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SELF-COVARIANT SOLUTIONS TO COOPERATIVE GAMES WITH TRANSFERABLE UTILITIES

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SELF-COVARIANT SOLUTIONS TO COOPERATIVE GAMES WITH TRANSFERABLE UTILITIES

A weakening of covariance property for solutions of cooperative games with transferable utilities – self-covariance – is defined. Self-covariant solutions are positively homogenous and satisfy a "restricted"translation covariance such that feasible shifts are only the solution vectors themselves and their multipliers. A description of all nonempty, efficient, anonymous, self-covariant, and single-valued solution for the class of two-person TU games is given. Among them the solutions admitting consistent extensions in the Davis–Maschler sense are found. They are the equal share solution, the standard solution, and the constrained egalitarian solution for superadditive two-person games. Characterizations of consistent extensions (Thomson 1996) of these solutions to the class of all TU games are given.

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Key words: cooperative game with transferable utilities, solution, self-covariance, consistent extensions, constrained egalitarianism.

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1 Introduction

Most solutions for cooperative games with transferable utilities (TU) are covariant with respect to positive linear transformations of individual utilities. However, this property does not take into account interpersonal comparisons of players' payoffs. The constrained egalitarian solution defined by Dutta and Ray (1989) for the class of convex TU games, being not covariant, served as a pretext for studying non-covariant solutions. One of the approaches consists in a weakening of covariance in such a manner that, together with some other properties, it could characterize new solutions or give new characterizations for some known ones.

In the paper a weakening of the translation covariance property is defined. A TU game solution is *self-covariant*, if for every TU game it is homogenous and satisfies a "restricted"translation covariance property such that feasible shifts are only the solution vectors themselves and their multipliers. On the one hand, all the most known TU game solutions verifies this property, and on the other one it permits to replace the stronger covariance property in axiomatizations of some solutions.

Evidently, the properties of solutions – positive homogeneity, weak covariance, and self covariance – jointly are weaker than covariance. The Dutta-Ray (DR)solution (Dutta, Ray 1989) on the class of convex TU games satisfies all them though it is not covariant.

It turns out that each of the well-known characterizations of the prenucleolus (Sobolev 1975) and of the Shapley value (Hart, Mas-Colell 1989) applied to the class of the convex games, under replacing covariance by weak covariance and self-covariance, gives three solutions: each of the two mentioned ones together with the Dutta–Ray solution and the equal share solution (Yanovskaya 2012).

In this paper we study efficient, self-covariant, and anonymous single-valued solutions for arbitrary two-person games that have consistent in the Davis–Maschler sense extensions to the class of all TU games. It turned out that only three such solutions for two-person games admit consistent extensions: they are the standard solution, the egalitarian solution, and the solution coinciding with the constrained egalitarian solution for superadditive games, and with the standard solution for subadditive games.

The paper is organized as follows. In Section 2 we give the definitions of some solutions for TU games and their properties, and formulate the known theorems characterizing the prenucleolus and the Dutta–Ray solution. A new property of TU game solutions – self-covariance that is a weakening of covariance – is introduced in Section 3. A complete characterization of single-valued, anonymous, and weak and self-covariant solutions for the class of all two-person games is given. Section 4 deals with consistent extensions of solutions studied in the previous section to the class of TU games with arbitrary sets of players. It turns out that only three such solutions admit consistent extensions. Section 5 poses two open problems connected with consistent solutions. The proofs of the main results are presented in Appendix.

2 Preliminaries

2.1 TU game solutions and theirs properties

A *cooperative game with transferable utilities (TU game)* is a pair (N, v), where N is a finite set of players, $v : 2^N \to \mathbb{R}$ is a *characteristic function* of the game assigning to every coalition $S \subset N$ a number $v(S)$ with a convention $v(\emptyset) = 0$). An *outcome* of the game is a payoff vector $x \in \mathbb{R}^N \in X(N, v)$, where

$$X(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq v(N)\}$$

is the set of *feasible* payoff vectors.

In the sequel we use the notation $x(S) = \sum_{i \in S} x_i$ for vectors $x \in \mathbb{R}^N$ and coalitions $S \subset N$.

A *solution* σ to a class $\mathcal{G}$ of TU games associates with every game $(N, v) \in \mathcal{G}$ a subset $\sigma(N, v) \subset X(N, v)$.

Denote by $X^*(N, v)$ the set of *efficient* payoff vectors or *preimputations*

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N)\}.$$

If for each game $(N, v) \in \mathcal{G}$ $|\sigma(N, v)| = 1$, then the solution σ is *single-valued* or a *value.*

For every set of players N denote by $\mathcal{G}_N$ the set of all TU games with players' set N. Let $\mathcal{N}$ be an arbitrary *universal* set of players. Then

$$(N, v) \in \mathcal{G}_{\mathcal{N}} \implies N \subset \mathcal{N}.$$

For every injection $\pi : N \to \mathcal{N}$ and every game $(N, v) \in \mathcal{G}_N$ define the game $(\pi(N), \pi v) \in \mathcal{G}_{\mathcal{N}}$ by $v(\pi(S)) = \pi v(S)$ for all $S \subseteq N$. If $x \in \mathbb{R}^N$ denote $y = \pi(x)$ the vector $y \in \mathbb{R}^{\pi(N)}$ such that $y_{\pi(i)} = x_i, i \in N$. The game (N', w) is *isomorphic* to the game (N, v), if there is an injection $\pi : N \to \mathcal{N}$ such that $\pi(N) = N'$ and $\pi v = w$.

A game (N, v) is *superadditive (subadditive)* if $v(S) + v(T) \leq (\geq) v(S \cup T)$ for every $S, T \subset N$, $S \cap T = \emptyset$. These properties are *strict*, if the inequalities in the previous definition are strict.

A game (N, v) is *convex (concave)*, if $v(S) + v(T) \leq (\geq) v(S \cup T) + v(S \cap T)$ for every $S, T \subset N$.

Given a game (N, v), the players $i, j \in N$ are *symmetric*, if $v)S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subset N \setminus \{i, j\}$.

Recall some well-known properties of cooperative game solutions that are applied for their characterizations. First we give them for an arbitrary subclass $\mathcal{G}'_N \subseteqq \mathcal{G}_N$.

