is an Equilibrium in Secure Strategies if and only if it is stable. This property provides a practical method for finding the EinSS. First, all BSR-profiles are to be found, then the unstable BSR-profiles are to be excluded.

The concept of the EinSS assumes that players make conjectures about the threats of other players. Implicitly this implies that players may choose their actions non-simultaneously. Therefore it would be interesting to find the game with minimal elements of dynamics which would mimic the reasoning of the players in a similar way to the EinSS. This investigation resulted in the concept of *a game with an uncertain insider*. Briefly it can be formulated in the following way. All players simultaneously choose their strategies in the original game and after that an "insider" is chosen randomly among them and has an opportunity to change his strategy. Nobody knows beforehand who is going to be the insider (even the insider himself). We prove that the EinSS in the game is the Nash equilibrium of the corresponding game with an uncertain insider, if all players resolve the uncertainty by the maximin criterion. However the set of equilibria in the game with an uncertain insider is wider than the set of Equilibria in Secure Strategies.

In order to illustrate the practical value and adequacy of the proposed concept we consider in this paper two classic games that fail to have Nash equilibria without using mixed strategies as was suggested by Dasgupta and Maskin (1986). The first one is the Hotelling's price game with a restricted reservation price and linear transportation costs (1929) on an infinite line.

We obtain solution in secure strategies for arbitrary distance between two players. The second model is the Tullock Contest (Tullock 1967, 1980) of two players. The EinSS for arbitrary values of the power parameter can be found. Depending on the power parameter there are three types of equilibria. Either it coincides with the Nash equilibrium found by Tullock (1980) or it corresponds to the newly discovered monopolistic solution or it represents an intermediate case of equilibrium of unequal or limited access.

The organization of the paper is as follows. In the next section the definitions of the EinSS are given. In Section 3 we compare the EinSS with the solution in objections and counter objections. In Section 4 we introduce the concept of the best secure response profile and investigate its relation to the EinSS. In Section 5 we introduce the concept of the game with an uncertain insider. In Section 6 we provide the modification of EinSS which allows to take into account the possibility of simultaneous and independent threats from many players. Finally in Sections 7 and 8 we consider the Hotelling's price game on an infinite line and the Tullock Contest of two players.

## 2. Equilibrium in Secure Strategies

We consider n-person non-cooperative game in the normal form $G = (i \in N, s_i \in S_i, u_i \in R)$. The concept of equilibria is based on the notion of threat and on the notion of secure strategy.

**Definition 1.** ***A threat*** *of player $j$ to player $i$ at strategy profile $s$ is a pair of strategy profiles $\{s, (s'_j, s_{-j})\}$ such that $u_j(s'_j, s_{-j}) > u_j(s)$ and $u_i(s'_j, s_{-j}) < u_i(s)$. The strategy profile $s$*

*is said **to pose a threat** from player $j$ to player $i$.*

**Definition 2.** *A strategy $s_i$ of player $i$ is **a secure strategy** for player $i$ at given strategies $s_{-i}$ of all other players if profile $s$ poses no threats to player $i$. A strategy profile $s$ is **a secure profile** if all strategies are secure.*

In other words a threat means that it is profitable for one player to worsen the situation of another. A secure profile is one where no one gains from worsening the situation of other players.

**Definition 3.** ***A secure deviation** of player $i$ with respect to $s$ is a strategy $s_i'$ such that $u_i(s_i', s_{-i}) > u_i(s)$ and $u_i(s_i', s_j', s_{-ij}) \geqslant u_i(s)$ for any threat $\{(s_i', s_{-i}), (s_i', s_j', s_{-ij})\}$ of player $j \neq i$ to player $i$.*

There are two conditions in the definition. In the first place a secure deviation increases the profit of the player. In the second place his gain at a secure deviation covers losses which could appear from retaliatory threats of other players. It is important to note that secure deviation does not necessarily mean deviation into secure profile. After the deviation the profile $(s_i', s_{-i})$ can pose threats to player $i$. However these threats can not make his or her profit less than in the initial profile $s$. We assume that the player with incentive to maximize his or her profit securely will look for secure deviations.

**Definition 4.** *A secure strategy profile is **an Equilibrium in Secure Strategies (EinSS)** if no player has a secure deviation.*

There are two conditions in the definition of EiSS. There are no threats in the profile and there are no profitable secure deviations[1]. The second condition implicitly implies maximization over the set of secure strategies.

Let us now formulate the first important property of the EinSS concept.

**Proposition 1.** *Any Nash-Cournot equilibrium is an Equilibrium in Secure Strategies.*

***Proof.*** Since Nash-Cournot equilibrium poses no threats so it is a secure profile. And no player in Nash-Cournot equilibrium can improve his or her profit using whatever deviation. Both conditions of the EinSS are fulfilled. □

First this means that a Nash equilibrium is always secure profile in terms of the proposed definitions. Second, the existence results can not be worse for EinSS than for Nash equilibrium. Whenever a Nash equilibrium exists an EinSS also exists. However for some practically important problems without Nash equilibrium (such as Hotelling's model and Tullock Contest which will be considered in this paper) the EinSS exists and provides an interesting interpretation.

For some games the reverse of Proposition 1 is true. For instance for strictly competitive games any EinSS is the Nash equilibrium. Indeed, suppose there is an EinSS in the strictly competitive game which is not a Nash equilibrium. Then there is at least one player who can increase his profit and there is

[1]Some details on previously used form of definitions are included in an Appendix for interested readers.

at least one player who will decrease his profit. Therefore the profile is not secure and can not be EinSS.

Let us now consider a simple matrix game example having no Nash equilibrium in order to illustrate the definitions introduced above:

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | (1,1) | (2,0) |
| $s_2$ | (2,2) | (0,3) |

One can find all threats in the game. First, the strategy profile $(s_2, t_1)$ poses a threat to player 1 as we move from payoffs $(2, 2)$ to payoffs $(0, 3)$. Second, the strategy profile $(s_1, t_2)$ poses a threat to player 1 as we move from payoffs $(2, 0)$ to payoffs $(1, 1)$. And finally profile $(s_2, t_2)$ poses a threat of player 1 to player 2 as we move from payoffs $(0, 3)$ to payoffs $(2, 0)$. In all three cases one player can make himself better off and another player worse off. These threats in the game can be visualized graphically in the following way:

$$\begin{array}{ccc} (1,1) & \leftarrow & (2,0) \\ & & \uparrow \\ (2,2) & \rightarrow & (0,3) \end{array}$$

The only secure profile in the game (which is secure for both players) is the profile $(s_1, t_1)$ with payoffs $(1, 1)$. If players were choosing best responses sequentially in the game they would end up in an infinite cycle so that there is no Nash equilibrium. This situation can change if we take into account the considerations of security. The profiles with payoffs $(2, 2)$, $(0, 3)$ and $(2, 0)$ can not be an equilibrium in secure strategies because they pose threats. The profile $(s_1, t_1)$ is the only secure profile in the game. The second player can not increase his profit by any deviation from it. There is a profitable deviation for the first player from this profile into the profile $(s_2, t_1)$ with payoffs $(2, 2)$. However it is not a secure deviation since the first player can lose more after the response deviation of the second player from the profile $(s_2, t_1)$ with payoffs $(2, 2)$ into the profile $(s_2, t_2)$ with payoffs $(0, 3)$. Therefore no player has in the profile $(s_1, t_1)$ a secure deviation and this profile is an EinSS. This means that a cautious player would prefer the guaranteed payment of 1 in the $(s_1, t_1)$ to the possibility of gaining 2 in $(s_2, t_1)$ accompanied by a high-risk to get zero in $(s_2, t_2)$.

Let us now add additional row and column to the matrix of the previous game.

| | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|
| $s_1$ | (1,1) | (2,0) | (-1,-1) |
| $s_2$ | (2,2) | (0,3) | (-1,-1) |
| $s_3$ | (-1,-1) | (-1,-1) | (0,0) |

Now we have a Nash equilibrium with payoffs $(0, 0)$ (perhaps, not a very good one!). It is also an EinSS according to Proposition 1. Threats in this game are the same as in the previous example. All newly added profiles are secure. Profiles with payoffs $(-1, -1)$ are secure only because they are the worst for both players, so they can not be EinSS. Equilibtium $(s_3, t_3)$ with payoffs $(0, 0)$ is Pareto dominated by all other profiles of the game. However there is a profile $(s_1, t_1)$ which is still an EinSS and dominates $(s_3, t_3)$. This example shows that there may be games with Nash equilibrium which nevertheless have another more reasonable solution given by an Equilibrium in Secure Strategies.

## 3. Solution in Objections and Counter Objections

Perhaps the most famous concept of "safety" applied to coalitions was developed by Aumann and Maschler (1961) for the cooperative games. The payoff configuration in cooperative game was called stable if any objection of one group of players against the other group can be answered by counter objection of the second group against the first one. Therefore the new concept of "security" raises a natural question: is the EinSS can be formulated in a way which somehow similar to the concept of objections and counter objections developed by Aumann and Maschler?

The concepts of the cooperative and non-cooperative games are very different by themselves in order to employ the approach of Aumann and Mashler for the non-cooperative games directly. An "objection" in cooperative game is an objection to a game profile whereas in our approach the "threat" is directed against a particular player. However it is possible to apply general logic of objections and counter objections to non-cooperative games. An "objection" to an equilibrium profile would be then a deviation of certain player which increases his profit. A "counter objection" would be a "counter deviation" of another player which effectively "punishes" the first player making his profit less than he had in the initial position. To define the "counter objection" more rigorously we introduce the following definition (which in a sense is opposite to the Definition 3).

