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A MODEL OF TACIT COLLUSION: NASH-2 EQUILIBRIUM CONCEPT

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A model of tacit collusion: Nash-2 equilibrium concept*

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Abstract

We examine the novel concept for repeated noncooperative games with bounded rationality: "Nash-2" equilibrium, called also "threatening-proof profile" in [16, Iskakov M., Iskakov A., 2012b]. It is weaker than Nash equilibrium and equilibrium in secure strategies: a player takes into account not only current strategies but also the next-stage responses of the partners to her deviation from the current situation that reduces her relevant choice set. We provide a condition for Nash-2 existence, criteria for a strategy profile to be the Nash-2 equilibrium in strictly competitive games, apply this concept to Bertrand and Hotelling game and interpret the results as tacit collusion.

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Keywords: Nash-2 equilibrium, secure deviation, Bertrand paradox, Hotelling model, tacit collusion.

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1 Motivation

Many economic applications of game theory require modelling repeated interactions among players. The common examples are two or several oligopolists competing in Cournot, Bertrand and Hotelling games. Somewhat similar is a repeated game between an employer and employee, a lender and a borrower, repeated common venture, etc. Intuitively many economists agree that quite often the realistic outcome of such game is a tacit collusion [31, Tirole, 1988, Chapter 6]; a sort of quasi-cooperative solution supported by *credible threats*. However, they disagree which formal game concept better describes this practical outcome.

According to Nash, any situation is an equilibrium if nobody can unilaterally increase her current payoff by changing own strategy, when other players do not react. This approach proved to be quite fruitful when the influence of each participant to the whole situation is negligible. However, its common applications to oligopoly rise serious doubts. Say, crazy undercutting one another in Bertrand duopoly is an inefficiency of theory, the same can be said about absence of Nash equilibrium in Hotelling game with prices. In our opinion, the old theoretical battle between Cournot and Bertrand modelling is useless (see [1, Amir, Jin, 2001], [25, Novshek, 1980], [26, Novshek, Chowdhury, 2003], [3, d'Aspremont, Dos Santos Ferreira, 2010], [29, Sidorov, Thisse, at al, 2014]). Both are too bad concepts for repeated games of few players from logic and realism viewpoints. By contrast, dynamic games and related Folk theorem approach (see [6, Benoit, Krishna, 1985], [27, Rubinstein, 1979]) look logically nice. However it is too complicated, both for studying and for players themselves. Indeed, can a theorist believe that two traders optimize in infinitely-dimensional space of all possible responses to all possible trajectories of their behavior? We prefer a bounded rationality concept: taking into account only the current and next step. This behavior displays moderate wisdom: not absolutely myopic as Nash concept and not infinitely wise as Folk theorem approach.

Accounting for strategic aspects of interaction among players can be implemented as a generalization of the Nash equilibrium concept. Since the pioneer concept of perfect and proper equilibria (see [28, Selten, 1975] and [22, Myerson, 1978], respectively) a number of further refinements has been done (for some of them see [30, Simon, Stinchcombe, 1995], [12, Guth, 2002], [9, Carbonell-Nicolau, McLean, 2013]).

A rational player can take in account reactions of other players when she makes a decision whether to deviate from the current strategy or not. Related discussion on the iterated strategic thinking process can be found in [7, Binmore, 1988]. The ideas of players' reflection are not new in non-cooperative games. Reflexive games with complicated hierarchy of reaction levels are developed in [8, Camerer, Ho, Chong, 2004], [24, Novikov, 2012]. This approach with asymmetric rationality of participants leads to rather complicated computations for agents, however some empirical studies supports the approach of k-level rationality ([8, Camerer, Ho, Chong, 2004], [21, Kawagoe, Takizawa, 2009]). We also must mention some farsighted solution concepts based on the idea of k-level rationality – they are the largest consistent set [10, Chwe, 1994], noncooperative farsighted stable set [23, Nakanishi, 2007], farsighted pre-equilibrium [19, Jamroga, Melissen, 2011]. The reasonable degree of farsightness is an open question.

More easy approach introduces *security* as an additional motivation for players' behavior. Two (the most closed to our ideas) second-stage-foreseeing concepts that have been proposed: bargaining set based on the notion of threats and counter-threats (for cooperative games, see [4, Aumann, Maschler, 1964]) and equilibrium in secure strategies (ESS, see [15, Iskakov M., Iskakov A., 2012a], [16, Iskakov M., Iskakov A., 2012b]). In fact, the latter paper introduces the idea of Nash-2 equilibrium as "threatening-proof profile", but it does not develop it, supposing less important than ESS (that differs in additional requirement: security of current profile).[1] The idea of both concepts is that players worry not only about own first-stage payoffs, but also about security against possible "threats" of the opponents, i.e., profitable responses that harms our player, and optimizing on secure set can bring additional stability, more equilibria. The main point of this paper is that for modelling oligopoly with 2-level rational agents, we actually do not need to additional security requirement. Nash-2 equilibrium concept means only absence of profitable deviations subject to the reaction of the opponent. The benefit of such weak concept is existence in most situations. The shortcoming is typical multiplicity of equilibria. So, to select among these equilibria in a specific application of NE-2, we need some additional game-specific considerations to predict a unique solution.

In the sequel, Section 2 defines Nash equilibrium, equilibrium in secure strategies and Nash-2 equilibrium in terms of deviations, threats, and security, it illustrates the concepts with Prisoner's Dilemma. It also provides a condition for existence of Nash-2 equilibrium in a two-person game. In Section 3 we give the complete characterization of Nash-2 equilibria in the class of strictly competitive games. In Sections 4 and 5 we apply our ideas to the classical Bertrand and Hotelling models, and show that Nash-2 concept can provide a strategic explanation for possible collusion between firms.