A solution σ for the class $\mathcal{G}'_N$ is

- *non-empty* or satisfies *nonemptiness (NE)*, if $\sigma(N, v)) \neq \emptyset$ for every game $(N, v) \in \mathcal{G}'_N$;
- *efficient (EFF)* or *Pareto optimal,* if $\sum_{i \in N} x_i(N, v) = v(N)$ for every $x \in \sigma(N, v)$ and for every $(N, v) \in \mathcal{G}'_N$;
- *single-valued (SV)* or a *value*, if for every game $(N, v) \in \mathcal{G}'_N$ $|\sigma(N, v)| = 1$;
- *positively homogeneous (PH)*, if for every $\alpha > 0$ and a game $(N, v) \in \mathcal{G}'_N$ it holds $(N, \alpha v) \in \mathcal{G}'_N$ and $\sigma(N, \alpha v) = \alpha \sigma(N, v)$;

– *translation covariant (TCOV)*, if for every game $(N,v) \in \mathcal{G}'_N$ and a number b $\langle N, v+b\rangle \in \mathcal{G}'_N$, and

$$x \in \sigma(N,v) \implies x \in \sigma(N, v+b),$$

where $(v+b)(S) = v(S) + b$ for all $S \subsetneqq N$, and $(v+b)(N) = v(N)$;

– *covariant (COV)*, if it is positively homogeneous and translation covariant;

– *weakly covariant (WCOV)*, if it is positively homogeneous and translation covariant with only respect to shift $b \in \mathbb{R}^N$ with equal coordinates;

– *anonymous (ANO),* if for every game $(N,v)) \in \mathcal{G}'_N$ and injection $\pi : N \to \mathcal{N}$ such that $(\pi N, \pi v) \in \mathcal{G}_\mathcal{N}$, the following equality holds: $\sigma(\pi N, \pi v) = \pi(\sigma(N,v))$. Here the function πv is defined by $\pi v(\pi S) = v(S)$ for all $S \subset N$;

– *symmetric* or satisfies the *equal treatment property (ETP)*, if $\varphi_i(N,v) = \varphi_j(N,v)$ as soon as i and j are symmetric in (N,v).

For games with variable set of players, i.e. those from some subclass $\mathcal{G}' \subset \mathcal{G}_\mathcal{N}$, define the *consistency property* connecting solutions of a game with those of games with smaller sets of players.

A solution σ for a class $\mathcal{G}'$ is

– *consistent (CONS)* , if, for every game $(N,v) \in \mathcal{G}'$, coalition $T \subset N$, and vector $x \in \sigma(N,v)$, the *reduced game* $(N\setminus T, v^x_{N\setminus T})$, obtained after leaving the game by players from the coalition T with payoffs $x_i, i \in T$, belongs to the class $\mathcal{G}'$, and the following equality holds:

$$x = (x_{N\setminus T}, x_T) \in \sigma(N,v) \implies x_{N\setminus T} \in \sigma(N\setminus T, v^\Phi_{N\setminus T}). \tag{1}$$

From definition (1) it follows that the consistency property can be determined for every class of games closed under reducing, i.e., such that with every game (N,v) and every its payoff vector x, it contains all its reduced games $(N\setminus T, v^x_{N\setminus T})$, $T \subset N$.

Note that in (1) the reduced games are not defined uniquely by a game and its payoff vector. There are different definition of the reduced games and of the corresponding different definitions of consistency. In this paper we consider the definition due to Davis and Maschler (1965):

The *reduced game* (S, v^x_S) of a game (N,v) on the player set S and w.r.t. the payoff vector x is the game defined by the characteristic function

$$v^x_S(T) = \begin{cases} v(N) - x(N\setminus S), & \text{if } T = S,\\ \max_{Q\subset N\setminus S}(v(T\cup Q) - x(Q)) & \text{for other coalitions .}\end{cases} \tag{2}$$

If we put the definition of a single-valued solution for a class of two-person games as an axiom, then some cooperative game solutions can be characterized by this property and by consistency. Among them there are two single-valued solutions[1] they are the equal share solution for the class of all TU games and the Dutta–Ray solution for the class of convex games.

[1] the Shapley value is also axiomatized by this two axioms, but consistency should be applied in another sense (Hart–Mas-Colell, 1987).

The prenucleolus has such a characterization only for the class of convex games, since on this class the prenucleolus coincides with the prekernel, and the prekernel is characterized as the maximum solution satisfied by standardness for two-person games, and by consistency (Peleg 1986). As for the class of all TU games, then the prenucleolus yet has no such a characterization, though there is no example of a consistent value being standard for two-person games and not coinciding with the prenucleolus.

Recall the definitions of these solutions.

The *equal share solution* for every TU game (N, v) divides the total gain $v(N)$ equally between all the players. It is efficient, single-valued, anonymous, and consistent.

Let (N, v) be an arbitrary game, $x \in X(N, v)$, $e(S, x) = v(S) - x(S)$ be the excess of a coalition $S \subsetneq N$ w.r.t. x, $\{e(S, x)\}_{S \subsetneq N}$ be the excess vector. Denote by $\theta(x) \in \mathbb{R}^{2^N}$ the vector whose components coincide with those of $\{e(S, x)\}_{S \subsetneq N}$, but disposed in a weakly decreasing manner:

$$\theta^t(x) = \max_{\substack{\mathcal{T} \subset 2^N \\ |\mathcal{T}| = t}} \min_{S \in \mathcal{T}} e(S, x). \tag{3}$$

Let $\geq_{lex}$ be the relation of lexicographic ordering of the space $\mathbb{R}^m$:

$$x \geq_{lex} y \iff x = y \text{ or } \exists 1 \leq k \leq m \text{ such that } x_k = y_k \text{ and } x_i > y_i \text{ for } i < k.$$

The prenucleolus $PN(N, v)$ of a game (N, v) is the unique efficient payoff vector on which the lexicographic minimum of the set of vectors $\theta(y)$, $y \in X(N, v)$ is attained:

$$\theta(y) \geq_{lex} \theta(PN(N, v)) \text{ for all } y \in X(N, v). \tag{4}$$

The prenucleolus is non-empty for every game. On the class of two-person games this solution coincides with the *standard solution (ST)*, defined for every two-person game $(\{i, j\}, v)$ as follows:

$$ST_i(N, v) = \frac{v(N)}{2} + \frac{v(\{i\})}{2} - \frac{v(\{j\})}{2}. \tag{5}$$

The *egalitarian Dutta–Ray solution (DR-solution)* (Dutta, Ray 1989) is defined on the class of all convex games. It associates with every convex game the unique payoff vector from the core that Lorenz dominates all other vectors from the core.

For two-person superadditive (convex) games the Dutta-Ray solution coincides with the *constrained egalitarian solution (CE)*:

$$CE(\{i, j\}, v) = \begin{cases} \left(\frac{v(N)}{2}, \frac{v(N)}{2}\right), & \text{if } v(\{i\}), v(\{j\}) \leq \frac{v(N)}{2}, \\ (v(\{i\}), v(N) - v(\{i\})), & \text{if } v(\{i\}) > \frac{v(N)}{2}, \\ (v(N) - v(\{j\}), v(\{j\})), \text{ if } v(j) > \frac{v(N)}{2}. \end{cases} \tag{6}$$

Let us give characterizations of the three solutions defined above.

Proposition 1 *The equal share solution is the unique solution for the class of all TU games with arbitrary universe of players satisfying non-emptiness, single-valuedness, consistency and coinciding with the equal share solution on the class of two-person games.*

Proof. Evidently, the equal share solution satisfies all the properties given in the Proposition.