**Definition 5.** *The deviation $s'_i$ of player $i$ from the profile $s$ such that $u_i(s'_i, s_{-i}) > u_i(s)$ is* **a contained deviation** *if there is a threat $\{(s'_i, s_{-i}), (s'_i, s'_j, s_{-ij})\}$ such that $u_i(s'_i, s'_j, s_{-ij}) < u_i(s)$.*

*This threat is* **a counter threat** *of player $j$ to deviation $s'_i$ from the profile $s$.*

There are several specific features of this definition worth a mention. First of all like in Definition 3 a counter threat is defined only with respect to the particular profile. This implies that profile $(s'_i, s_{-i})$ by itself may pose threats to player $i$ which are not counter threats if $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$. Therefore profile $(s'_i, s_{-i})$ without counter threats is not necessarily secure profile. Secondly, a counter threat is understood as an answer to a deviation which may or may not be a threat in the initial profile $s$. And finally note that a counter threat (like a threat) is directed from one particular player to another particular player rather than being directed against a strategy profile.
Now we are ready to provide two equivalent formulations of a threatening-proof profile:

**Definition 6.** *A strategy profile is **a threatening-proof profile** if any deviation from it that increases the profit of some player is contained by a counter threat.*

**Definition 6′.** *A strategy profile is **a threatening-proof profile** if no player can make a secure deviation.*

Implicitly the last definition implies maximization over the set of secure strategies. However to obtain more meaningful concept of equilibrium it would be better to maximize the utility functions over the set of the threatening-proof profiles themselves. Let now $M$ be a set of all threatening-proof profiles in the game.

**Definition 7.** *A profile $s^*$ is* ***a solution in objections and counter objections***

$$\text{if} \quad s^* \in M \quad \text{and for all } i: \quad u_i(s^*) = \max_{(s_i, s^*_{-i}) \in M} u_i(s_i, s^*_{-i}).$$

The comparison of Definitions 4, 6′ and 7 immediately identifies the following relationship between the EinSS and the solution in objections and counter objections.

**Proposition 2.** *Any Equilibrium in Secure Strategies is a threatening-proof profile. Any Nash-Cournot equilibrium is a solution in objections and counter objections*

The difference however can be clearly seen from the comparison of Definitions 4 and 6′. An EinSS is a secure profile, i.e. there are no threats in it. In a threatening-proof profile threats are allowed, however they are not secure deviations. In other words these threats are *contained by counter threats* which is reflected in the term of the solution in objections and counter objections. Let us consider the following matrix game example.

| | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|
| $s_1$ | (2,2) | (3,3) | (8,0) |
| $s_2$ | (3,3) | (4,4) | (5,5) |
| $s_3$ | (0,8) | (5,5) | (6,6) |

There are no Nash equilibria in the game. In all profiles with $s_3$ or $t_3$ it is profitable for one player to worsen the situation of another. So these profiles are insecure. All other profiles do not pose threats. Therefore they are secure. The profile $(s_1, t_1)$ with payoffs $(2, 2)$ allows secure deviations into profiles with payoffs $(3, 3)$ which in turn allow secure deviation into profile $(s_2, t_2)$

with payoffs $(4, 4)$. There are profitable deviations from $(s_2, t_2)$ into profiles with payoffs $(5, 5)$. However they are not secure deviations since players lose more after the response deviation of the competitor into the profile $(s_3, t_1)$ or $(s_1, t_3)$ with payoffs $(0, 8)$ and $(8, 0)$ correspondingly. Therefore the profile $(s_2, t_2)$ is an EinSS. The threat of deviations into profiles $(s_1, t_3)$ and $(s_3, t_1)$ prevent cautious players from choosing profile $(s_3, t_3)$ with the highest possible mutual payoffs $(6, 6)$. However these deviations by themselves are contained by counter threats. The deviation from $(s_3, t_3)$ into $(s_1, t_3)$ is contained by the counter threat to deviate into $(s_1, t_2)$ and deviation into $(s_3, t_1)$ is contained by the counter threat to deviate into $(s_2, t_1)$. Therefore profile $(s_3, t_3)$ with payoffs $(6, 6)$ is a threatening-proof profile and a solution in objections and counter objections. This solution in objections and counter objections is different from the EinSS profile $(s_2, t_2)$ with payoffs $(4, 4)$. The concept of EinSS describes more cautious behavior of players when they want to insure themselves against all threats. Whereas solution in objections and counter objections is more risky and ignores the threats which are contained by counter threats.

Here there is a general theoretical problem connected with the different levels of reflection of threats. For instance if a threat can be contained by a counter threat why a counter threat can not be also contained, i.e invalid in some sense? And is the initial threat valid in this case? In the Definition 5 we employ in fact two-level reflection. We take into account here only threats and counter threats.

## 4. Best Secure Response

The definition of EinSS implicitly implies maximization of payoff functions over the set of secure strategies. We can therefore expect that the EinSS is an analogue of the Nash equilibrium on a narrower set of strategies (the secure ones). In this case the EinSS would be a profile in which the "secure strategy" of each player is the best one in the same way as the Nash equilibrium is a profile in which strategy of each player is the best response. In order to clarify this question let us start with the rigorous definition of the best secure response. Denote by $V_i(s_{-i})$ the set of secure strategies of player $i$ at given strategies $s_{-i}$ of all other players. Notice that $V_i(s_{-i})$ can be empty if all strategies of player $i$ are insecure at $s_{-i}$.

**Definition 8.** *A strategy $s_i^*$ of player $i$ is* ***a Best Secure Response*** *to strategies $s_{-i}^*$ of all other players if*

$$s_i^* \in V_i(s_{-i}^*) \quad \textit{and} \quad u_i(s^*) = \max_{s_i \in V_i(s_{-i}^*)} u_i(s_i, s_{-i}^*).$$

*A profile $s^*$ is* ***the Best Secure Response profile*** *(BSR-profile) if all strategies are Best Secure Responses.*

Let us now consider the following matrix game example.

| | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|
| $s_1$ | (0,0) | (2,2) | (2,2) |
| $s_2$ | (2,2) | (1,3) | (3,1) |
| $s_3$ | (2,2) | (3,1) | (1,3) |

There are no Nash equilibria and no EinSS in the game. The profiles $(s_2, t_1)$, $(s_2, t_3)$, $(s_3, t_1)$, $(s_3, t_2)$ are insecure for the first player. The profiles $(s_1, t_2)$, $(s_1, t_3)$, $(s_2, t_2)$, $(s_3, t_3)$ are insecure for the second player. Therefore $(s_1, t_1)$ is the only and the best secure profile in the game. Are there any secure deviations from it? If the first player for example deviates from the profile $(s_1, t_1)$ with payoffs $(0, 0)$ into profile $(s_2, t_1)$ or $(s_3, t_1)$ with payoffs $(2, 2)$ his new position will be subjected to threat of the second player. However the expected loss from these threats (equal to 1) does not exceed the gain obtained at deviation from $(s_1, t_1)$ (equal to 2). Therefore deviations from $(s_1, t_1)$ are secure for players since no threats can make their payoffs less than zero payoffs in $(s_1, t_1)$. Hence the profile $(s_1, t_1)$ can not be stable situation in the game and it is not the EinSS. Based on this example one can establish the following relationship between the BSR-profile and the EinSS.

**Proposition 3.** *Any Equilibrium in Secure Strategies is a BSR-profile. A BSR-profile may not be an Equilibrium in Secure Strategies.*

***Proof.*** An EinSS is a secure profile by definition. And it must be the best secure response for each player since otherwise there is a player who can increase his profit by secure deviation. Therefore an EinSS is a BSR-profile. The reverse is not true. In the above example the profile $(s_1, t_1)$ is a BSR-profile. However it is not an EinSS. □

Let us now take the previous matrix game example and increase payoffs in the profile $(s_1, t_1)$ and strengthen the threats:

| | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|
| $s_1$ | (1,1) | (2,2) | (2,2) |
| $s_2$ | (2,2) | (0,3) | (3,0) |
| $s_3$ | (2,2) | (3,0) | (0,3) |

Now the BSR-profile $(s_1, t_1)$ with payoffs $(1, 1)$ is an EinSS since deviations from it pose threats to receive less payoffs than in the profile $(s_1, t_1)$. These examples demonstrate the property of BSR-profiles which makes the difference. In order to be EinSS the BSR-profile has to satisfy an additional condition which we can define as *stability*. More precisely,

**Definition 9.** *A BSR-profile is* ***stable*** *if there is no player $i$ and deviation $s'_i$ such that $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_i(s'_i, s'_j, s_{-ij}) \geqslant u_i(s)$ for any threat $\{(s'_i, s_{-i}), (s'_i, s'_j, s_{-ij})\}$ of player $j \neq i$ to player $i$.*

In the unstable BSR-profile at least one player has non-secure alternatives with threats which in all cases are more profitable for him than staying in the initial BSR-profile. From the above definitions 4, 8 and 9 it follows:

**Proposition 4.** *A BSR-profile is an Equilibrium in Secure Strategies if and only if it is stable.*

Propositions 3 and 4 provide a practical method for finding EinSS. The BSR-profile is a Generalized Nash Equilibrium concept and all the corresponding results and algorithms of maximization can be applied to finding BSR-profiles. When all BSR-profiles are found, then unstable BSR-profiles are to be excluded.