2 Basic notions and equilibrium concepts

Consider a 2-person non-cooperative game in the normal form

$$G = (i \in \{1, 2\}; s_i \in S_i; u_i : S_1 \times S_2 \to R).$$

Let us give the formal definition of the Nash equilibrium in terms of deviations. This will help us to explain our modification of this equilibrium concept more clearly.

[1] During preparing this paper we have known that simultaneously Iskakov M. and Iskakov A. have returned to studying "threatening-proof profile" renamed as equilibrium contained by counter-threats [18, Iskakov M., Iskakov A., 2014].

Definition 1. A *profitable deviation* of player i at strategy profile $s = (s_i, s_{-i})$ is a strategy s'_i such that $u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$. A strategy profile s is a *Nash Equilibrium* (NE) if no player has a profitable deviation.

Note that inequality is strict so players can not deviate to a situation with the same payoff. The condition on the deviation is rather weak in the sense that too many deviations are allowed. So a game can have no stable profiles. Another shortcoming is that there are games where a Nash equilibrium exists but doesn't seem a reasonable outcome for many repeated games.

In [15, Iskakov M., Iskakov A., 2012a] authors propose a refinement with the notion of security. Here we slightly reformulate the main definitions from [15, Iskakov M., Iskakov A., 2012a].

Definition 2. A (credible) *threat* of player i to player $-i$ at strategy profile s is a strategy s'_i such that

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{and} \quad u_{-i}(s'_i, s_{-i}) < u_{-i}(s_i, s_{-i}).$$

The strategy profile s is said to *pose a threat* from player i to player $-i$. A strategy profile s is *secure* for player i if s poses no treats from other players to i.

Definition 3. A profitable deviation s'_i of player i at s is *secure* if for any threat s'_{-i} of player $-i$ at profile (s'_i, s_{-i})

$$u_i(s'_i, s'_{-i}) \geq u_i(s_i, s_{-i}).$$

Definition 4. A strategy profile is an *equilibrium in secure strategies* (ESS) if

- it is secure,
- no player has a profitable secure deviation.

The crucial suggestion of our paper is that the condition of security may be omitted. This case was mentioned in [16, Iskakov M., Iskakov A., 2012b] and such situations were named threatening-proof. However, these profiles do not satisfy the security condition and the authors didn't study such profiles as equilibria and added the explicit condition of the profit maximization on the set of threatening-proof profiles (solution in objections and counter objections).

We argue that a threatening-proof profile itself is stable enough be viewed as a possible equilibrium concept whereas any additional condition should be motivated by additional information about the specific game modelled. Moreover, we can simplify the definition since we need not the notion of threats to characterize a deviation as secure. The reflexive idea of accounting the responses of the opponent seem to be sufficient for our purposes. Thus, secure deviation in fact matters only on 2-level rationality.

Definition 5. [alternative: profitable secure deviation] A profitable deviation s'_i of player i at $s = (s_i, s_{-i})$ is *secure* if for any profitable strategy s'_{-i} of player $-i$ ($u_{-i}(s'_i, s'_{-i}) > u_{-i}(s'_i, s_{-i})$) our player i is not worse off:

$$u_i(s'_i, s'_{-i}) \geq u_i(s_i, s_{-i}).$$

Definition 6. A strategy profile is a *Nash-2 equilibrium* (NE-2) if no player has a profitable secure deviation.

Here we consider only equilibria in pure strategies, but one can consider mixed extension. Mixed strategies (see [11, Dasgupta, Maskin, 1986]) is an alternative approach to deal with non-existence of pure NE, but we expect that a motivated solution in pure strategies can be also useful for economic applications.

Thus, NE-2 concept seems to be a reasonable outcome in a repeated game. Such 2-stage rationality is one of the possible compromises between zero rationality of NE and infinite rationality of Folk theorem. Obviously, NE-2 may be not secure. Indeed, when NE-2 includes a threat from one player to another nobody actualizes his/her threats because they are not secure deviations for him/her.

Example 1 (Prisoner's Dilemma). Consider the classical non-repeated Prisoner's dilemma.

	Cooperate	Defect
Cooperate	(1,1)	(-1,2)
Defect	(2,-1)	(0,0)

The only NE is for both players to defect. ESS is the same and moreover it is the only secure profile in the game. But one can easily check that in addition to this low-profit equilibrium the mutual cooperation is NE-2, and the profit $(1, 1)$ is more desirable for both players. So we observe the strategic motivation for cooperative solution without explicit modeling repeated game structure; NE-2 is an appropriate description of tacit collusion in such situations whereas NE and ESS are not.

Proposition 1 (Iskakov & Iskakov, 2012)**.** *Any NE is an ESS. Any ESS is a NE-2.*

This claim just follows from definitions and rises the question: is ESS the most "natural" refinement of NE-2? Some refinement would be good for predictions because NE-2 set in some games appears to be large (even continuum, see sec. 4). We can regard it as an analogue of the bargaining set in cooperative games. Note that accounting for the responses of the opponents play a role of tacit communication between players. The result of such approach looks like a tacit collusion between agents. Maximization of joint payoff is one of the ways to choose the concrete equilibrium from the rich set of NE-2, but not the unique one and actually depends on the specific game under consideration. Alternative way is, for instance, to restrict the set of admissible strategies to the set of NE-2 profiles and than play NE, in such a reduced game.

In our "profitable secure deviation" a player takes care about all profitable responses of the opponent. An interesting modification would be to take in account only the opponent's best responses. Papers [13, Halpern, Rong, 2010], [5, Bazenkov, Korepanov, 2014] introduce the similar concepts of equilibrium: cooperative equilibrium and equilibrium in double best responses. One can expect they to be a very close concept to NE-2. Indeed, for some games double best response provides the same equilibrium set, but not always. At least because of computational reasons for analytic solution it is sometimes more practical to apply NE-2 concept as more simple one.