Let now Φ be an arbitrary solution satisfying all these properties. Let (N, v) ne an arbitrary game, $x = \Phi(N, v)$. Let $(\{i,j\}, v^x_{i,j})$ be the reduced game on the players set $\{i, j\}$ w.r.t. x. Then, by consistency of Φ, $(x_i, x_j) = \Phi(\{i,j\}, v^x_{i,j})$, and, hence, $x_i = x_j = \frac{x_i+x_j}{2}$. Since the last equality fulfils for every $i, j \in N$, we obtain $x_i = x_j$ for all $i, j \in N$. By the definition of the reduced games $v^x_{i,j}(\{i,j\}) = v(N) - \sum_{k\neq i,j} x_k = x_i + x_j$. The last equality shows efficiency of Φ and we obtain that $\Phi(N, v) = (x, x, ..., x)$, where $x = \frac{v(N)}{n}$. □

Theorem I [Sobolev 1985] *The unique solution for the class of all TU games with infinite universal set of players satisfying non-emptiness, single-valuedness, anonymity, covariance, and consistency is the prenucleolus.*

Theorem II [Dutta 1990] *The unique solution for the class of convex games with arbitrary universal set of players, satisfying nonemptiness, single-valuedness, consistency, and coinciding with the CE solution on the class of two-person games is the Dutta–Ray solution.*

The Dutta–Ray solution is anonymous, but it is not covariant, it satisfies only weak covariance. For the class of convex games the prenucleolus has the axiomatization similar to Theorem II:

Theorem I^{con}. *The unique solution for the class of convex games with arbitrary universal set of players, satisfying nonemptiness, single-valuedness, consistency, and coinciding with the standard solution on the class of two-person games is the prenucleolus.*

The proof follows from the coincidence of the prenucleolus with the prekernel for the class of convex games and from Peleg's characterization of the prekernel (Peleg 1986).

Thus, we obtain the characterizations of the three values – the equal share solution, the prenucleolus and the Dutta–Ray solution (the two last ones only for convex games) – by the unique manner with the help of consistency and the definition of the solution for two-person games. This fact can be treated as the existence of consistent extensions for the three two-person games solutions to convex games with arbitrary sets of players. In the paper we try to find another single-valued anonymous solutions for both superadditive and subadditive two-person games, possessing a weakening of covariance and admitting the consistent extensions to the class of all TU games.

3 Self-covariance of TU game solutions

3.1 Definition

The DR-solution to the class of convex games does not satisfy covariance, it is only a weakly covariant solution. Since the homogeneity axiom is a part of the both mentioned axioms, they differ only in the second part of them: translation covariance means invariance of the solution w.r.t. arbitrary translation vector, and weak translation covariance admits only translations with equal components. The last property, together with nonemptiness, single-valuedness, anonymity and consistency, is insufficient for characterizing the Dutta–Ray solution to the class of convex games.

Let us introduce one more weakening of the translation covariance property, intermediate between these two ones. Let $\mathcal{G}$ be an arbitrary class of TU games closed under summation of characteristic functions with arbitrary vectors such that if $(N, v) \in \mathcal{G}$, then $(N, v + a) \in \mathcal{G}$ for every vector $a \in \mathbb{R}^N$.

Definition 1 A nonempty single-valued solution φ for the class $\mathcal{G}$ is called *self-covariant (self-COV)*, if it is homogeneous and for every number $A \geq -1$ the equalities

$$\varphi(N, v + A\varphi(N, v)) = (A + 1)\varphi(N, v) \tag{7}$$

hold for all games $(N, v) \in \mathcal{G}$.

In Definition 1 translations of characteristic functions are permitted only for multipliers of the solution vectors such that they would not change the signs of the solution vectors in both parts of equality (7).

Proposition 2 *The DR-solution verifies self-covariance on the class of convex games.*

Proof. Let $(N, v) \in \mathcal{G}^{\rfloor}$, $x = DR(N, v)$. Then the vector вектор x can be represented as follows (Dutta 1990):

$$x = (\underbrace{a_1, \ldots, a_1}_{T_1}, \underbrace{a_2, \ldots, a_2}_{T_2}, \ldots \underbrace{a_m \ldots, a_m}_{T_m}), \tag{8}$$

where $a_1 = \max_{S \subset N} \dfrac{v(S)}{|S|} = \dfrac{v(T_1)}{|T_1|}$, $a_j = \max_{S \subset N \setminus \cup_{i=1}^{j-1} T_i} \dfrac{v^j(S)}{|S|} = \dfrac{v^j(T_j)}{|T_j|}$, $j = 2, \ldots, m$, and

$$v^j(S) = v\left(\bigcup_{i=1}^{j-1} T_i \cup S\right) - v\left(\bigcup_{i=1}^{j-1} T_i\right) \text{ for } S \subset N \setminus \bigcup_{i=1}^{j-1} T_i. \tag{9}$$

Here coalitions $T_1, T_2, ...$ are the maximal in inclusion coalitions among those satisfying (8) and (9).

Inequalities $\frac{v(T_1)}{|T_1|} \geq \frac{v(S)}{|S|}$ for all $S \subset N$ and $a_1 > a_j$ for all $j = 2, ..., m$, imply $DR_{T_1}(N, v) = (a_1, ..., a_1)$. Since the DR-solution belongs to the core and $A \geq -1$, we obtain the inequalities

$$(A + 1) \cdot a_1 \geq (A + 1) \cdot \frac{DR(N, v)(S)}{|S|} \geq (A + 1)\frac{v(S)}{|S|}$$

for all coalitions $S \subset N$, implying the relation

$$T_1 \in \arg\max_{S \subset N} \frac{v(S) + A \cdot DR(N, v)(S)}{|S|} \quad \forall S \subset N. \tag{10}$$

Consider the games $(N \setminus T_1, v^1)$, $(N \setminus T_1, (v + A \cdot DR(N, v))^1)$. By definition (9) we obtain

$$\begin{aligned} v^1(S) &= v(T_1 \cup S) - a_1|T_1|, \\ (v + A \cdot DR(N, v))^1(S) &= v(T_1 \cup S) + A \cdot a_1|T_1| \quad + A \cdot DR(N, v)(S) - (1 + A) \cdot a_1|T_1| = \\ v(T_1 \cup S) + A \cdot DR(N, v)(S) - a_1|T_1| &\quad = v^1(S) + A \cdot DR(N, v)(S). \end{aligned} \tag{11}$$

From equalities (11) it follows that

$$T_2 \in \arg\max_{S \subset N \setminus T_1} \frac{v^1(S) + A \cdot DR(S)(N, v)}{|S|}, \tag{12}$$

and T_2 is the maximal in inclusion coalition satisfying equality (12).

The remaining proof is fulfiled by the evident induction in the numbers of coalitions $T_i, i = 1, 2, ..., m$ in representation (8).

$\square$

3.2 Self-covariant solutions for two-person games

In this subsection we characterize all nonempty, efficient, single-valued, anonymous, weak- and self-covariant solutions for the class of all two-person games with arbitrary universe of players. Since we will consider only anonymous solutions for this class, it suffices to restrict their definition by games with a fixed pair of players $\{i, j\}$. In the sequel we denote $v(\{i\}) = v_i, v(\{j\}) = v_j$. For simplicity, we will denote a two-person game by a letter v, and for a two-dimensional vector (v_i, v_j) we will apply the notation $\bar{v}$.