## 5. Game with Uncertain Insider

Players in the EinSS make conjectures about threats of other players. Implicitly it implies that the players may choose their actions non-simultaneously. In Economics the assumption of simultaneous and independent decision making by players is indeed a very strong one. This raises natural questions about the relationship of the EinSS concept with dynamic games (especially if there are more than two players). Let us try to find a game with minimal elements of dynamics which would reproduce the reasoning of players in a similar way as in the EinSS. Let us suppose that after the players simultaneously choose their strategies an "insider" is chosen randomly among them and has an opportunity to change his strategy. Nobody knows beforehand who is going to be the insider (even the insider himself). Let us suppose also that players adopt a cautious behavior with respect to the actions of the insider.

Let us provide now a rigorous formulation. Take a non-cooperative game in the normal form $G = \{N = \{1, ..., n\}, s_i \in S_i, u_i(s) \in R\}$. We define an associated sequential game. During the first stage all players select simultaneously their strategies $s = (s_1, ..., s_n)$. At the second stage Nature chooses randomly the insider player number $j_0 \in N$. Then finally player $j_0$ either keeps the same strategy $s_{j_0}$ or choose another one that would increase his profit: his new strategy $\tilde{s}_{j_0}(s) \in \Theta_{j_0}(s) = \{s_{j_0}\} \cup \{s'_{j_0} \in S_{j_0} : u_{j_0}(s'_{j_0}, s_{-j_0}) > u_{j_0}(s)\}$. The final payoffs of players are $u_i(\tilde{s}_{j_0}, s_{-j_0})$.

We assume further that all players adopt at the beginning a cautious behavior and resolve uncertainty by the maximin criterion so that their payoff functions can be written as

$\hat{u}_i(s) = \min\limits_{j\in N,\ j\neq i,\ \tilde{s}_j\in\Theta_j(s)} u_i(\tilde{s}_j, s_{-j}) = \min\limits_{j\in N,\ \tilde{s}_j\in\Theta_j(s)} u_i(\tilde{s}_j, s_{-j})$. This defines a game $\hat{G} = \{N, s_i \in S_i, \hat{u}_i \in R\}$ that we call *a game with an uncertain insider.*[2] The following proposition establishes basic relationship between an EinSS and the corresponding game with uncertain insider.

**Proposition 5.** *An Equilibrium in Secure Strategies of the game $G$ is a Nash equilibrium of the corresponding game $\hat{G}$ with an uncertain insider.*

***Proof.*** Let $s^*$ be the EinSS of the game $G$. By the definition of EinSS there are no threats in the profile $s^*$. Thus no deviation $\tilde{s}_j \in \Theta_j(s^*)$ of player $j$ can decrease the profit of other players, i.e. $\min\limits_{j\in N,\ \tilde{s}_j\in\Theta_j(s^*)} u_i(\tilde{s}_j, s^*_{-j}) \geqslant u_i(s^*)$. Besides $s^*_j \in \Theta_j(s^*)$ and $\min\limits_{j\in N,\ \tilde{s}_j\in\Theta_j(s^*)} u_i(\tilde{s}_j, s^*_{-j}) \leqslant u_i(s^*)$. Therefore for all $i$ we have $u_i(s^*) = \hat{u}_i(s^*)$.
Assume there is a deviation $s'_i$ of player $i$ such that $\hat{u}_i(s'_i, s^*_{-i}) > \hat{u}_i(s^*)$, i.e. $\min\limits_{j\in N, j\neq i, s'_j\in\Theta_j} u_i(s'_i, s'_j, s^*_{-ij}) > \hat{u}_i(s^*) = u_i(s^*)$. In particular this implies that $u_i(s'_i, s^*_{-i}) > u_i(s^*)$ and $u_i(s'_i, s'_j, s^*_{-ij}) > u_i(s^*)$ for any threat $\{(s'_i, s^*_{-i}), (s'_i, s'_j, s^*_{-ij})\}$ of player $j \neq i$ to player $i$. According to the definitions 3 and 4 the

[2]In our model we assume that the uncertain insider either increases his profit or keeps the same strategy. An alternative assumption, that insider instead would strictly maximize his profit, would reduce the number of threats taken into account by other players. This would allow more risky strategies. However when we model the situation of cautious behavior it seems more natural to assume that players would insure against a wider variety of threats rather than expecting absolutely rational behavior of the insider.

player $i$ can increase his profit by secure deviation $s'_i$ and $s^*$ is not the EinSS. This is a contradiction, and therefore our assumption was wrong. $\hat{u}_i(s'_i, s^*_{-i}) \leqslant \hat{u}_i(s^*)$ for all $i$ and deviations $s'_i$, i.e. $s^*$ is the Nash equilibrium of the game $\hat{G}$ with an uncertain insider. □

However the set of Nash equilibria in $\hat{G}$ is wider than the set of EinSS in $G$. This can be seen if we come back to the matrix game from the previous section (without the EinSS):

| | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|
| $s_1$ | (0,0) | (2,2) | (2,2) |
| $s_2$ | (2,2) | (1,3) | (3,1) |
| $s_3$ | (2,2) | (3,1) | (1,3) |

The profiles $(s_1, t_2)$, $(s_1, t_3)$, $(s_2, t_1)$, $(s_3, t_1)$ are Nash Equilibria in the corresponding game $\hat{G}$ with uncertain insider but are not even secure in the original game $G$. The following proposition sets the necessary and sufficient conditions for a profile to be an EinSS.

**Proposition 6.** *A profile $s^*$ is an EinSS if and only if*

$$\begin{cases} \text{(a) } s^* \text{ is a secure profile ;} \\ \text{(b) } s^* \text{ is a Nash equilibrium of the game } \hat{G} \text{ ;} \\ \text{(c) } \forall i, \forall s'_i : \ \hat{u}_i(s'_i, s^*_{-i}) = \hat{u}_i(s^*) \ => \ u_i(s'_i, s^*_{-i}) = u_i(s^*) \end{cases}$$

***Proof.*** Let $s^*$ be an EinSS. By definition $s^*$ is a secure profile. According to Proposition 5 $s^*$ is a Nash equilibrium of the game $\hat{G}$. Let us assume there is a player $i$ and his strategy $s'_i$ such that $\hat{u}_i(s'_i, s^*_{-i}) = \hat{u}_i(s^*)$. Like in the previous proposition for the secure profile $s^*$ we can prove that for all $i$: $u_i(s^*) = \hat{u}_i(s^*)$ and obviously $\hat{u}_i(s'_i, s^*_{-i}) \leqslant u_i(s'_i, s^*_{-i})$. Therefore $u_i(s^*) \leqslant u_i(s'_i, s^*_{-i})$ Let us assume that $u_i(s^*) < u_i(s'_i, s^*_{-i})$. Then $s'_i$ is a secure deviation of player $i$ from $s^*$ and profile $s^*$ can not be EinSS. This is a contradiction, and therefore our assumption was wrong and $u_i(s^*) = u_i(s'_i, s^*_{-i})$. The condition (c) is proven.

Let us prove the opposite direction. Like in the previous proposition for the secure profile $s^*$ we can prove that for all $i$: $u_i(s^*) = \hat{u}_i(s^*)$. Let us suppose there is a deviation $s'_i$ of player $i$ such that $u_i(s'_i, s^*_{-i}) > u_i(s^*)$. Then from (b) and (c) it follows that $\hat{u}_i(s'_i, s^*_{-i}) < \hat{u}_i(s^*)$ and there exist a deviation $s'_j \in \Theta_j(s'_i, s^*_{-i})$ (obviously $j \neq i$) such that $u_i(s'_i, s'_j, s^*_{-ij}) < \hat{u}_i(s^*) = u_i(s^*)$. The pair of strategy profiles $\{(s'_i, s^*_{-i}), (s'_i, s'_j, s^*_{-ij})\}$ is a threat according to the Definition 1. Therefore $s'_i$ is not a secure deviation according to the Definition 3. Thus no player can make in the profile $s^*$ a secure deviation. The profile $s^*$ is an Equilibrium in Secure Strategies. □

**Corollary.** *If a secure profile $s^*$ in the game $G$ is a strict Nash equilibrium of the game $\hat{G}$ then $s^*$ is an Equilibrium in Secure Strategies in the game $G$.*

General relationship between EinSS, BSR-profiles in the game $G$ and Nash equilibria in the game $\hat{G}$ with uncertain insider can be summarized by a diagram in Fig.1.

The model of the game with uncertain insider gives another (dynamic) approach to the concept of EinSS. The EinSS is an equilibrium in the game of cautious players with the minimal dynamics introduced by the possibility to change randomly the strategy of one player, unknown in advance.

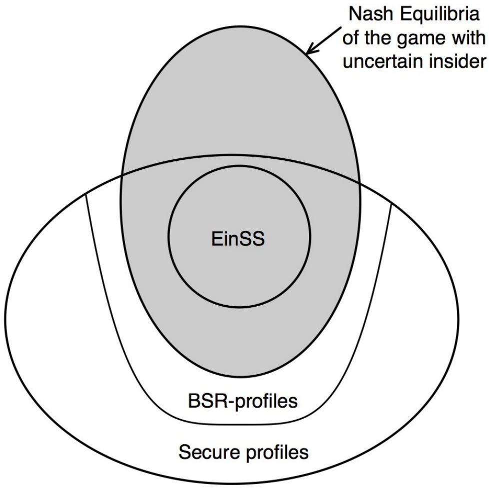


Fig. 1: General relationship between EinSS, BSR-profiles and Nash equilibria in the game with uncertain insider.

## 6. Games with more than two players

Definitions 1-4 of EinSS are based on the analysis of pairwise interaction of players. Nevertheless the threats in the game may take more complicated forms. This raises the question if the EinSS can adequately describe the collective behavior of many players.