Now we show that NE-2 equilibrium really exists for most games and fails to exist only in degenerate (in some sense) cases. For this purpose we introduce the notion of "secure cycle".

Definition 7. A path of profiles $\{(s_i^t, s_{-i}^t)\}_{t=1,\dots,T}$ is called a *secure path* if each its arc $(s_i^t, s_{-i}^t) \to (s_i^{t+1}, s_{-i}^{t+1}) = (s_i^{t+1}, s_{-i}^t)$ contains a secure profitable deviation s_i^{t+1} for some player i. This path is called a *secure cycle* if it is closed: $(s_i^1, s_{-i}^1) = (s_i^T, s_{-i}^T)$.

Using this notion one can easily check the following theorem providing the criterion for absence of NE-2 in some game.

Proposition 2. *The game does not have NE-2 if and only if it contains a secure cycle and there is a secure path from any profile to some secure cycle.*

Let us, without loss of generality, assume that in a secure cycle player 1 deviates at odd steps while player 2 deviates at even ones. It helps us to formulate an important observation that secure cycles are very special: all nodes where player 1 deviates (say, odd ones) should have *same* payoff for this player $(u_1(s_1^{2t+1}, s_2^{2t+1}) = u_1(s_1^{2t+3}, s_2^{2t+3})\ \forall t)$, the same is true for even nodes and player 2: $(u_2(s_1^{2t}, s_2^{2t}) = u_2(s_1^{2t+2}, s_2^{2t+2})\ \forall t)$ – see Example 3 (game "Heads or Tails").

Corollary. Whenever a game does not have NE-2, any perturbation of payoffs that breaks equality in the secure cycles yields NE-2 existence.

Thereby, in essence, we have proven that NE-2 exists "almost always" without strictly defining this notion. Proposition 2 hence aims to demonstrate the existence of NE-2 but not the optimal algorithm of finding it in arbitrary game.

One can try to define NE-2 for several players. Here the question arise, how my partners respond to my deviation from the current strategies: simultaneously or one by one. In [15, Iskakov M., Iskakov A., 2012a], [16, Iskakov M., Iskakov A., 2012b] the latter suggestion was proposed, but we prefer to consider the possibility when several players are allowed to react at the same moment on someone's deviation. So we need the extension of our notion of "profitable secure deviation".

Consider an n-person non-cooperative game in the normal form $G = (i \in N = \{1, 2, \dots, n\}; s_i \in S_i; u_i : S_1 \times \ldots \times S_n \to R)$.

Definition 8. A *profitable deviation* of player i at strategy profile $s = (s_1, \dots, s_i, \dots, s_n)$ is a strategy s_i' such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$. A profitable deviation s_i' of player i at s is *secure* if for any subset of another players $K \subseteq N \setminus \{i\}$ and for any their profitable simultaneous deviation $s_K' = \{s_j'\}_{j \in K}$ $(u_j(s_i', s_K', s_{-iK}) > u_j(s_i', s_{-i})\ \forall j \in K)$ our player i is not worse off:

$$u_i(s_i', s_K', s_{-iK}) \geq u_i(s).$$

The definition of NE-2 equilibrium is exactly the same as definition 6.

Now we turn to particular cases.

3 Nash-2 equilibrium in strictly competitive games

In this section we deal with the class of strictly competitive games for which NE and ESS coincide and therefore often fail to exist. By contrast, NE-2 concept typically provides existence and even a wide range of equilibria.

Definition 9. A two-person game G is *strictly competitive* if for every two strategy profiles s and s'

$$u_i(s) \geq u_i(s') \quad \Longrightarrow \quad u_{-i}(s) \leq u_{-i}(s').$$

Examples are zero-sum games, constant-sum games. Moreover, when we we confine ourselves to pure strategies, strictly competitive games are equivalent to zero-sum games. Still, to compare our propositions with the next proposition, we prefer terminology of strictly competitive games.

Proposition 3 (Iskakov & Iskakov, 2012)**.** *Any ESS in a strictly competitive game is a NE.*

By contrast, NE-2 is more rich than ESS in this class of games and we are going to show this, giving also the necessary and sufficient conditions characterizing NE-2 situations in terms of guaranteed payoffs.

In order to apply NE-2 concept to strictly competitive games, let us introduce the notation.

Denote the guaranteed gain of player 1 by

$$\underline{V_1} = \max_{s_1} \min_{s_2} u_1(s_1, s_2).$$

When maxmin is attained on strategy profile $s^I = (s_1^I, s_2^I)$, we denote the corresponded gain of player 2 by $\overline{V_2} = u_2(s^I)$.

By analogy we denote the guaranteed gain of player 2 by

$$\underline{V_2} = \max_{s_2} \min_{s_1} u_2(s_1, s_2)$$

and the corresponding gain of player 1 by $\overline{V_1}$. The interval $[\underline{V_i}, \overline{V_i}]$ being called further *attainable interval.*

Theorem 1 (necessary condition of NE-2 in SC games)**.** *If strategy profile s is a NE-2 in a strictly competitive game, then payoffs belong to the attainable interval:*

$$u_i(s) \in [\,\underline{V_i}\,, \overline{V_i}\,], \quad i = 1, 2.$$

Proof. Let us consider a NE-2 $\hat{s} = (\hat{s}_1, \hat{s}_2)$. Assume that $u_1(\hat{s}_1, \hat{s}_2) > \overline{V_1}$. It means that $u_2(\hat{s}_1, \hat{s}_2) < \underline{V_2}$.