Every class of two-person games consists of subclasses of additive, strictly superadditive, and strictly subadditive games. Since feasible transformations of the individual utilities applied in the definitions of weak and self covariance do not turn a game from one of the three classes out of it, we will define weak and self-covariant solutions separately for every class. Weak covariance of solutions permits to define them only for games with zero value of the grand coalition.

To begin with, consider the class $\mathcal{G}_2^{0ad}$ of additive games with zero total payoff $v(\{i, j\}) = 0$.

Proposition 3 *There are two single-valued solutions for the class $\mathcal{G}_2^{0ad}$ that verify axioms NE, EFF, ANO, PH, and self-COV. They are the equal share solution, giving zero payoffs to both players for every games, and the solution $\phi_{ind}(v) = \bar{v}$.*

Proof. Evidently, both solutions verify all the axioms.

Let ϕ be an arbitrary single-valued solution for the class $\mathcal{G}_2^{0ad}$ verifying all the axioms. If $\phi(v) = 0$ for some game $v \neq (0, 0)$, then by positive homogeneity and anonymity of ϕ $\phi(v) = (0, 0)$ for all games $v \in \mathcal{G}_2^{0ad}$.

It remains to consider the case when $\phi(v) \neq (0, 0)$ for all games $v \neq (0, 0)$. Then $\phi(v) = (x, -x) = \alpha\bar{v}$ for some $x, \alpha \neq 0$, and positive homogeneity and anonymity of ϕ implies that $\phi(v) = \alpha\bar{v}$ for all $v \in \mathcal{G}_2^{0ad}$. By self-covariance of ϕ $\phi(v + \alpha\bar{v}) = 2\phi(v) = \phi(2v)$, and by positive homogeneity $\phi(v + \alpha\bar{v}) = (1 + \alpha)\phi(v)$. The two last equalities imply $\alpha(1 + \alpha)\bar{v} = \alpha(2\bar{v})$. Hence, $\alpha = 1$, and $\phi(v) = \bar{v}$ for all $v \in \mathcal{G}_2^{0ad}$, i.e., $\phi = \phi_{ind}$. □

These two solutions can be easily extended to the class of additive games with arbitrary values $v(\{i, j\})$ with the help of weak covariance, Then the equal share solution is defined for every additive two-person games by $ES(v) = \left(\frac{v(\{i,j\})}{2}, \frac{v(\{i,j\})}{2}\right)$. The solution $\phi_{ind}(v) = \bar{v}$ does not change.

Now consider the class $\mathcal{G}_2^{0sp}$ of strictly superadditive two-person games with the set of players $\{i, j\}$, and with zero total gain of players $v(\{i, j\}) = 0$. Let us define for this class a collection of one-parametric non-empty, efficient, single-valued, and anonymous solutions $\varphi_k, k \in (-1, 1]$. Because of anonymity of solutions it suffices to define them only for games v with $v_i < v_j$.

$$\varphi_k(v) = \begin{cases} (0, 0), & \text{if } v_i = v_j, \text{ or } v_j < kv_{i,}, \\ \left(\left(\dfrac{kv_i - v_j}{1 + k}\right)_i, \left(\dfrac{v_j - kv_i}{1 + k}\right)_j\right), & \text{if } v_j \geq kv_i. \end{cases} \tag{13}$$

Evidently, for every $k \in (-1, 1]$ φ_k is an efficient, nonempty, and anonymous solution.

For $k = 0$ the solution φ_0 coincides with the constrained egalitarian solution;

For $k = 1$ φ_1 is the standard solution;

For $k = -1$ the solution φ_{-1} is not defined by (13) for strictly superadditive games. However, when $k \to -1$, then for every game $v \in \mathcal{G}^{0sp}$, the domain $v_j \geq kv_i$, $v_i + v_j < 0$ transforms into a half-line $v_i + v_j = 0, v_i < 0$, and for every game v $\lim_{k \to -1} \varphi_k(v) = (0, 0)$.

So, we will define φ_{-1} for the class of strictly superadditive gams as the equal share solution, when both players obtain zero.

Let us depict solutions φ_k with the help of the level lines:

1. $k > 0$.

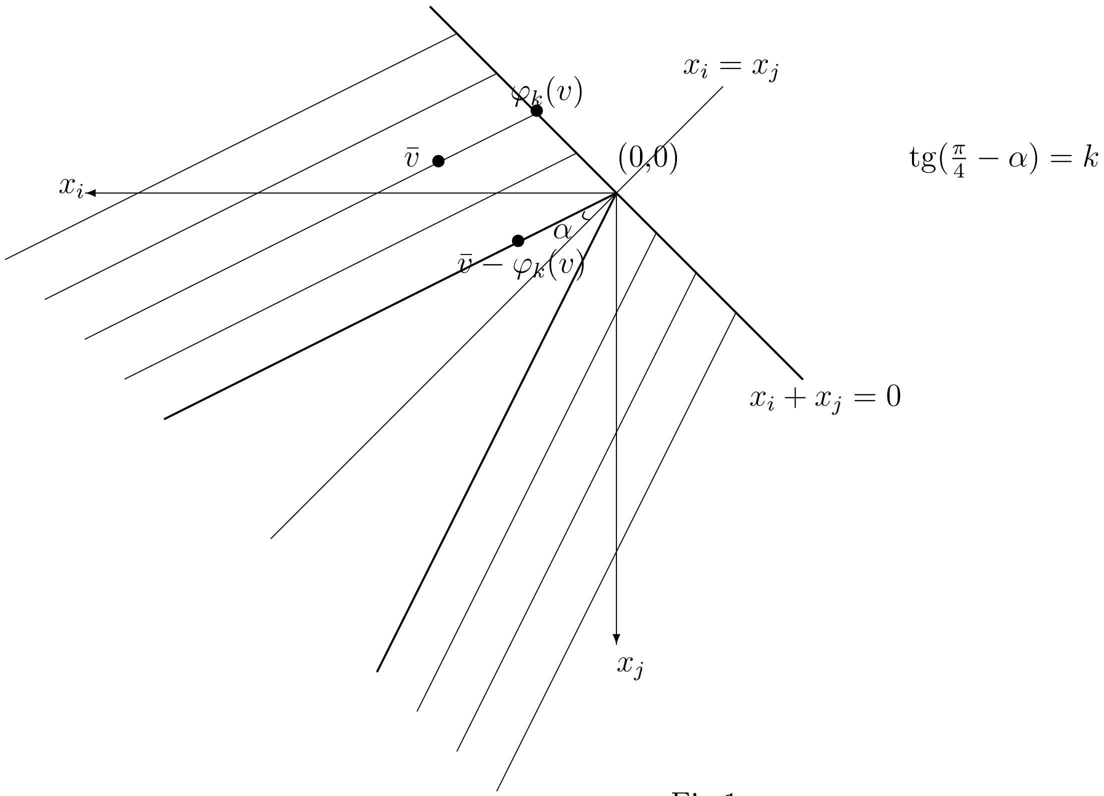

Fig.1

2. $k < 0$.