In order to illustrate the specific effects arising in the games with more than two players let us first consider the following matrix game example. The first player chooses the matrix ($r_1$ or $r_2$), the second chooses the row ($s_1$ or $s_2$) and the third chooses the column ($t_1$ or $t_2$).

$r_1$ :

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | (1,1,1) | (0,0,0) |
| $s_2$ | (1,1,1) | (0,0,0) |

$r_2$ :

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | (0,0,0) | (0,0,0) |
| $s_2$ | (2,2,0.5) | (0,0,0) |

There are three EinSS in the game: $(r_1, s_1, t_1)$, $(r_2, s_2, t_1)$, $(r_2, s_1, t_2)$ (which are also Nash equilibria). Let us consider the profile $(r_1, s_1, t_1)$. The second player can deviate from it into $(r_1, s_2, t_1)$ without changing his profit. The new profile is secure for him and he does not threaten other players. However this deviation creates a threat of the first player to the third player which brings the game into another Nash equilibrium $(r_2, s_2, t_1)$. In comparison with the initial profile $(r_1, s_1, t_1)$ the first and the second players increase their profits at the expense of the third player. This example demonstrates the case of externality when seemingly "indifferent" deviation of one player creates threats between the other players. Moreover the realization of these threats could be profitable for "indifferent" player. In the given matrix game the threat "in two moves" can be arranged by two different players who can win at the expense of the third player. The concept of EinSS in our current formulation obviously does not take into account this effect properly. In fact some EinSS positions might be considered as unstable in the result of an appropriate consideration of these more "complex" threats. However in this case we shall assume that players without any collusion are able to play such "complex" combinations, which implies that they can calculate the possible actions and conjectures of each other.

The next matrix game example illustrates the case of simultaneous and independent threats of several players to one player. Each of this threats is small enough to be ignored but taken together they become crucial.

$r_1$:

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | (0,1,0) | (0,1,0) |
| $s_2$ | (0,4,0) | (0,2,1) |

$r_2$:

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | (0,1,0) | (0,1,0) |
| $s_2$ | (1,2,0) | (1,0,1) |

There are following threats in the game: $(r_1, s_2, t_1) \to (r_1, s_2, t_2)$, $(r_1, s_2, t_1) \to (r_2, s_2, t_1)$, $(r_2, s_2, t_1) \to (r_2, s_2, t_2)$, $(r_1, s_2, t_2) \to (r_2, s_2, t_2)$, $(r_2, s_2, t_2) \to (r_2, s_1, t_2)$. There are three EinSS in the game: $(r_2, s_1, t_1)$, $(r_1, s_1, t_2)$ and $(r_2, s_1, t_2)$. But it is interesting to consider also profile $(r_1, s_1, t_1)$. The second player can deviate from it into profile $(r_1, s_2, t_1)$ where he is threatened by players 1 and 3. Each of these threats taken separately can decrease the profit of the second player but it will be still higher than his profit in $(r_1, s_1, t_1)$. So profile $(r_1, s_1, t_1)$ is not formally EinSS since the second player can increase his profit by a secure deviation. However if players 1 and 3 would apply their threats in profile $(r_1, s_2, t_1)$ simultaneously and bring the game into profile $(r_2, s_2, t_2)$ the payoff of player 2 would be less than in $(r_1, s_1, t_1)$ and his deviation into $(r_1, s_2, t_1)$ would not be secure. Therefore if player 2 would take into account the possibility of two simultaneous and independent threats from players 1 and 3 then he would consider profile $(r_1, s_1, t_1)$ as an equilibrium in the generalized sense.

The concept of EinSS in our current formulation takes into account only individual deviations and hence can not treat this effect properly. Perhaps in its current formulation it also can not describe properly games with multiple players creating small threats which can be ignored individually but taken together become crucial. However these threats which arise as the result of simultaneous and independent actions of many players could be taken into account by an appropriate extension of the concept of secure deviation.

**Definition** $3'$. ***A secure deviation*** *of player* $i$ *with respect to* $s$ *is a strategy* $s'_i$ *such that* $u_i(s'_i, s_{-i}) > u_i(s)$ *and, whenever* $u_l(s'_i, s'_l, s_{-il}) > u_l(s'_i, s_{-i})$ *for all* $l$ *in some set* $N' = \{j, ..., k\},\ i \notin N'$ *then* $u_i(s'_i, s'_j, ..., s'_k, s_{-ij...k}) \geqslant u_i(s)$.

The Definition $3'$ sets more restrictive conditions for the secure deviation and corresponds to a more cautious behavior. This modification reduces the number of secure deviations for a given profile. All EinSS according to the Definitions $1-4$ would still be EinSS after changing the Definition 3 for the Definition $3'$. However some "new" EinSS appear which correspond to the possibility of threats from simultaneous and independent actions of other players. It is important to notice that these other players according to Definition $3'$ do not take into account the behavior of each other. Therefore their behavior represents rather the behavior of a crowd than a collusion in a group of players. Indeed the crowd behavior plays an important role in many practical situations when analyzing security which justifies our modification.

Let us finally consider the following slight modification of the previous matrix game example:

$r_1$ :

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | (1,0,0) | (1,0,0) |
| $s_2$ | (1,0,0) | (1,0,0) |

$r_2$ :

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | (2,0,0) | (2,0,1) |
| $s_2$ | (2,1,0) | (-1,-1,-1) |

To make this example more obvious we will call it "the game with rescue boat at shipwreck". The first player (the boat captain) has two strategies: $r_1$ - keep the boat for himself and $r_2$ - provide place in the rescue boat for other players. Players 2 and 3 have two strategies: $s_1$, $t_1$ - avoid the rescue boat and $s_2$, $t_2$ - try to get place in the rescue boat. The rescue boat sinks with all players at profile $(r_2, s_2, t_2)$ when all players get place in the boat. There are no threats in the game according to a formal definition. There are three EinSS: $(r_1, s_2, t_2)$, $(r_2, s_1, t_2)$, $(r_2, s_2, t_1)$ which are also Nash equilibria. If we assume that players 2 and 3 take their actions sequentially then the game will be set either in the equilibrium profile $(r_2, s_1, t_2)$ or in the equilibrium profile $(r_2, s_2, t_1)$ which implies two players on board and one player trying to save his life by himself. Let us now consider deviation of the first player from $(r_1, s_1, t_1)$ into $(r_2, s_1, t_1)$ as we move from payoffs $(1, 0, 0)$ to payoffs $(2, 0, 0)$. Formally it is a secure deviation for the first player since there are no individual threats of other players to him in his new position. The profile $(r_1, s_1, t_1)$ is not formally an EinSS. However if we assume that players 2 and 3 would take their actions simultaneously and independently (which is probably the case at the shipwreck) they will end up in the profile $(r_2, s_2, t_2)$ and would not only do harm to player 1 but also to themselves. Therefore if player 1 takes into account the possibility of simultaneous and independent actions of players 2 and 3 then he would not consider deviation $(r_1, s_1, t_1) \to (r_2, s_1, t_1)$ as a secure one.

This example shows that in order to take into account more "complex" threats one should make particular assumptions about the simultaneous or sequential nature of deviations of other players. This raises again the question of the relationship of the proposed concept to dynamic games.

## 7. EinSS in the Hotelling's Model

To illustrate the concept of EinSS we examine the classic model of spatial competition between two players formulated by Hotelling (1929). The principal theoretical problem of this model is that for a great variety of transportation cost functions no price equilibrium exists. In particular, D'Aspremont et al. (1979) showed that in the original Hotelling's game with linear transportation costs there is no price equilibrium when duopolists choose locations too close to each other. The following matrix game example can be considered as an illustration for the Hotelling's game:

| | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|
| $s_1$ | (2,2) | (3,3) | (8,0) |
| $s_2$ | (3,3) | (4,4) | (5,5) |
| $s_3$ | (0,8) | (5,5) | (6,6) |

There is no Nash equilibrium in this matrix game as well. The incentive to maximize profits impels players to choose strategies with a higher number. However if at least one player chooses the third strategy it makes profitable for his competitor to choose the first strategy and leave the first player with zero profit. The first player can in turn choose the strategy with a lower number and get positive profit again. The highest possible secure profits are reached at the profile with payoffs $(4,4)$ which is the EinSS.

This corresponds to the situation in the Hotelling's price game when one player can undercut his rival's price and take away his entire business with profit to himself. However the player pressed out of the market can decrease his price and regain some positive profit. Although the Hotelling's game has no Nash price equilibrium when players choose locations too close to each other it does have just like the above matrix game an equilibrium in secure strategies. The solution of the Hotelling's price game in secure strategies in the original setting was presented in M.Iskakov and A.Iskakov (2012). In the particular case of the discrete Hotelling's problem an equilibrium concept which coincides with the EinSS was published in Shy (2002). In this paper we provide solution in secure strategies of the Hotelling's price game with a restricted reservation price on an infinite line.