On the other hand, consider another strategy s_2^{II} of player 2 that guarantees him/her at least $\underline{V_2}$. It is easy to see that the deviation s_2^{II} of player 2 from strategy profile $\hat{s}$ is profitable and secure. Thus, $\hat{s}$ is not NE-2. □

Theorem 2 (sufficient condition of NE-2 in SC games)**.** *If a strategy profile $\hat{s} = (\hat{s}_1, \hat{s}_2)$ in a strictly competitive game is such that for each player $i = 1, 2$ the payoff is strictly inside the attainable interval:*

$$u_i(\hat{s}) \in (\,\underline{V_i}\,, \overline{V_i}\,),$$

then s is a NE-2.

Proof. Assume that player 1 has a profitable deviation s_1^* at $\hat{s}$: $u_1(s_1^*, \hat{s}_2) > u_1(\hat{s}_1, \hat{s}_2)$. Lets us show that it is not secure. Consider the strategy s_2^I of player 2. Then

$$u_1(s_1^*, s_2^I) \leq \max_{s_1} u_1(s_1, s_2^I) = \overline{V_1}.$$

So $u_1(s_1^*, s_2^I) \leq \overline{V_1} < u_1(\hat{s}_1, \hat{s}_2)$ and, thus, deviation s_1^* is not secure. □

Thus, for NE-2 existence it is sufficient that the game would be "rich" enough, i.e. have intermediate situations in which each players gets the payoff inside the attainable interval.

Now let us consider the boundary situation when one player gets exactly his/her lower guaranteed gain. Next two theorems complete the full classification of NE-2 in class of strictly competitive games.

Theorem 3 (criterion 1 of NE-2)**.** *Assume a strictly competitive game $\underline{V_i} < \overline{V_i}$, and a strategy profile $s^* = (s_i^*, s_{-i}^*)$ that brings minimal payoff $u_i(s^*) = \underline{V_i}$ for some player i. Then s^* is NE-2 if and only if for any strategy $s_i \in \tilde{S}_i \equiv \{s_i : \min_{s_{-i}} u_i(s_i, s_{-i}) = \underline{V_i}\}$ bringing the same payoff under optimal partner's behavior, s_i yields the same payoff under current behavior:*

$$u_i(s_i, s_{-i}^*) = \underline{V_i}.$$

Proof. Consider NE-2 profile $s^* = (s_i^*, s_{-i}^*)$, for which $u_i(s^*) = \underline{V_i}$. Assume that there exists a strategy $s_i \in \tilde{S}_i$ such that $u_i(s_i, s_{-i}^*) > \underline{V_i}$. Then the deviation s_i at s^* is profitable and secure. This proves the necessity.

Sufficiency. If $u_i(s^*) = \underline{V_i}$, then $u_{-i}(s^*) = \overline{V_{-i}}$. Assume that for any $s_i \in \tilde{S}_i$ $u_i(s_i, s_{-i}^*) = \underline{V_i}$. Let us show that no player has a profitable secure deviation.

Consider any profitable deviation s_{-i} of player $-i$ at s^*. It means that $u_{-i}(s_i^*, s_{-i}) > u_{-i}(s_i^*, s_{-i}^*)$. Chose a strategy s_i of player i that minimizes $u_{-i}(s_i, s_{-i})$. Then

$$u_{-i}(s_i, s_{-i}) \le \max_{s_{-i}} \min_{s_i} u_{-i}(s_i, s_{-i}) = \underline{V_{-i}} < \overline{V_{-i}} = u_{-i}(s^*).$$

So the deviation s_{-i} is not secure.

Consider now a profitable deviation s_i of player i at s^*. If $s_i \notin \tilde{S}_i$, than there exists a strategy s_{-i} such that $u_i(s_i, s_{-i}) \le \underline{V_i} = u_i(s^*)$.

The deviation s_i is secure only if $\min_{s_{-i}} u_i(s_i, s_{-i}) = \underline{V_i}$, i.e. $s_i \in \tilde{S}_i$. By the hypothesis for all $s_i \in \tilde{S}_i$ $u_i(s_i, s_{-i}^*) = \underline{V_i} = u_i(s^*)$, that means that the deviation s_i is not profitable. This completes the proof. □

Theorem 4 (criterion 2 of NE-2)**.** *Assume a strictly competitive game with degenerate admissible interval $\underline{V_i} = \overline{V_i} = V_i^*$, and a strategy profile $s^* = (s_i^*, s_{-i}^*)$, such that $u_i(s^*) = V_i^*$, $i = 1, 2$. A strategy profile s^* is NE-2 if and only if for any $s_i \in \tilde{S}_i = \{s_i : \min_{s_{-i}} u_i(s_i, s_{-i}) = V_i^*\}$ equality $u_i(s_i, s_{-i}^*) = V_i^*$ holds for both $i = 1, 2$.*

Proof. The proof of necessity is the same as in Theorem 3. Let us prove the sufficiency. Consider a profitable deviation s_i of player i at s^*: $u_i(s_i, s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$. Then $s_i \notin \tilde{S}_i$. This means that there exists a strategy s_{-i} such that $u_i(s_i, s_{-i}) < V_i^*$. Thereby, the deviation s_i is not secure. □

If we restrict the class of our strictly competitive games with continuous and "connected" ones, we immediately obtain the existence theorem for such type of games. Recall the definition of a path-connected space.

Definition 10. The topological space X is said to be *path-connected* if for any two points $x, y \in X$ there exist a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Example: convex set in $\mathbb{R}^n$.

One can easily prove the following theorem.

Theorem 5 (NE-2 existence in continuous SC games)**.** *Let G be a two-person strictly competitive game. Assume that strategy sets S_1 and S_2 are compact and path-connected, payoff functions u_1 and u_2 are continuous. Then there exists a pure NE-2 in G.*

Thus, we have shown very mild conditions for NE-2 existence in competitive games. Now, to show the difference between NE and NE-2, consider a few examples of concrete games.

Example 2 (NE $\neq$ NE-2).