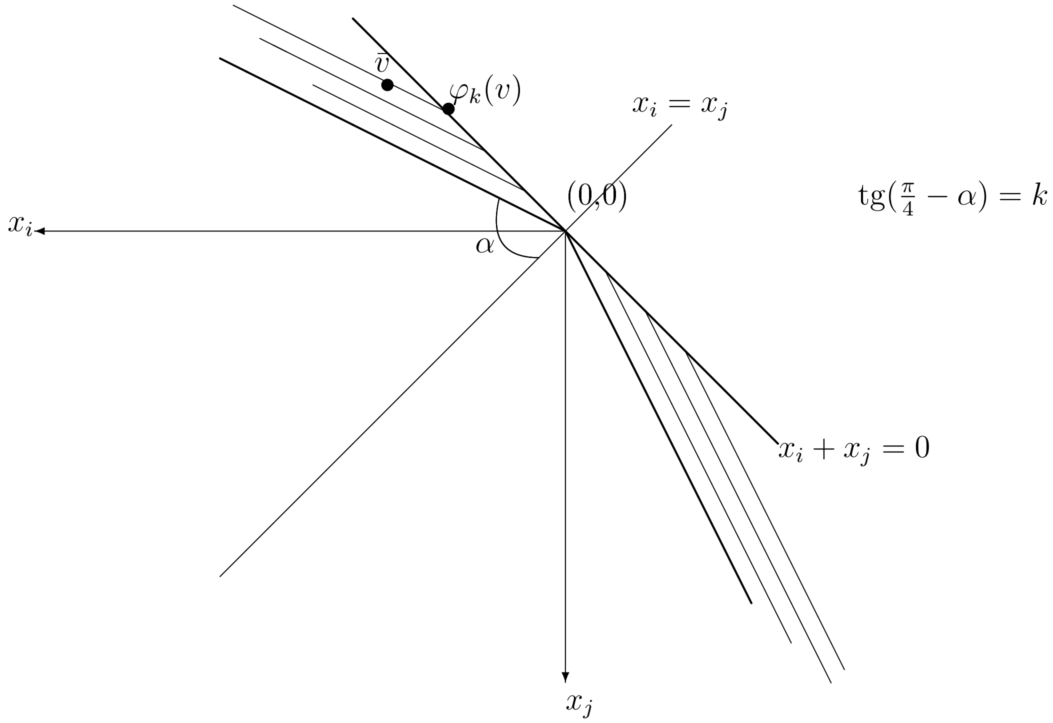

Fig.2

If the point $\bar{v}$, is placed inside the angle α, then $\varphi_k(v) = (0,0)$

Note that if $k \geq 0$, then the solutions φ_k are individually rational, and if $k < 0$, then they do not.

By the same manner we can define solutions φ_k for all strictly superadditive two-person games with arbitrary $v(\{i,j\})$. However, without other conditions, the parameter k in the extension of such solutions to the whole class of superadditive two-person games may depend on $v(\{i,j\})$ that seems too much complicated and unreasonable for studying. Thus, in this section we will consider the self-covariance property only jointly with the weak covariance one.

Let us extend the solutions φ_k to the class of all strictly superadditive two-person games with the help of translation covariance, i.e., by putting for arbitrary strictly superadditive two-person game v

$$\varphi_k(v) = \varphi_k(v - e_{v(\{i,j\})}) + e_{v(\{i,j\})},$$

where $e_{v(\{i,j\})} = \left(\frac{v(\{i,j\})}{n}, ..., \frac{v(\{i,j\})}{n}\right)$. The complete formula for $\varphi_k(v)$, $k \in (-1,1]$ and for $v_i \leq v_j$ follows from (13):

$$\varphi_k(v) = \begin{cases} \left(\dfrac{v(\{i,j\})}{2}, \dfrac{v(\{i,j\})}{2}\right), & \text{if } v_i = v_j, \text{ or } \quad v_j - kv_i < (1-k)\frac{v(\{i,j\})}{2}, \\ \left(\dfrac{v(\{i,j\}) - v_j + kv_i}{k+1}, \dfrac{kv(\{i,j\}) + v_j - kv_i}{k+1}\right) & \text{otherwise.} \end{cases} \qquad (14)$$

From (14) it follows that $\varphi_k(v) = \bar{v}$ for all additive games and for all $k \in (-1,1]$.

For strictly superadditive games $x_i \neq v_i$, we obtain the formula for the parameter k :

$$k = \frac{(\varphi_k(v))_j - v_j}{(\varphi_k(v))_i - v_i}. \tag{15}$$

Thus, the parameter k equals the tangent of the angle between the horizontal ax and the direct line passing through the points $\bar{v}$ and $((\varphi_k(v))_i, (\varphi_k(v)_j) \neq \left(\frac{v(\{i,j\})}{2}, \frac{v(\{i,j\})}{2}\right)$.

For $k = -1$, as in the case $v(\{i,j\}) = 0$, we put $\varphi_{-1}(v) = \left(\frac{v(\{i,j\})}{2}, \frac{v(\{i,j\})}{2}\right)$ for all strictly superadditive games. Now the solutions φ_k have been defined for the whole class $\mathcal{G}_2^{sp}$ of strictly superadditive two-person games, and these solutions are anonymous and weakly covariant.

Lemma 1 *The solutions φ_k are self-covariant in the class $\mathcal{G}_2^{sp}$ for all $k \in [-1, 1]$.*

Let us give an axiomatization of φ_k solutions for all $k \in [-1, 1]$.

Theorem 1 *If a solution φ on the class of strictly superadditive two-person games $\mathcal{G}_2^{sp}$ satisfies axioms NE, EFF, ANO, wCOV , and self-COV, then it is a φ_k solution for some $k \in [-1, 1]$. If, moreover, it is individually rational, then $k \in [0, 1]$.*

Remark 1 The solutions φ_k are defined by (14) for additive two-person games as well. However, when we describe the set of all efficient, single-valued, anonymous, and self-covariant solutions for the class of two-person games, we should consider all three subclasess of super-additive, sub-additive and additive games separately, since feasible transformations of individual utilities applied in the definition of self-covariance, do not move a game from one class to another.

Let us turn to the class $\mathcal{G}_2^{sb}$ of strictly subadditive two-person games.

A two-person game v is strictly subadditive, if $v_i + v_j > v(\{i,j\})$. An extension of the constrained egalitarian solution to the class of two-person subadditive games is an analogue of the *equal awards rule* for cost allocation problems. The first attempt to its definition was due to Arin and Iñarra (2001). For every subadditive two-person game v the constrained egalitarian solution is defined by

$$CE(v) = \begin{cases} \left(\frac{v(\{i,j\})}{2}, \frac{v(\{i,j\})}{2}\right), & \text{if } v_i, v_j \geq \frac{v(\{i,j\})}{2}, \\ (v_i, v(\{i,j\}) - v_i), & \text{if } v_i \leq \frac{v(\{i,j\})}{2} < v_j, \\ (v(\{i,j\}) - v_j, v_j), & \text{if } v_j \leq \frac{v(\{i,j\})}{2} < v_i. \end{cases} \tag{16}$$

The CE solution for subadditive games (16) protects the "poor"player (with a smaller value of the characteristic function) as well as this solution does for superadditive games (6). In fact, solution (16) either gives to both players equal losses, or the “poor” player saves his value, and another, ("rich"player) receives his marginal value, which is smaller then his individual value.