On an infinite line two sellers of a homogeneous product with zero production cost are located at the distance $\delta$ from each other. The sellers maximize profits by setting prices $p_1, p_2$ noncooperatively. Customers are evenly distributed with a unit density along the line. When buying from one of the sellers the consumer bears a transportation cost which is linear in the distance. The transportation rate is $t$. A customer purchases from the seller who quotes the lower full price (including transportation). In contrast to the original version of Hotelling's model we assume that the customer refrains from buying if the full price exceeds his reservation price $v$. The sold quantities are equal respectively to the lengths of intervals with the customers choosing the corresponding seller. Therefore the profit functions of the firms are:

$$(1)\quad \tilde{u}_1(\tilde{p}_1,\tilde{p}_2) = \begin{cases} \tilde{u}_1^I = 2\tilde{p}_1\left(v-\tilde{p}_1\right)/t, & \tilde{p}_1 < \tilde{p}_2 - \delta t \\ \tilde{u}_1^{II} = \tilde{p}_1(v-\tilde{p}_1+ & \\ \quad +\min\{v-\tilde{p}_1, \frac{\delta t+\tilde{p}_2-\tilde{p}_1}{2}\})/t, & |\tilde{p}_1-\tilde{p}_2| \leqslant \delta t \\ 0, & \tilde{p}_1 > \tilde{p}_2 + \delta t \end{cases}$$

$$\tilde{u}_2(\tilde{p}_1,\tilde{p}_2) = \begin{cases} \tilde{u}_2^I = 2\tilde{p}_2\left(v-\tilde{p}_2\right)/t, & \tilde{p}_2 < \tilde{p}_1 - \delta t \\ \tilde{u}_2^{II} = \tilde{p}_2(v-\tilde{p}_2+ & \\ \quad +\min\{v-\tilde{p}_2, \frac{\delta t+\tilde{p}_1-\tilde{p}_2}{2}\})/t, & |\tilde{p}_1-\tilde{p}_2| \leqslant \delta t \\ 0, & \tilde{p}_2 > \tilde{p}_1 + \delta t \end{cases}$$

These expressions can be simplified if we make the following change of variables:

$$(2)\qquad u = \tilde{u}t/v^2, \quad p = \tilde{p}/v, \quad d = \delta t/v$$

In the dimensionless form profit functions (1) can be written as:

$$(3)\quad u_1(p_1,p_2) = \begin{cases} u_1^I = 2p_1\left(1-p_1\right), & p_1 < p_2 - d \\ u_1^{II} = p_1(1-p_1+ & \\ \quad +\min\{1-p_1, \frac{d+p_2-p_1}{2}\}), & |p_1-p_2| \leqslant d \\ 0, & p_1 > p_2 + d \end{cases}$$

$$u_2(p_1,p_2) = \begin{cases} u_2^I = 2p_2\left(1-p_2\right), & p_2 < p_1 - d \\ u_2^{II} = p_2(1-p_2+ & \\ \quad +\min\{1-p_2, \frac{d+p_1-p_2}{2}\}), & |p_1-p_2| \leqslant d \\ 0, & p_2 > p_1 + d \end{cases}$$

Dimensionless prices and payoffs depend upon only one free parameter $d$ instead of three parameters $\delta, v, t$. In order to find equilibria in secure strategies in the dimensionless Hotelling's game (3) one can first analyze the threats and identify the secure profiles. Then one can find the Best Secure Response functions, identify BSR-profiles and select the stable ones. According to Propositions 3 and 4 they will correspond to the Equilibria in Secure Strategies. The obtained result is summarized in the following proposition.

**Proposition 7.** *The dimensionless Hotelling's price-setting game $\{i \in \{1,2\}, p_i \in [0,1], u_i(p_1,p_2) \in R\}$ on an infinite line with a restricted reservation price and the profit functions (3) has the following solution in secure strategies depending on the distance $d$ between the sellers:*

$$\text{(4a)} \qquad \textit{when } d \in \left[0, \frac{10\sqrt{10}-14}{67}\right] \approx [0, 0.263]:$$

$$p_1^* = p_2^* = p^* = \frac{2+7d-\sqrt{17d^2-4d+4}}{4},$$
$$u_1^* = u_2^* = 2(p^*-d)(1-p^*+d);$$

$$\text{(4b)} \qquad \textit{when } d \in \left[\frac{10\sqrt{10}-14}{67}, \frac{6}{7}\right]:$$

$$p_1^* = p_2^* = p^* = \frac{2+d}{5},\ u_1^* = u_2^* = \frac{3}{2}p^{*2};$$

$$\text{(4c)} \qquad \textit{when } d \in \left[\frac{6}{7}, 1\right] \textit{ - multiple solutions}:$$

$$\max\left\{\frac{1}{2}, \frac{10}{7}-d\right\} \leqslant p_1^* \leqslant \min\left\{\frac{4}{7}, \frac{3}{2}-d\right\},$$
$$p_2^* = 2-d-p_1^*,\ u_i^* = 2p_i^*(1-p_i^*),\ i \in \{1,2\};$$

$$\text{(4d)} \qquad \textit{when } d \geqslant 1:\ p_i^* = u_i^* = 0.5, i \in \{1,2\}.$$

***Proof.*** See Appendix. □

**Corollary.** *The Hotelling's price-setting game* $\{i \in \{1,2\}, \tilde{p}_i \in [0,v], \tilde{u}_i(\tilde{p}_1, \tilde{p}_2) \in R\}$ *on an infinite line with a restricted reservation price* $v$ *and the profit functions (1) has the following solution in secure strategies depending on* $\delta, v, t$*:*

$$\tilde{u}(\delta, v, t) = \frac{v^2}{t} u\left(\frac{\delta t}{v}\right), \quad \tilde{p}(\delta, v, t) = vp\left(\frac{\delta t}{v}\right) \tag{5}$$

*where* $u(d)$ *and* $p(d)$ *is given by (4).*

***Proof*** is given by inverse change of variables in relation to (2). □

The dependence (4) of the equilibrium prices and profits from the distance between the stores is shown in Fig.2 for dimensionless price game. The shaded area corresponds to the multiple solutions. The analysis of the price competition on a line in secure strategies allows to distinguish four qualitative cases of interaction between competitors. When they are situated very close (the area $BC$ in Fig.2) both players are limited by the threat of mill-price undercutting. The corresponding Equilibrium in Secure Strategies (4a) can be interpreted as *the Bilateral Containment equilibrium (or BC-Equilibrium).* Under the threat of being driven out of the market by undercutting the equilibrium prices in the BC area are much lower as compared Hotelling price equilibrium. In the second area (the area $H$ in Fig.2) the Nash equilibrium (4b) found by Hotelling is realized. One can call it *the Hotelling Equilibrium (or H-Equilibrium).* In the third area (the area $B$ in Fig.2) the competition reaches the multiple Nash Equilibria (4c). They can be called *Borderline Equilibria or B-Equilibria* and interpreted as a division of spheres of influence on the border of the trade zones of players. And

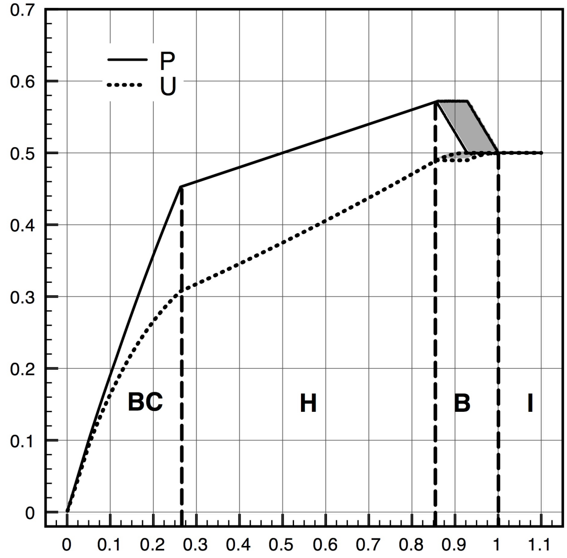


Fig. 2: The equilibrium secure prices (P) and profits (U) in the price Hotelling's game on an infinite line depending on the distance $d$ between the stores.

finally in the fourth area when $d > 1$ the local monopoly (4d) is realized when trade zones of players are not intersected. One can call it *the Independent Price Equilibrium (or I-Equilibrium).*

Let us consider multiple price equilibria in the area $B$. The full price on the border of trade zones of players reaches in B-equilibrium exactly the reservation price. The equilibrium payoff function can be calculated through equilibrium price according to (4c) as $u^* = 2p^*(1 - p^*)$. It is not profitable for player to raise the price and break away from trade zone of the rival when $\frac{\partial u}{\partial p}|_{p=p^*+0} = 2(1-2p^*) < 0$, i.e. when $p^* \geqslant \frac{1}{2}$. It is not profitable for player to lower the price and take market share from the rival when $\frac{\partial u}{\partial p}|_{p=p^*-0} = \frac{4-7p}{2} > 0$, i.e. when $p^* \leqslant \frac{4}{7}$. Hence inside the price interval $\frac{1}{2} \leqslant p^* \leqslant \frac{4}{7}$ we obtain multiple price equilibrium solutions for both players.

## 8. EinSS in the Tullock Contest[3]

In the Tullock Contest n players compete for a prize and each player exerts effort $x_i$ so as to increase his probability of winning $x_i / \sum_{j=1}^{n} x_j$ (Tullock, 1967, 1980). Scaperdas (1996) suggested a more generalized form of the game with the expected profits of players $x_i^\alpha / \sum_{j=1}^{n} x_j^\alpha - x_i,\ \alpha > 0$. The detailed analysis of the game in terms of secure strategies will be provided in our new publication (M.Iskakov, A.Iskakov, A.Zaharov, 2012). Here we consider the Tullock Contest of two players to illustrate the EinSS concept. The players exert efforts $x_1$ and $x_2$. The contest is supposed to be fair and the payoff functions of players are

[3] The results of this section were obtained with participation of Alexey Zakharov.

taken as:

$$u_1 = \frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha} - x_1,\ u_2 = \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha} - x_2,\ \alpha > 0 \tag{6}$$

This game reaches the unique Nash equilibrium $(\alpha/4, \alpha/4)$ when $\alpha \leqslant 2$ and there is no equilibrium when $\alpha > 2$.