	R	L
T	1	-1
B	0	0

This is a very degenerate example in the sense that $\underline{V} = \overline{V} = 0$. However, the game has unique NE and two NE-2 providing the same zero profit to both players. Namely, strategy profile (B,R) is NE and NE-2, and profile (B,L) is NE-2, but not NE. So both boundary situations here are NE-2.

Example 3 (Heads or Tails: NE-2 does not exist).

	R	L
T	1	-1
B	-1	1

This is an example of a game in which $\underline{V} = -1$, $\overline{V} = 1$, and its is “poor” in the sense that no intermediate profile exist. In this game any boundary profile fails to be NE-2.

Example 4 (SC game with an intermediate situation, Bazenkov, 2014[2]).

	R	L
T	(2/3, 1/3)	(-1, 2)
C	(1/2, 1/2)	(1, 0)
B	(1, 0)	(0, 1)

Here $\underline{V_1} = 1/2$, $\overline{V_1} = 1$, $\underline{V_2} = 0$, $\overline{V_2} = 1/2$. NE-2 set consists of two strategy profiles: boundary situation (C,R) and intermediate situation (T,R) with profits (1/2, 1/2) and (2/3, 1/3), respectively. Thereby, not every boundary situation is NE-2.

Note that the assumption of strict competitiveness is essential. For instance, we can slightly relax this requirement and look at a unilaterally competitive game (where only one player harms her partner by improving her payoff, see [20, Kats, Thisse, 1992]). Then the statement of Theorem 1 need not hold.

Example 5 (UC game, Iskakov A., 2014).

	R	C	L
T	(-1, 3)	(2, -1)	(1, 2)
B	(1, 0.5)	(0, 1)	(2, 0)

Here $\underline{V_1} = 0$, $\overline{V_1} = 1$, $\underline{V_2} = 0.5$, $\overline{V_2} = 1$.

The profile (T,L) is NE-2 (not unique) with profits (1, 2). However, related payoff is not in the admissible interval: $2 \notin [\underline{V_2}, \overline{V_2}]$.

Now we turn to economic applications of NE-2 to show why it is a fruitful concept.

[2]Examples 4 and 5 have been proposed by Nikolay Bazenkov and Alexey Iskakov in private collaboration.

4 The Bertrand duopoly

Consider the classical Bertrand model with two firms producing a homogeneous product. Let m_c be the constant marginal costs (equal for both firms), p_M being the monopoly price level, and p_1 and p_2 being the price levels of firms 1 and 2, respectively. Well known is that the only NE in this model is equal prices $p_1 = p_2 = m_c$ for both firms, that leads to zero profits. It is just the situation of Bertrand paradox, criticized as a bad description of the real-life behavior. It can't be resolved by using a concept of equilibrium in secure strategies because ESS here coincide with NE. However, applying the concept of NE-2 allows to obtain an equilibrium with any price level $p_1 = p_2 \in [m_c, p_M]$ yielding positive firms' profits. In particular, the highest of the NE-2 profiles establishes the monopoly price level. This outcome is exactly what can be regarded as a tacit collusion between the firms (explicit cooperation is not permitted).

Thus, the classical paradox is resolved by NE-2 concept without changing the model or its timing. Though we have in mind a repeated game, its simple one-shot form is sufficient for modelling. As far as we can judge, the problem of choosing the appropriate outcome out of the NE-2 set is deeply connected with the problem of stability and failure of collusion in a long-run perspective. One of possible approaches is to introduce the financial power of firms affecting the collusion stability, for instance, see [32, Wiseman, 2014].

5 The Hotelling price game under symmetric locations

Let us compare the concepts NE-2 and ESS in the simple version of the Hotelling price game (with exogenous locations). The consumers are uniformly distributed along the unit line. Two firms producing the homogeneous product are located equidistant from the ends of the interval and at distance $d \in [0; 1]$ from each other. Production costs are zero for both firms. Transportation costs are linear, one unit per unit of distance, being covered by consumers. The demand is absolutely inelastic that means that irrespectively of its price a unit quantity of the product must be consumed by a buyer from each point of the interval. A buyer chooses a firm with a lower final price (accounting the delivery cost).

In case when no firm proposes inadequately high price (high enough to drop out), the market is divided into two parts, presented in the Fig. 1.

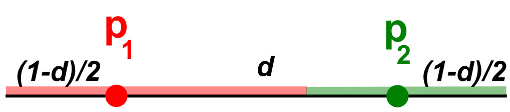

Figure 1: Linear city model

Here firm 1 assigns price p_1 that is lower than p_2, so more consumers prefer to buy from firm 1. The interval of consumers who buy from firm 1 is marked with red color, whereas the firm 2's consumers are marked with green one.

Let location d has already been chosen and now is unchangeable. Now we look only on the price-setting game, the location problem being beyond our consideration. The price strategy of firm i is to propose price $p_i \in [0, \infty)$, $i = 1, 2$. The profit functions π_i, $i = 1, 2$, are given by formulae:

$$\pi_i(p_i, p_{-i}) = \begin{cases} p_i(1 + p_{-i} - p_i)/2, & if \;\; |p_i - p_{-i}| \leq d, \\ p_i, & if \;\; p_i < p_{-i} - d, \\ 0, & if \;\; p_i > p_{-i} + d, \end{cases} \tag{1}$$

When $\bar{p}_2$ is fixed, the discontinuous shape of firm's 1 profit function $\pi_1(p_1, \bar{p}_2)$ is presented in Fig. 2.