Thus, formulas (6), (16) and the equal share solution for additive games define the constrained egalitarian solution for the class of all two-person games. On Fig.3 by the thick line the locus of the CE solution is depicted for a particular pair of values (v_i, v_j), $v_j > v_i$ and all values $v(\{i,j\})$.

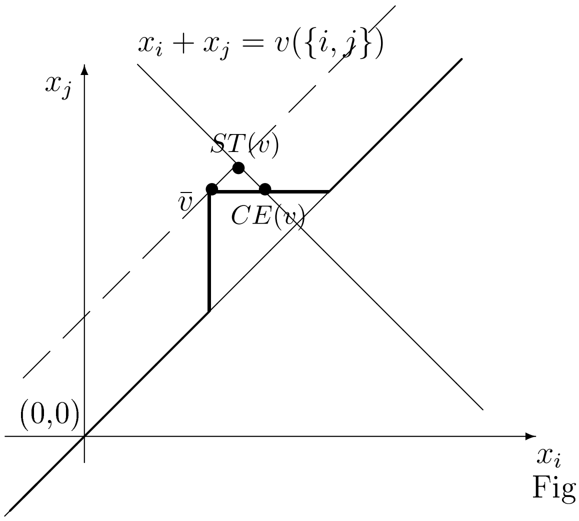

Fig.3

The locus for the standard solution is depicted by the dotted line. Since both the CE and the standard solutions are anonymous, on the second half-plane $v_i > v_j$ they are depicted similarly.

Similarly to solutions φ_k, define now a one-parametric family of anonymous values for two-person strictly subadditive games. Every value ψ_l from the family is defined by a parameter $l \in (-\infty,-1) \cup [1,\infty)$ such that for a game v with $v_j > v_i$

$$\psi_l(v) = \begin{cases} \left(\frac{v(\{i,j\})}{2}, \frac{v(\{i,j\})}{2}\right), & \text{if } v_j - lv_i < \frac{(1-l)v(\{i,j\})}{2}, \\ \left(\frac{v(\{i,j\}) - v_j + lv_i}{l+1}\right)_i, \left(\frac{lv(\{i,j\}) + v_j - lv_i}{l+1}\right)_j, & \text{otherwise.} \end{cases} \tag{17}$$

Note that formula (17) almost coincides with that (14). The difference is only in the domain of parameters k, l and characteristic functions: $k \in [-1,1]$, $l \in (-\infty,-1) \cup [1,\infty)$; and $v_i + v_j \leq v(\{i,j\})$ in (14), $v_i + v_j \geq v(\{i,j\})$ in (17).

The equality $(\psi_l(v))_i = v_i$ is possible only for additive games, thus, from (17) it follows

$$l = \frac{(\psi_l(v))_j - v_j}{(\psi_l(v))_i - v_i}. \tag{18}$$

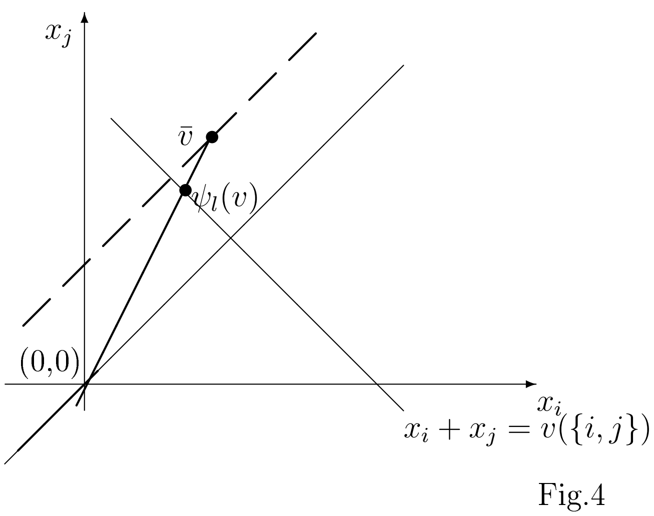

Fig.4

On Fig.4 the thick piece-wise line depicts the locus of the solutions ψ_l, $l > 1$ for variable values $v(\{i,j\}) \leq v_i + v_j$. The parameter l equals the tangent of the angle between the horizontal ax and the ray from the point (v_i, v_j) to the solution vector $\psi_l(v)$.

The angle $\frac{\pi}{2}$ corresponds to $\lim_{l\to\pm\infty} \psi_l$ and coincides with the solution of the constrained egalitarianism. Thus, we put $\psi_{\pm\infty}(v) = CE(v)$.

For $l = 1$ ψ_l is the standard solution,
If $l \to -1$, then $\lim_{l\to-1} \psi_l(v) = \left(\frac{v(\{i,j\})}{2}, \frac{v(\{i,j\})}{2}\right)$ for every strictly subadditive game v, so we put $\psi_{-1}(v) = \left(\frac{v(\{i,j\})}{2}, \frac{v(\{i,j\})}{2})\right)$.

By anonymity, formula (17) and its additions for $l = -1, l = \infty$ completely determines the solutions ψ_l for $l \in [-\infty, -1] \cup [1, \infty]$.

The solutions ψ_l on the class $\mathcal{G}_2^{sb}$ satisfy the same axioms as those characterizing the solutions φ_k for the class of superadditive two-person games in Theorem 1: they are non-empty, efficient, single-valued, anonymous, and weakly covariant. Moreover, by rewriting the proof of Lemma 1 adapted to solutions ψ_l, it is easy to show that these solutions are self-covariant on the class $\mathcal{G}_2^{sb}$ for all $l \in [-\infty, -1] \cup [1, \infty]$.

Let us formulate an analogue of Theorem 1:

Theorem 2 *If a solution ψ to the class of strictly subadditive two-person games $\mathcal{G}_2^{sb}$ satisfies axioms NE, EFF, SV, ANO, WCOV u self-COV, then it is a ψ_l solution for some $l \in [-\infty, -1] \cup [1, \infty]$.*

Now let us unite the solutions φ_k and ψ_l, and define with their help a family of non-empty solutions to the class of all non-additive two-person TU games. Such a union is possible, since the transformations of games admitted for the weak covariance property save super- and sub-additivity of games.

Then we obtain solutions ϕ_{kl} defined for every non-additive two-person game $(\{i,j\}, v)$ by

$$\phi_{kl}(v) = \begin{cases} \varphi_k(v) & \text{if } v_i + v_j < v(\{i,j\}),\ k \in (-1, 1], \\ \psi_l(v) & \text{if } v_i + v_j > v(\{i,j\}),\ l \in [-\infty, -1), \cup [1, \infty], \end{cases} \tag{19}$$

and $\phi_{-1-1} = \varphi_{-1} = \psi_{-1}$. The last solution – the equal share solution – does not depend on super or subadditivity of games.

For additive games $\phi_{kl}(v) = \phi_{ind}(v) = (v_i, v_j)$ for all $k, l \neq -1$, and $\phi_{-1-1}(v)$ equals the equal share solution. Note that these two solutions can be combined arbitrarily with solutions for non-additive games (19) and ϕ_{-1-1}.