The following matrix game example can be considered as an illustration for the Tullock Contest of two players.

| | $t_1$ | $t_2$ | $t_3$ |
|---|---|---|---|
| $s_1$ | (0,0) | (0,4) | (0,3) |
| $s_2$ | (4,0) | (2,2) | (-1,-1) |
| $s_3$ | (3,0) | (-1,-1) | (-2,-2) |

There is Nash equilibrium $(s_2, t_2)$ in this game when players get equal payoffs $(2, 2)$. There are also two EinSS $(s_1, t_3)$ and $(s_3, t_1)$ in which one player gains 3 and the other player has to be content with zero payoff to avoid losses. Formally the first player could deviate from $(s_3, t_1)$ into $(s_2, t_1)$ increasing his payoff from 3 to 4. But he would prefer not to do it since it is not secure deviation and the other player would in turn bring the game into the Nash equilibrium $(s_2, t_2)$ with equal payoffs $(2, 2)$. This example shows that even if there is a unique Nash equilibrium (which seems to complete the study of the game) there may be additional equilibria in secure strategies which significantly alter the overall picture. In the given case there are three stable profiles which have different values for players. Which of them will be realized in the game is not predetermined and each player is interested in the profile favorable to him (like in the game of battle of the sexes).

In the Tullock Contest the equilibria from the above matrix game correspond to the EinSS of the two possible types. One of them coincides with the Nash Equilibrium found by Tullock (1980). The other two equilibria correspond to the monopolistic EinSS. In these equilibria the winning monopolist fixes high enough payment for the rent to create the entrance barrier for the other player making him unprofitable to participate in the competition.

The general algorithm of finding solution in secure strategies is following. First the set of secure profiles is found as well as the best secure responses of players. Then the BSR-profiles are found as an intersection of the best secure responses of players plotted in the plane of strategies $(x_1, x_2)$. And finally the conditions of the EinSS are checked for these BSR-profiles.

The best response and the best secure response of the second player are plotted in the plane of strategies $(x_1, x_2)$ in Fig.3. The shaded (gray) area corresponds to the profiles secure for the second player. The dashed line represents the best response and the solid line represents the best secure response. Secure profiles, best response and best secure response for the first player are symmetric. The analysis of the Tullock Contest of two players in terms of secure strategies can be summarized by the following proposition.

**Proposition 8.** *When $0 < \alpha < 1$ the Tullock Contest (6) of two players reaches the following unique equilibrium in secure strategies (which is also Nash equilibrium):*

$$\{\alpha/4, \alpha/4\}. \tag{7}$$

*When $1 \leqslant \alpha \leqslant 2$ the Tullock Contest (6) reaches the following*

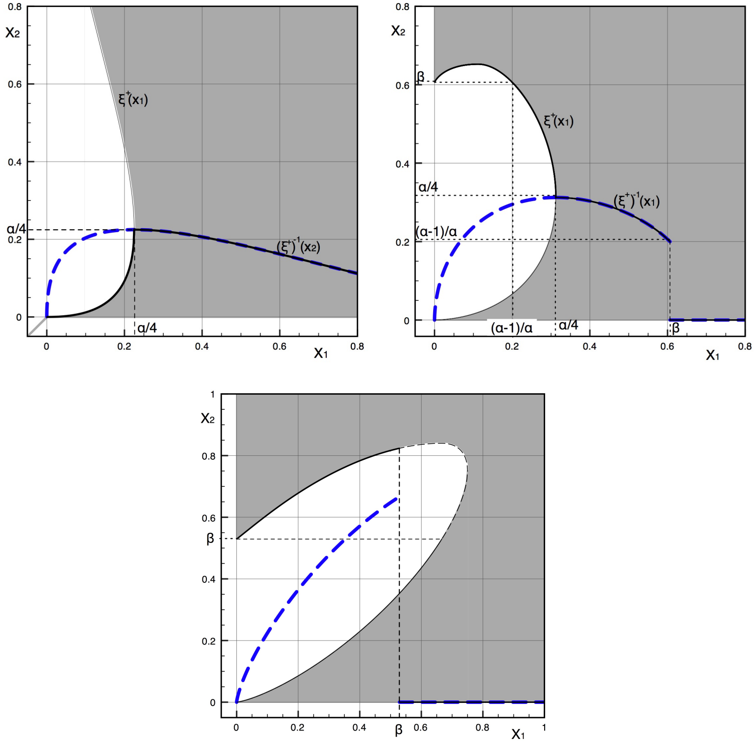


Fig. 3: Secure profiles (gray area), best response (dashed line) and best secure response (solid line) for the player 2 depending on the parameter $\alpha$: $\alpha < 1$ (left), $1 \leqslant \alpha \leqslant 2$ (right) and $\alpha > 2$ (center). $\beta \equiv \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}}$

*equilibria in secure strategies:*

$$\{\alpha/4, \alpha/4\} \cup \{(0, \bar{x})\} \cup \{(\bar{x}, 0)\}, \tag{8}$$
$$\text{where} \quad \bar{x} = \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}, \quad \alpha > 1 \quad \text{and} \quad \bar{x} = 1, \quad \alpha = 1$$

*and all other equilibria in secure strategies lie on the curve:*

$$\left\{(x_1,\ \xi^+(x_1)) : \ \frac{\alpha-1}{\alpha} \leqslant x_1 \leqslant \frac{\alpha}{4}\right\} \cup$$
$$\cup \left\{(\xi^+(x_2),\ x_2) : \ \frac{\alpha-1}{\alpha} \leqslant x_2 \leqslant \frac{\alpha}{4}\right\}, \tag{9}$$
$$\xi^+(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2}\left(\alpha - 2x_i + \sqrt{\alpha^2 - 4\alpha x_i}\right)\right)^{1/\alpha},$$
$$\max\left\{0, \frac{\alpha^2-1}{4\alpha}\right\} \leqslant x_i \leqslant \alpha/4.$$

*When $\alpha > 2$ the Tullock Contest (6) reaches only two monopolistic equilibria in secure strategies:*

$$\left\{(0,\ \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}})\right\} \cup \left\{(\frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}},\ 0)\right\}. \tag{10}$$

***Proof.*** See in (M.Iskakov, A.Iskakov, A.Zaharov, 2012). □

***Remark.*** Our numerical computations showed that all points on the curve (9) are in fact multiple equilibria in secure strategies.

The concept of EinSS allows to discover new type of equilibria in the Tullock game of the rent-seeking. In these equilibria the player prefer to fix his or her secure monopolistic position rather than to participate in the competition. Moreover when power parameter $\alpha > 2$ the monopolistic situation is the only stable position in the game in terms of secure strategies. The logic of the best responses can not reveal the possibility of such kind of equilibria since it does not take into account the security considerations and assumes the player would choose the most profitable but insecure and possibly eventually not-profitable for him strategy.

The power parameter $\alpha$ can be interpreted as the stiffness of the competition at rent-seeking. There is an egalitarian distribution of rent at $\alpha \leqslant 1$, i.e. the probability to win for the player paying less is more than proportional to his contribution. Following the classification of North, Wallis and Weingast (2009) we can interpret the corresponding Nash equilibrium as *an equilibrium of an open access.* If $\alpha \geqslant 2$ then the competition rules are strongly differentiating. The chances to win for players contributing small payments are much less than proportional to their contributions. The only possible equilibrium in this case can be interpreted as an *equilibrium of the privileged monopoly* which fixes access to resources or institutions to one player. In the intermediate case of $1 \leqslant \alpha \leqslant 2$ the rules of competition are weakly differentiating and there are possibilities both for the equilibrium of an open access and for the equilibrium of the privileged monopoly. Furthermore there are also intermediate equilibria which could be interpreted as *equilibria of unequal or limited access.*

## Conclusion

The article presents a new concept of the Equilibrium in Secure Strategies. Although it was first published in (Iskakov, 2005, 2008) here is the first time we formulate it in the intuitively clear way. We discuss in detail its connection with the other closely related theoretical models such as the solution in objections and counter objections, the profile of the best secure responses and the game with uncertain insider. We illustrate our concept by the set of matrix game examples. And finally we apply it to analyze two well-known games that fail to have Nash equilibria. In fact all sections of this paper have arisen in the process of active discussions at the conferences and seminars often as answers to the posed questions. The obtained results confirm the practical value and adequacy of the proposed concept and lay a firm ground for the future research.

## Acknowledgements

We are deeply indebted to C.d'Aspremont who helped us to find an elegant and intuitively clear formulation of the EinSS concept and inspired us to write this paper. We are especially thankful to F.Aleskerov and D.Novikov for regular discussions on the subject during seminars at Moscow High School of Economics and at V.A.Trapeznikov Institute of Control Sciences RAS. The results on the Tullock Contest were obtained with participation of Alexey Zakharov. We wish to thank leaders of the seminars V.Burkov, V.Danilov, V.Polterovich and J.-F.Thisse. We wish to thank A.Chkhartishvili, M.Gubko, S.Izmalkov, N.Korgin, S.Kokovin, N.Kukushkin, S.Mishin, P.Pavlov, A.Savvateev, K.Sonin, A.Vasin and E.Zhelobodko for the fruitful discussions. Our work was supported by the grant of the RFBR (No.10-07-00063-a).