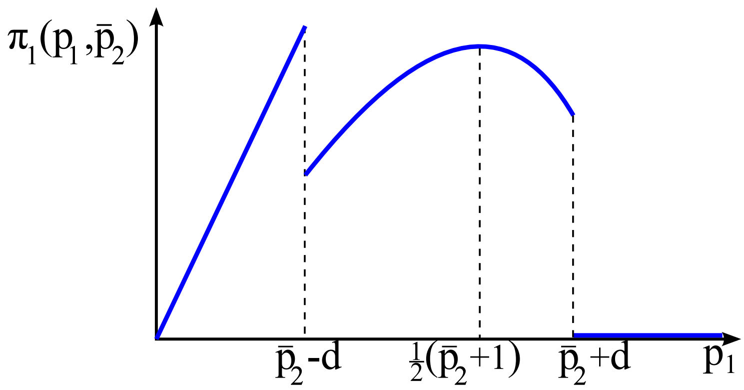

Figure 2: Gain function $\pi_1(p_1, \bar{p}_2)$

Simplifying a theorem from [2, d'Aspremont, Gabszewicz, Thisse, 1979], we claim that pure Nash equilibrium need not exist for all locations.

Theorem 6 (see [2]). *Consider the Hotelling price-setting game $H = \{i \in \{1, 2\}, p_i \in \mathbb{R}, \pi_i(p_1, p_2) : \mathbb{R}^2 \to \mathbb{R}\}$, where profits π_i are given by (1).*

- *For $d \in [\frac{1}{2}, 1]$ a unique NE exists: equilibrium prices are $p_1^* = p_2^* = 1$ and equilibrium profits are $\pi_1 = \pi_2 = 1/2$.*
- *For $d = 0$ the unique NE is $p_1^* = p_2^* = 0$. $\pi_1 = \pi_2 = 0$.*
- *For $d \in (0, \frac{1}{2})$ NE does not exist.*

In such a game the concept of ESS can solve the problem of absent equilibria as follows [17, Iskakov M., Iskakov A., 2013].

Theorem 7 (see [17]). *In the Hotelling game H there exists a unique ESS for all locations*

- *For $d \in [\frac{1}{2}; 1]$ the ESS is $p_1^* = p_2^* = 1$. $\pi_1 = \pi_2 = 1/2$.*
- *For $d \in [0; \frac{1}{2})$ the ESS is $p_1^* = p_2^* = 2d$. $\pi_1 = pi_2 = d < 1/2$.*

Note that all situations with high profits (we mean prices p_i higher then $(p_{-i} + 1)/2)$ are excluded as non-secure. But this situations are obviously the most profitable for players if they are rational enough to collude, i.e., to abstain from sharply undercutting prices or locally decreasing them. Applying the NE-2 concept we obtain all these "collusion outcomes" as a reasonable equilibria, in addition to sharper competition.

The simulation in Figures 3-6 demonstrates various outcomes depending on the parameters. Yellow areas are NE-2.

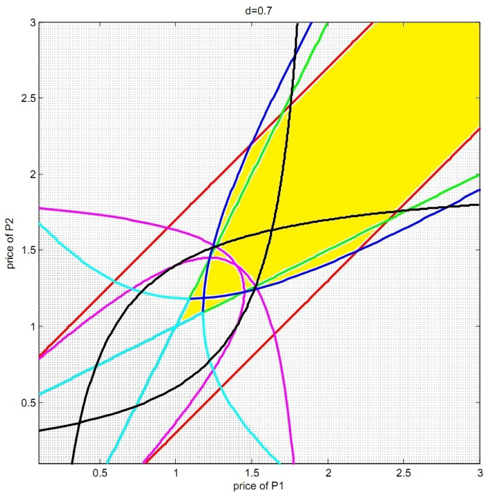

Figure 3: $d = 0.7$. (1, 1) is NE.

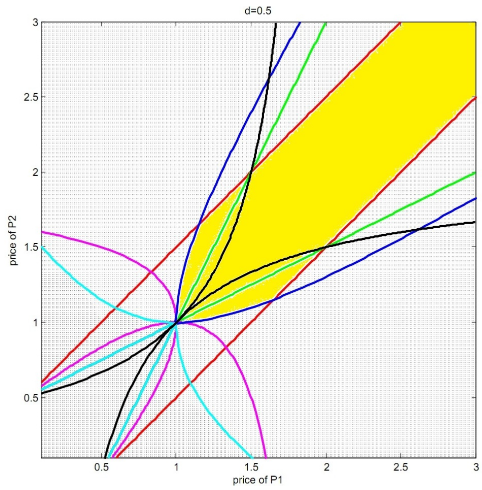

Figure 4: $d = 0.5$.(1, 1) is NE.

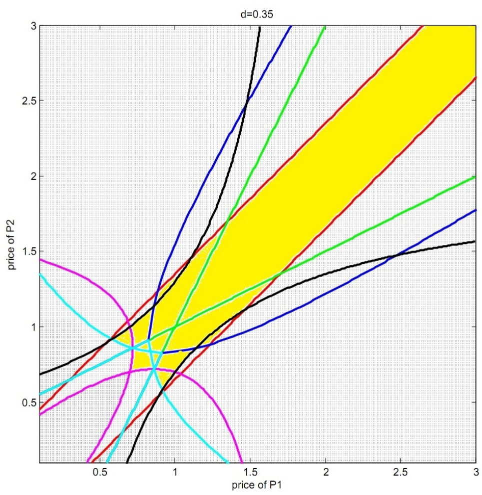

Figure 5: $d = 0.35$. (2d, 2d) is ESS.

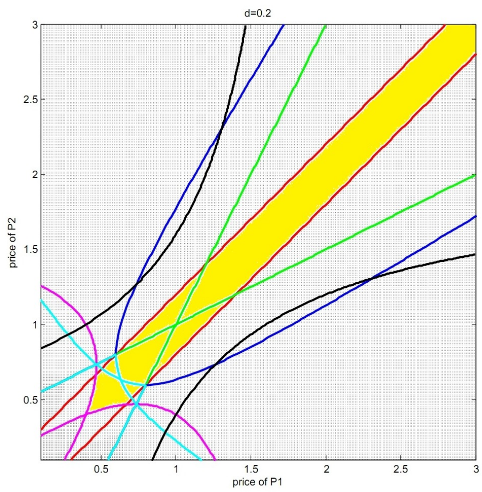

Figure 6: $d = 0.2$. (2d, 2d) is ESS.