The locus of a solution ϕ_{kl} for some values (v_i, v_j), $v_i < v_j$, $k \neq -1, 0, 1$, $l \neq \infty, -1, 1$ and for variable $v(\{i,j\})$ is depicted by the thick piece-wise line on Fig 5.

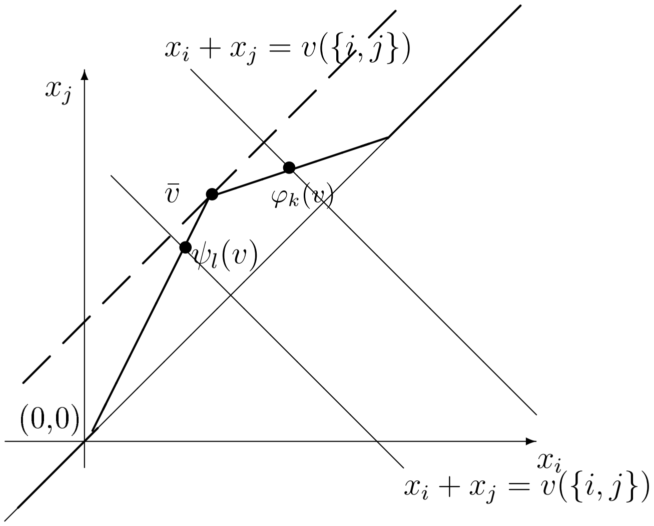

Fig.5

The next Theorem is the union of the statements of Theorems 1 and 2.

Theorem 3 *If a solution ϕ to the class of all non additive two-person games $\mathcal{G}_2$ satisfies axioms NE, SV, EFF, ANO, w-COV and self-COV, then it is a ϕ_{kl} solution for $k \in (-1,1], l \in [-\infty,-1), [1,\infty]$ and $k = l = -1$.*

Note that both solutions for additive two-person games characterized in Proposition 3, may be considered jointly with every solution from Theorem 3 and in this way we obtain the set of all solutions to the class of two-person games verifying all the axioms of Theorem 3.

4 Consistent extensions of self- and weak covariant solutions for two-person games

In this section we find which parameters k, l admit consistent extensions of the solutions ϕ_{kl}, i.e., for what values k, l there exist consistent and non-empty solutions ϕ for the class of all TU games such that on the class of two-person games they coincide with ϕ_{kl}.

Because we consider only non-empty solutions for the whole class of TU games, in order to exclude some values k, l it suffices to show their impossibility at least for one game.

Proposition 4 *Both solutions for additive two-person games defined in Proposition 3 admit consistent extensions. The unique parameters k, l which can have consistent extensions of the solutions ϕ_{kl} to the class of all games with arbitrary set of players are: $k = \{0,1\}$, $l = 1$, and $k = l = -1$.*

The following Theorem shows that the consistent extensions of the two-person game solutions given in Proposition 4 really exist.

Theorem 4 *The unique efficient, single-valued, anonymous, and self-covariant solutions to the class of all two-person games, admitting consistent extensions to games with arbitrary set of players are: 1) the standard solution ϕ_{11} 2) the constrained egalitarian solution ϕ_{01} for superadditive games and the standard solution for subadditive games; 3)the equal share solution ϕ_{-1-1}, giving the equal gains/losses to all players.*

Remark 2 Note that here we apply the solutions ϕ_{11}, ϕ_{01} for the class of additive games as well, see Remark 1. In fact, it is impossible to extend a solution ϕ_{kl} for non-additive two-person games together with the equal share solution for additive ones even to games with more than two players (see the proof of Theorem 4).

Since parameters k, l have expressions in excesses of the corresponding solutions in two-person games: $k, l = \frac{e_j}{e_i}$ if $x_j > x_i$, and since the maximal surpluses of n-person games

$$s_{ij}(x) = \max_{S \ni i, S \not\ni j} (v(S) - x(S)) \tag{20}$$

do not change under reducing, the second case in Theorem 4 may be formulated as follows:

Corollary 1 *Let ϕ be a consistent extension of the solution ϕ_{01} to the class of all TU games. Then for every game (N, v) and $x \in \phi(N, v)$ $s_{ij}(x)s_{ji}(x) \geq 0$, for all $i, j \in N$ and*

$$\text{either } s_{ij}(x) = s_{ji}(x) > 0, \ \text{ or, if } x_j > x_i, \ \text{ then } s_{ji}(x) = 0, s_{ij}(x) \leq 0. \tag{21}$$

Proof. Let a solution ϕ satisfy all the properties given in the statement of the Theorem, (N, v) be an arbitrary game, $x \in \phi(N, v)$. The definitions of surpluses and values of the parameters k, l defining the solution ϕ for the reduced two-person games imply that if $x \in \phi(N, v)$, then

$$k \text{ or } l = \frac{s_{ji}(x)}{s_{ij}(x)}. \tag{22}$$

Since $k = 0$ or 1, and $l = 1$, equality (22) proves the corollary. □

Let us take the assertion of Corollary 1 as the definition of a solution for cooperative games.

Definition 2 An efficient solution Φ on the class $\mathcal{G}$ of all TU games is the *egalitarian prekernel (EPK),* if for every game (N, v) and $x \in \Phi(N, v)$ $s_{ij}(x)s_{ji}(x) \geq 0$ for all $i, j \in N$, and for arbitrary $i, j \in N$ either $s_{ij}(x) = s_{ji}(x) > 0$, or $x_i = x_j$, or else $x_j > x_i$ implies $s_{ji}(x) = 0, s_{ij}(x) \leq 0$.

The name 'egalitarian prekernel' is conditioned by the facts that the payoff vector $x \in \Phi(N, v)$ belongs to the prekernel, $x \in PK(N, v)$, if $s_{ij}(x) > 0$ for all $i, j \in N$, and to the egalitarian set solution (Arín, Iñarra 2001), if $s_{ij}(x) \leq 0$ for all $i, j \in N$.

Theorem 4 and Corollary 1 imply that Φ verifies all the properties of the solution ϕ in Corollary 1.

Definition 2, as well as those of the prekernel and of the egalitarian set solution, determines a solution with the help of some equality and inequality relations between the components of the solution vector and between the corresponding maximal surplus values. Since the maximal surplus function $s_{ij}(x)$ (20) does not change under reducing of the game, all such solutions are consistent. Moreover, they possess the converse consistency property. Let us show converse consistency of the egalitarian prekernel.

Proposition 5 *The egalitarian prekernel possesses the converse reduced game property.*

Proof. Let (N, v) be an arbitrary game, $x \in X^*(N, v)$, and for all two-person reduced games w.r.t. x their solution $(x_i, x_j) = EPK(\{i, j\}, v^x_{i,j})$ satisfies (21). Since the values $s_{ij}(x)$ are the same in two-person reduced games and in the initial game, we obtain that x satisfies (21) as well. □

Let (N, v) be an arbitrary TU game, $PC(N, v)$ be its positive core. It is not empty and compact for every TU game (Orshan, Sudhölter 2010).