***Comment on the definitions of EinSS***

In order to keep consistency with the previously used form of definitions of EinSS we prove here that *the definitions of EinSS 1-4 are equivalent to the definitions published in (M.Iskakov and A.Iskakov, 2012).*

***Proof.*** Below we provide the definitions of the EinSS published in M.Iskakov and A.Iskakov (2012). The definition of threat and the definition of secure profile are the same. The definitions 3 and 4 were formulated in the following form:

**Definition 3*.** *A set $W_i(s)$ of preferable strategies secured against threats* is a set of strategies $s_i'$ of player $i$ at a given $s$ such that $u_i(s_i', s_{-i}) \geqslant u_i(s)$ and provided that $u_i(s_i', s_j', s_{-ij}) \geqslant u_i(s)$ for any threat $\{(s_i', s_{-i}), (s_i', s_j', s_{-ij})\}$ of player$j \neq i$ to player $i$.

**Definition 4*.** A strategy profile $s^*$ is *an Equilibrium in Secure Strategies* (EinSS) if and only if for all $i$ we have that

$$W_i(s^*) \neq \varnothing, \quad s_i^* \in \arg \max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*).$$

Any strategy from *the set $W_i(s)$ of preferable strategies secured against threats* according to the definition 3* is either **(a)** a secure strategy $s_i'$ such that $u_i(s_i', s_{-i}) = u_i(s)$ or **(b)** *secure deviation* according to the Definition 3. If $s^*$ is an EinSS according to 4* then for all $i$ $s_i^* \in W_i(s^*)$. $s_i^*$ can not be **(b)** *secure deviation* according to the Definition 3. Therefore $s_i^*$ must be **(a)**, i.e a secure strategy in the profile $s^*$. Since all strategies $s_i^*$ are secure then the profile $s^*$ is a secure profile.

If some player can increase his profit by secure deviation then $s_i^* \notin \arg\max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*)$. Therefore no player in $s^*$ can make *a secure deviation* (according to the Definition 3). $s^*$ is an EinSS according to the Definition 4. Now let $s^*$ is the EinSS according to the Definition 4. As $s^*$ is a secure profile so for all $i$ $s_i^* \in W_i(s^*)$ and $W_i(s^*) \neq \varnothing$. As no player in $s^*$ can increase his profit by secure deviation so $s_i^* \in \arg\max_{s_i \in W_i(s^*)} u_i(s_i, s_{-i}^*)$. And $s^*$ is an EinSS according to the definition 4*. □

### *Proof of the Proposition 7*

We will use below the following notation according to (3). As $u_i^I(p_i)$ we denote the payoff function of player $i$ in the domain $p_i < p_{-i} - d$ where player $i$ captures the whole market and his payoff function depends only upon his own price. As $u_i^{II}(p_i, p_{-i})$ we denote the payoff function of player $i$ in the domain $|p_i - p_{-i}| \leqslant d$ where price competition between two players takes place. In order to find equilibria in secure strategies in the dimensionless Hotelling's game (3) let us first identify the secure profiles and prove the following Lemma.

**Lemma.** *The profile $(p_1, p_2)$ in the dimensionless price-setting game $\{i \in \{1,2\}, p_i \in [0,1], u_i(p_1,p_2)\}$ with the profit functions (3) is a secure strategy profile if and only if*

*1) when $d < 1$ :*

$$
\begin{cases}
|p_1 - p_2| \leqslant d & (\Pi.1a) \\
p_i \leqslant \arg\max\limits_p u_i^{II}(p, p_{-i}),\ i \in \{1,2\} & (\Pi.1b) \\
\textit{if } p_{-i} > d, \quad u_i^I(p_{-i} - d) \leqslant u_i^{II}(p_i, p_{-i}),\ i \in \{1,2\} & (\Pi.1c)
\end{cases}
$$

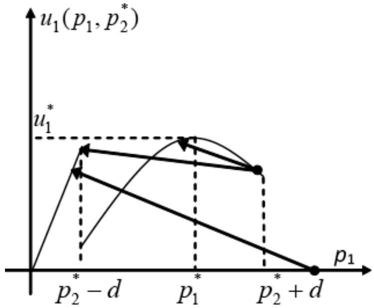


Fig. 4: The first player increases profit either by shifting price in area $I$ or in area $II$.

*2) when* $d \geqslant 1$ :

(П.2)

$$\begin{cases} |p_1 - p_2| \leqslant d \\ p_i \leqslant \arg\max_p u_i^{II}(p, p_{-i}),\ \ i \in \{1, 2\} \end{cases} \cup \begin{cases} |p_1 - p_2| \leqslant d - 1 \\ p_1 + p_2 > 2 - d \end{cases}$$

***Proof of Lemma.*** When $p_1 < p_2 - d$ player 2 gets zero profit and there is always a threat to player 1 that player 2 will decrease his price till $\hat{p}_2 < p_1$ and will get positive profit. Since the trade zones of players are in contact then the market share and the profit of player 1 will decrease. Therefore the profile $(p_1, p_2)$ is not secure for player 1. In a similar way when $p_2 < p_1 - d$ the profile $(p_1, p_2)$ is not secure for player 2. Hence all secure profiles lie in the area $|p_1 - p_2| \leqslant d$. The condition (П.1a) and the first condition in (П.2) are proven.

Let us consider the existing threats to player 2 when $|p_1 - p_2| \leqslant d$. According to (3) the payoff function of the first player $u_1(p_1)$ in each price area $I$ ($p_1 < p_2 - d$) and $II$ ($|p_1 - p_2| \leqslant d$) is concave and one-picked (see Fig.4). Player 1 can increase his profit only in two ways: either by shifting price in the price area $I$ or by moving closer to the pick in the price area $II$. The first situation is possible when $\max_{p\in(0,p_2-d)} u_1^I(p) > u_1^{II}(p_1,p_2)$ and always produces a threat to player 2 to be driven out of the market. The corresponding security condition is $\max_{p\in(0,p_2-d)} u_1^I(p) \leqslant u_1^{II}(p_1,p_2)$. In the second situation player 2 keeps his security in two cases. Either the first player can not increase his profit by reducing price, i.e. $p_1 \leqslant \arg\max_p u_1^{II}(p,p_2)$. Or he can increase his profit by reducing price but even at maximum reduction of his price profitable to him his trade zone will not get in contact with the trade zone of player 2. For the profit functions (3) the last condition can be written as $1/2 < p_1 \leqslant d-1+p_2$. Therefore the security condition of player 2 against both types of threats can be written as:

$$\begin{cases} \text{if } p_2 > d, \quad \max_{p\in(0,p_2-d)} u_1^I(p) \leqslant u_1^{II}(p_1,p_2) & (1*) \\ p_1 \leqslant \arg\max_p u_1^{II}(p,p_2) \quad \text{or} \quad 1/2 < p_1 \leqslant d-1+p_2 & (2*) \end{cases}$$

According to (1) at $|p_1-p_2| \leqslant d$ we have $u_1^{II}(p_1,p_2) \leqslant u_1^I(p_1)$ for all $p_2$. Then it follows from the $(1*)$ that $\max_{p\in(0,p_2-d)} u_1^I(p) \leqslant u_1^I(p_1)$, i.e. the concave function $u_1^I(p)$ reaches maximum at $p > p_2-d$ and therefore $\max_{p\in(0,p_2-d)} u_1^I(p) = u_1^I(p_2-d)$. Then the first condition $(1*)$ can be written in a more convenient form as $u_1^I(p_2-d) \leqslant u_1^{II}(p_1,p_2)$. The security condition for the profile $(p_1,p_2)$ can be written then as:

$$\begin{cases} |p_1-p_2| \leqslant d \\ p_i \leqslant \arg\max_p u_i^{II}(p,p_{-i}) \quad \text{or} \quad 1/2 < p_i \leqslant d-1+p_{-i}, \quad i\in\{1,2\} \\ \text{if } p_{-i} > d, \quad u_i^I(p_{-i}-d) \leqslant u_i^{II}(p_i,p_{-i}), \quad i\in\{1,2\} \end{cases} \qquad (*)$$

Now let us assume that for the secure profile $(p_1, p_2)$ at least one of the conditions $1/2 < p_i \leqslant d - 1 + p_{-i}$ is satisfied. For example $1/2 < p_1 \leqslant d - 1 + p_2$ which implies $p_2 > 3/2 - d$ and $p_1 + p_2 > p_2 + (1 - p_1) \geqslant 2 - d$.
If $p_2 \leqslant 1/2$ then $3/2 - d < p_2 \leqslant 1/2 \implies d > 1$.
If $p_2 > 1/2$ and $p_1 + p_2 > 2 - d \implies p_2$ must be on the descending part of the function $u_2^{II}(p_1, p)$, i.e. $p_2 > \arg\max_p u_2^{II}(p_1, p)$ $\implies$ the condition $p_2 \leqslant d - 1 + p_1$ must be satisfied $\implies$ $p_1 + p_2 \leqslant 2d - 2 + p_1 + p_2 \implies d \geqslant 1$.
Therefore it is proven that if $d < 1$ then neither of the conditions $1/2 < p_i \leqslant d - 1 + p_{-i},\ i \in \{1, 2\}$ is satisfied for the secure profile $(p_1, p_2)$. Therefore the formula (П.1) is proven.