The following claim characterizes the shape of the equilibria sets displayed in our figures.

Theorem 8. *In the Hotelling price-setting game H the boundary curves of the NE-2 area have the following form:*

Red: $|p_1 - p_2| = d$

Green: $p_1 = (p_2 + 1)/2$ *and vice versa.*

Pink: $2(p_1 - d) = p_2(1 + p_1 - p_2)$ *and vice versa.*

Dark blue: $p_1 = \frac{1+p_2}{2} + \sqrt{\left(\frac{1+p_2}{2}\right)^2 - 2d - p_2(1-p_2)}$ *and vice versa.*

Light blue: $p_2 = \frac{1+p_1}{2} - \sqrt{\left(\frac{1+p_1}{2}\right)^2 - 2d - p_1(1-p_1)}$ *and vice versa.*

Black: $p_2 = 2\left(1 - \frac{1-d}{p_1}\right)$ *and vice versa.*

Proof is in Appendix.

Comparing the outcomes under various locations, we note that too close locations of firms are not good for them under ESS, because they get low revenue. However, under NE-2 where security is not required, the location doesn't affect profit too much.

An interesting feature of NE-2 concept is that there exist asymmetric price equilibria under symmetric locations and costs. We can also observe that NE-2, being insecure, provides higher profits to firms than secure equilibria (ESS).[3] Such non-secure situation with higher profits can be treated as a tacit collusion, whereas ESS may be regarded as a strict competition.

6 Conclusion

We have reformulated rather novel NE-2 concept for repeated games, provided conditions for its existence and comparison with NE and ESS equilibrium concepts. The main goal is application of NE-2 to various oligopoly situations, because NE-2 concept provides a good strategical explanation for the origins of tacit collusion.

Appendix

Proof of theorem 8.

1. For any $d \in [0,\, 1]$: $|p_1 - p_2| \le d$. If it is not the case then one firm gets all the market and it is profitable and secure for another firm for instance to undercut. Hereinafter we assume this condition to be held.

It is to be noted that for any firm undercutting is never a profitable secure deviation. Hence we can test on security only deviations that preserve sharing the market.

2. If both firms propose prices $p_i \ge (p_{-i} + 1)/2$, $i = 1, 2$, then (p_1, p_2) is NE-2. Consider the profitable deviation $p'_i = p_i - \varepsilon$ of the firm i with $\varepsilon \in (0, 2p_i - p_{-i} - 1)$. It is not secure because of the firm $-i$ profitable deviation $p'_{-i} = p'_i - \varepsilon$.

3. Assume now that for both firms: $p_i \le (p_{-i} + 1)/2$, $i = 1, 2$.

[3] Actually, *any* profit level can be achieved because of inelastic demand and absence of choke-price in the model. This possibility has already been mentioned in the original paper of Hotelling [14, Hotelling, 1929].

If at least for one firm $i = 1$ or $i = 2$: $2(p_{-i} - d) < p_i(1 + p_{-i} - p_i)$, then (p_1, p_2) is not NE-2. Indeed, the firm $-i$ has the profitable secure deviation $p'_{-i} = p_{-i} + 0$ to slightly increase it's price.

If for both firms $i = 1, 2$: $2(p_{-i} - d) > p_i(1 + p_{-i} - p_i)$, then (p_1, p_2) is NE-2. The deviation $p'_i = p_i + \varepsilon$ is profitable for the firm i for $\varepsilon \in (0, p_{-i} + 1 - 2p_i)$. The undercutting $p'_{-i} = p'_i - d$ for the firm $-i$ is profitable, so the initial deviation p'_i of the firm i is not secure.

4. Consider the remaining case: $p_i \geq (p_{-i} + 1)/2$ and $p_{-i} \leq (p_i + 1)/2$. In spite of symmetry let $i = 1$. There are two possibilities for the profile (p_1, p_2) not to be NE-2.

Let the firm 1 has a profitable secure deviation p'_1: $p_1 > p'_1 > 1 + p_2 - p_1$. The firm 2 shouldn't benefit from undercutting: $2(p'_1 - d) \leq p_2(1 + p'_1 - p_2)$.

The boundary of the area is given by the system of equations:

$$\begin{cases} p'_1 = 1 + p_2 - p_1, \\ 2(p'_1 - d) = p_2(1 + p'_1 - p_2). \end{cases}$$

This yields the black curve $p_2 = 2\left(1 - \frac{1-d}{p_1}\right)$.

Also if p'_1 is small enough ($p_2 > (p'_1 + 1)/2$) then p'_1 should remain profitable for the firm 1 even the firm 2 maximum decreases its price $p'_2 = 1 + p'_1 - p_2 + 0$: $p_1(1 + p_2 - p_1) \leq p'_1(1 + p'_2 - p'_1)$.

The system

$$\begin{cases} 2(p'_1 - d) = p_2(1 + p'_1 - p_2), \\ p'_2 = 1 + p'_1 - p_2, \\ p_1(1 + p_2 - p_1) = p'_1(1 + p'_2 - p'_1), \end{cases}$$

leads to the equation of the dark blue boundary:

$$p_1 = \frac{1 + p_2}{2} + \sqrt{\left(\frac{1 + p_2}{2}\right)^2 - 2d - p_2(1 - p_2)}.$$

Another possibility is that the firm 2 has a profitable secure deviation $p'_2 = 1 + p_1 - p_2 - 0$. It should remains profitable in case of decreasing price by the firm 1, and undercutting shouldn't give benefits to the firm 1. Similarly to the above these two conditions results in the equation of the light blue curve $p_2 = \frac{1+p_1}{2} - \sqrt{\left(\frac{1+p_1}{2}\right)^2 - 2d - p_1(1 - p_1)}$. This completes the proof. □

References

[1] Amir R., Jin J. Y. *Cournot and Bertrand equilibria compared: substitutability, complementarity and concavity* // International Journal of Industrial Organization. 2001. Vol. 19. No. 3. P. 303-317.

[2] d'Aspremont C., Gabszewicz J., Thisse J.-F. *On Hotelling's "Stability in Competition"* // Econometrica. 1979. Vol. 47. No. 5. P. 1145-1150.

[3] d'Aspremont C., Dos Santos Ferreira R. *Oligopolistic competition as a common agency game* // Games and Economic Behavior. 2010. Vol. 70. No. 1. P. 21-33.

[4] Aumann R.J., Maschler M. *The bargaining set for cooperative games* // Advanced in game theory. 1964. Vol. 52. P. 443-476.

[5] Bazenkov N., Korepanov V. *Double best response as a network stability concept* // Proceedings of the International Conference on Network Games, Control and Optimization NETGCOOP2014. Trento, October 29-31, 2014. P. 201-207.

[6] Benoit J.-P., Krishna V. *Finitely repeated games* // Econometrica. 1985. Vol. 53. No. 4. P. 905-922.

[7] Binmore K. *Modeling rational players: Part II* // Economics and Philosophy. 1988. Vol. 4. No. 01. P. 9-55.

[8] Camerer C. F., Ho T. H., Chong J. K. *A cognitive hierarchy model of games* // The Quarterly Journal of Economics. 2004. P. 861-898.

[9] Carbonell-Nicolau O., McLean R. P. *Approximation results for discontinuous games with an application to equilibrium refinement* // Economic Theory. 2013. Vol. 54. No. 1. P. 1-26.

[10] Chwe M. S. Y. *Farsighted coalitional stability* // Journal of Economic Theory. 1994. Vol. 63. No. 2. P. 299-325.

[11] Dasgupta P., Maskin E. *The Existence of Equilibrium in Discontinuous Economic Games, II: Applications* // The Review of Economic Studies. 1986. P. 27-41.

[12] Guth W. *On the inconsistency of equilibrium refinement* // Theory and decision. 2002. Vol. 53. No. 4. P. 371-392.

[13] Halpern J. Y., Rong N. *Cooperative equilibrium* // Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems: volume 1-Volume 1. International Foundation for Autonomous Agents and Multiagent Systems. 2010. P. 1465-1466.

[14] Hotelling H. *Stability in competition* // The Economic Journal. 1929. Vol. 39. No. 153. P. 41-57.

[15] Iskakov M., Iskakov A. *Equilibrium in secure strategies* // CORE Discussion Paper 2012/61.

[16] Iskakov M., Iskakov A. *Equilibrium in secure strategies – intuitive formulation* // Working paper WP7/2012/06. Math. methods for decision making in economics, business and politics. 2012.

[17] Iskakov M., Iskakov A. *Complete solution of the Hotelling problem: equilibrium in secure strategies for the price subgame* // Journal of New Economic Association. 2013. No. 1(13). P.10-33. (in Russian)

[18] Iskakov M., Iskakov A. *Equilibium contained by counter-threats and complex equilibrium in secure strategies* // Large scale systems control. 2014. Vol. 51. P. 130-157. (in Russian)

[19] Jamroga W., Melissen M. *Doubtful deviations and farsighted play* // Progress in Artificial Intelligence. Springer Berlin Heidelberg. 2011. P. 506-520.

[20] Kats A., Thisse J.-F. *Unilaterally competitive games* // International Journal of Game Theory. 1992. Vol. 21. P. 291-299.

[21] Kawagoe T., Takizawa H. *Equilibrium refinement vs. level-k analysis: An experimental study of cheap-talk games with private information* // Games and Economic Behavior. 2009. Vol. 66. No. 1. P. 238-255.

[22] Myerson R. B. *Refinements of the Nash equilibrium concept* // International journal of game theory. 1978. Vol. 7. No. 2. P. 73-80.

[23] Nakanishi N. *Purely noncooperative farsighted stable set in an n-player prisoners' dilemma*. Graduate School of Economics, Kobe University. 2007.

[24] Novikov D. *Models of strategic behavior* // Automation and Remote Control. 2012. Vol. 73. No. 1. P. 1-19.

[25] Novshek W. *Cournot equilibrium with free entry* // Review of Economic Studies. 1980. Vol. 47. No. 3. P. 473-486.

[26] Novshek W., Chowdhury P. R. *Bertrand equilibria with entry: limit results* // International Journal of Industrial Organization. 2003. Vol. 21. No. 6. P. 795-808.

[27] Rubinstein A. *Equilibrium in supergames with overtaking criterion* // Journal of Economic Theory. 1979. Vol. 21. No. 1. P. 1-9.

[28] Selten R. Reexamination of the perfectness concept for equilibrium points in extensive games // International journal of game theory. 1975. Vol. 4. No. 1. P. 25-55.

[29] Sidorov A. V., Thisse J.-F., Zhelobodko E. V. *Revisiting the relationships among Cournot, Bertrand and Chamberlin: On Tendency to Perfection* // Papers rep. at XV Apr. Intern. Acad. Conf. on Economic and Social Development, Moscow, April 1–4, 2014. Available on-line http://conf.hse.ru/en/2014/prog [date 29.10.2014]

[30] Simon L. K., Stinchcombe M. B. *Equilibrium refinement for infinite normal-form games* // Econometrica: Journal of the Econometric Society. 1995. P. 1421-1443.

[31] Tirole J. *The theory of industrial organization*. MIT press. 1988.

[32] Wiseman T. *When Does Predation Dominate Collusion?* // W.p. 2014.

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