Let us prove that the $(*)$ is equivalent to (П.2) when $d \geqslant 1$. The conditions (П.1c) and $(1*)$ together with the threat of mill-price undercutting disappear when $d \geqslant 1$ since in this case we have $p_i \leqslant 1 \leqslant d,\ i \in \{1, 2\}$.
For the profiles $(p_1, p_2)$ which satisfy the condition $p_1 + p_2 \leqslant 2 - d$ the conditions (П.2) and $(*)$ are obviously equivalent (since for these profiles the second conditions in $(2*)$ are not satisfied).
For the profiles $(p_1, p_2)$ which satisfy the condition $p_1 + p_2 > 2 - d$ we obtain $\arg\max_p u_i^{II}(p, p_{-i}) = \min\{\frac{2+d+p_{-i}}{6}, \max\{2 - d - p_{-i}, 1/2\}\} = \frac{1}{2}$ and the conditions (П.2) and $(*)$ take the following forms:

$$\{(p_1, p_2) : p_1 \leqslant 1/2, p_2 \leqslant 1/2\} \cup \{(p_1, p_2) : |p_1 - p_2| \leqslant d - 1\}$$

$$\text{and} \qquad \begin{cases} |p_1 - p_2| \leqslant d \\ p_1 \leqslant 1/2 \quad \text{or} \quad 1/2 < p_1 \leqslant d - 1 + p_2 \\ p_2 \leqslant 1/2 \quad \text{or} \quad 1/2 < p_2 \leqslant d - 1 + p_1 \end{cases}$$

The equivalence of these conditions at $d \geqslant 1$ for the profiles satisfying $p_1 + p_2 > 2 - d$ can be proven by straightforward verification. □

Now we are ready to prove the Proposition.
According to Lemma the set of secure strategies in the price Hotelling's game at $d \geqslant 1$ is given by (Π.2). Substituting into this system the expressions (3) for the payoff functions we obtain:

$$\begin{cases} \text{if} \quad p_1 + p_2 \leqslant 2 - d, \quad |p_1 - p_2| \leqslant d \\ \text{if} \quad p_1 + p_2 > 2 - d, \quad |p_1 - p_2| \leqslant d - 1 \\ p_{-i} \leqslant \min\left\{\frac{2+d+p_i}{6}, \max\{2 - d - p_i, 1/2\}\right\}, i \in \{1,2\} \end{cases}$$

The best secure responses of players at $|p_1 - p_2| \leqslant d$ take the following form ($i \in \{1,2\}$):

$$p_{-i} = \max\left\{1 - d + p_i, \min\left\{\frac{2 + d + p_i}{6}, \max\{2 - d - p_i, 1/2\}\right\}\right\}.$$

which has at $d \geqslant 1$ the unique solution (4d).

The set of secure strategies in the price Hotelling's game at $d < 1$ according to Lemma is given by the system (Π.1). Substituting into this system the expressions (1) for the payoff functions we obtain:

$$\begin{cases} |p_1 - p_2| \leqslant d & (*a) \\ p_{-i} \leqslant \min\left\{\dfrac{2 + d + p_i}{6}, \max\{2 - d - p_i, 1/2\}\right\}, i \in \{1,2\} & (*b) \\ p_{-i} \leqslant \max\left\{d, d + \dfrac{4 - p_i}{8} - \right. \\ \quad \left. - \sqrt{\left(d + \dfrac{4 - p_i}{8}\right)^2 - \dfrac{p_i}{4}(2 + d - 3p_i) - d(d+1)}\right\} & (*c) \end{cases}$$

In the last inequality we take into account that $(*b) \Rightarrow 5p_{-i} \leqslant 2 + d + p_i - p_{-i} \leqslant 2 + 2d \Rightarrow p_{-i} \leqslant \frac{2+2d}{5} < \frac{4+7d}{9} \Rightarrow 8p_{-i} < 4 + 8d - d - p_{-i} \leqslant 4 + 8d - p_i \Rightarrow p_{-i} < d + \frac{4-p_i}{8} \Rightarrow$ the second branch of the solution of the quadratic inequality $(*c)$ is not realized.

Under the found conditions $(*a, *b, *c)$ the function $u_i^{II}(p_i, p_{-i})$ increases by $p_i$ and hence the Best Secure Response (BSR) of players at $|p_1 - p_2| < d$ takes the following form $(i \in \{1, 2\})$:

$$(*) \quad p_{-i} = \min\left\{\frac{2 + d + p_i}{6}, \max\left\{2 - d - p_i, 1/2\right\}, \max\left\{d,\right.\right.$$

$$\left.\left. d + \frac{4 - p_i}{8} - \sqrt{\left(d + \frac{4 - p_i}{8}\right)^2 - \frac{p_i}{4}(2 + d - 3p_i) - d(d + 1)}\right\}\right\}.$$

These equations define the plots of the best secure responses of players in the plain $(p_1, p_2)$ at $|p_1 - p_2| < d$. The intersection of these plots is the point of the BSR-profile. According to the Proposition 3 any EinSS is the BSR-profile, i.e. it must satisfy $(*)$. From the other side any solution of $(*)$ is the EinSS. Indeed any deviation of player from $(*)$ in the direction of lower price decreases his profit. And any deviation from $(*)$ in the direction of higher price either decreases his profit or creates the threat of being undercut throughout the whole market, i.e. no player can increase his profit by secure deviation.

The solution of the system $(*)$ corresponds to the first three cases in the Proposition 7. Indeed this solution shall be symmetric about a line $p_1 = p_2$ and shall by of two types. The multiple solutions of $(*)$ lie in the interval of the line

$p_1 + p_2 = 2 - d$ which is the common place for the BSR of both players. Checking the limit conditions provides the solution (4c). The solutions of (∗) of another type are located on line $p_1 = p_2 \equiv p$. When $d \geqslant 6/7$ the solution is defined by the following equation:

$$p = \min\left\{\frac{2+d+p}{6}, \max\{2-d-p, 1/2\}\right\},$$

When $6/7 \leqslant d \leqslant 1$ this solution takes the form $p_1 = p_2 = 1 - d/2$ which is a special case of the multiple solution (4c). Finally the solution at $d \leqslant 6/7$ is defined by the equation:

$$p = \min\left\{\frac{2+d+p}{6}, \max\left\{d,\right.\right.$$

$$\left.\left. d + \frac{4-p}{8} - \sqrt{\left(d + \frac{4-p}{8}\right)^2 - \frac{p}{4}(2+d-3p) - d(d+1)}\right\}\right\},$$

which gives solutions (4a) and (4b). The Proposition is proven.
□

## REFERENCES


1. D'Aspremont C., Gabszewicz J., Thisse J.-F. (1979), On Hotelling's 'Stability in Competition', *Econometrica*, **47**(5), 1145-1150.

2. Aumann R.J., Maschler M. (1964), The bargaining set for cooperative games, Advances in game theory, *Ann. Math. Studies*, **52**, Princeton Univ. Press, pp. 443–476.

3. Dasgupta P., Maskin E. (1986), The existence of equilibrium in discontinuous economic games, II: Applications, *Review of Economic Studies*, **LIII**, pp. 27-41

4. Eaton B.C., Lipsey R. (1978), Freedom of Entry and the Existence of Pure Profits, *The Economic Journal*, **88** (351), 455-469.

5. Hotelling H. (1929), Stability in Competition, *The Economic Journal*, **39**(153), 41-57.

6. Iskakov M.B. (2005), Equilibrium in Safe Strategies, *Automation and Remote Control*, **66**, 3, 465-478.

7. Iskakov M.B. (2008), Equilibrium in Safety Strategies and equilibriums in objections and counter objections in noncooperative games, *Automation and Remote Control*, **69**, 2, 278-298.

8. Iskakov M., Iskakov A. (2012), Solution of the Hotelling's game in secure strategies, *Economics Letters*, **117**, 115-118.

9. Iskakov M., Iskakov A., Zakharov A. (2012), Solution of the Tullock Contest in Secure Strategies, *Work Paper*, Moscow High School of Economics (in print).

10. Facchinei F., Kanzow C. (2007), Generalized Nash equilibrium problems. *4OR - A Quarterly Journal of Operations Research*, **5**, 173-210.

11. North D.C., Wallis J.J., and Weingast B.R. (2009), Violence and Social Orders: A Conceptual Framework for Interpreting Recorded Human History, New York: Cambridge University Press.

12. Shy O. (2002), A quick-and-easy method for estimating switching costs. *International Journal of Industrial Organization* **20**, 71-87.

13. Skaperdas S. (1994), Contest success functions. *Economic Theory*, **7**, 283-290.

14. Tullock G. (1967), The welfare costs of tariffs, monopoly and theft, *Western Economic Journal*, **5**, 224–232.

15. Tullock G. (1980), Efficient rent seeking. In James M. Buchanan, Robert D. Tollison, Gordon Tullock, (Eds.), Toward a theory of the rent-seeking society. College Station, TX: Texas A&M University Press, 97-112.

**Искаков М., Искаков А.** Развитие концепции равновесий в безопасных стратегиях : препринт WP7/2012/06 [Текст] / М. Искаков, А. Искаков ; Нац. исслед. ун-т «Высшая школа экономики». – М. : Изд. дом Высшей школы экономики, 2012 (на англ. яз.).

Представлена новая концепция равновесия в безопасных стратегиях (РБС) в некооперативных играх. РБС совпадает с равновесием Нэша, когда равновесие Нэша существует, и предполагает стремление игроков максимизировать свой выигрыш при условии безопасности относительно действий других игроков. Новая концепция проиллюстрирована набором матричных игр. В работе проведено ее сравнение с другими близкими теоретическими моделями. Доказано существование равновесий в безопасных стратегиях для двух классических игр, в которых нет равновесий Нэша. Получено решение в безопасных стратегиях для классической игры выбора цен Хотеллинга (1929) на бесконечной прямой с ограниченной резервационной ценой и линейными транспортными тарифами. Обнаружены новые монополистические равновесия в безопасных стратегиях в состязании Таллока (1967, 1980) двух игроков.

*Препринт WP7/2012/06*
*Серия WP7*
Математические методы анализа решений
в экономике, бизнесе и политике

Искаков Михаил, Искаков Алексей

**Развитие концепции равновесий в безопасных стратегиях**

(*на английском языке*)