Recall the definition of Lorenz domination. Let $x, y \in \mathbb{R}^n$. Denote by $\theta(x), \theta(y) \in \mathbb{R}^n$ the vector whose components coincide with those of x, y respectively, but arranged in a weakly increasing manner. The vector x *Lorenz dominates* y, if there is $k = 1, ..., n$ such that $\sum_{i=1}^k \theta_i(x) > \sum_{i=1}^k \theta_i(y)$, and $\sum_{i=1}^j \theta_i(x) = \sum_{i=1}^j \theta_i(y)$ for all $j = 1, ..., k-1$. A payoff vector x is *Lorenz maximal* in $PC(N, v)$, if $x \in PC(N, v)$, and there is no $z \in PC(N, v)$ that Lorenz dominate x.

Since the positive core is compact, the set of maximal Lorenz preimputations $PC_{Lor}(N, v)$ is non-empty for every game (N, v).

Thus, we can consider the set $PC_{Lor}(N, v)$ as the solution set for a non-empty solution PC_{Lor} for the class of all TU games.

Proposition 6 *$PC_{Lor}(N, v) \subset EPK(N, v)$ for every TU game (N, v).*

Proof. Let (N, v) be and arbitrary game, $x \in PC_{Lor}(N, v)$. . By the definition of the positive core the equalities $s_{ij}(x) = s_{ji}(x)$ hold for all $i, j \in N$ such that $s_{ij}(x) \geq 0$.

Assume that $s_{ij}(x) < 0$, if $x_i < x_j$. Then by the definition of Lorenz domination there are no transfers (y_i, y_j) such that $y_i + y_j = x_i + x_j$, $(x||y_i, y_j) \in PC(N, v)$ and $(x||y_i, y_j)$ Lorenz dominates x. This can happen only if $s_{ji}(x) = 0$ that proves $PC_{Lor}(N, v) = \phi_{0,1}(N, v)$ for all two-person games, or, that is the same, $PC_{Lor}(N, v) \subset EPK(N, v)$.

Proposition 7 *The solution PC_{Lor} is consistent on the class of all TU games.*

Proof. Since the positive core possesses the reconfirmation property (RCP) (Orshan, Sudhölter 2010), the proof coincides with that of Lemma 2 in (Hougaard, Peleg, Thornlund 2001). □

Propositions 6 and 7 show that the solution PC_{Lor} is a non-empty consistent subsolution of the egalitarian prekernel. Moreover, it is a non-empty consistent extension of the solution ϕ_{01}.

5 Concluding remarks

The results of sections 3 and 4 imply open problems concerning solutions being consistent extensions of the solution $\phi_{1,0}$. For the class of convex TU games there is the unique consistent extension of the constrained egalitarian solution being the Dutta–Ray solution. For the class of all TU games the maximum consistent extension of the constrained egalitarian solution is the egalitarian prekernel. Its subsolution PC_{Lor} has at least two single-valued consistent and weakly covariant selectors. For every game (N, v) they are the maximum of the lexmin relation and the minimum of the lexmax relation on the set $PC_{Lor}(N, v)$. The proof of this fact coincides with that for balanced games (Yanovskaya 1997). Thus, a problem arises to describe the class of TU games, not being a subclass of the convex ones, such that for every game from that class the egalitarian prekernel would have the unique consistent selector. Such a class could be considered as an extension of the class of convex games.

The well-known single-values solutions – the Shapley value and the Dutta–Ray solution – can be characterized by the definition of the value for two-person games, and by consistency – the Sobolev consistency for the Shapley value, and the Davis–Maschler consistency for the Dutta–Ray solution. However, the prenucleolus has no such an axiomatization. In fact, yet it is not known whether the covariance property is independent of standardness, ETP, and consistency.

6 Appendix

Proof of Lemma 1. It suffices to prove the Lemma only for the class $\mathcal{G}_2^{0sp}$ with the fixed player set $\{i, j\}$. Evidently the equal share solution φ_1 is self-covariant. Let $v \in \mathcal{G}_2^{sp}$ be an arbitrary game. If $\varphi(v) = (0,0)$ then the Lemma is true. Let now $\varphi_k(v) \neq (0,0)$. Then, by (14)

$$\varphi_k(v) = \left(\frac{-v_j + kv_i}{k+1}, \frac{v_j - kv_i}{k+1}\right).$$

Let us calculate $\varphi_k(v + A\varphi_k(v)$. Denote $v + A\varphi_k(v) = (m_i, m_j)$.

From (14) it follows that $(\varphi_k(v))_i \leq \varphi_k(v))_j$ for all $k \in [0,1]$. Hence, for $A > -1$ $m_i < m_j$, and we have the equality

$$\Big(\varphi_k(v+A\varphi_k(v)\Big)_i = \frac{-m_j + km_i}{k+1} = \frac{(-v_j + kv_i)}{k+1}\Big(1+\frac{A}{k+1}+\frac{Ak}{k+1}\Big) = (1+A)\frac{-v_j + kv_i}{k+1} = (1+A)(\varphi_k(v))_i.$$

Similarly it can be checked that $\Big(\varphi_k(v + A\varphi_k(v)\Big)_j = (A+1)(\varphi_k(v))_j$. □

Proof of Theorem 1. Evidently, the φ_k solutions satisfy axioms NE, EFF, SV, ANO and wCOV. Lemma 1 shows that they satisfy the axiom self-COV as well.

Let now φ be an arbitrary solution for the class $\mathcal{G}_2^{sp}$ satisfying all the axioms stated in the Theorem. Because of weak covariance of φ it suffices to prove that for every game $v \in \mathcal{G}_2^{0sp} \subset \mathcal{G}_2^{sp}$ $\varphi(v) = \varphi_k(v)$ for some $k \in [-1,1]$.

Let $v \in \mathcal{G}_2^{0sp}$ be an arbitrary game. Then $v_i + v_j \leq 0$.

If $\varphi(v) = (0,0)$ for all $v \in \mathcal{G}_2^0$, then $\varphi = \varphi_{-1}$.

Let now there exist a game $v \in \mathcal{G}_2^{0sp}$ such that $\varphi(v) \neq (0,0)$. From definition (7) of self-covariance it follows that $\varphi(w) = \varphi(v)$ for all games w such that

$$w = \beta v + (1-\beta)\varphi(v) \quad \text{for some } \beta \geq 0. \tag{23}$$

Let us show that the parameter $k(v) = \frac{\varphi_j(v) - v_j}{\varphi_i(v) - v_i}$ is the same for all games v' with $v'_i < v'_j$ and $\varphi(v') \neq (0,0)$. Assume that there are two games $v^1, v^2 \in \mathcal{G}_2^{0sp}$ with $\varphi(v^1), \varphi(v^2) \neq (0,0)$ such that $k(v^1) \neq k(v^2)$. Let $k(v^1) > k(v^2)$.

Suppose that $\varphi(v^2)_i > \varphi(v^1)_i$. Then the rays from the points $\varphi(v^1)$ and $\varphi(v^2)$ through v^1, v^2 respectively, intersect in a point (u_i, u_j), $u_i < u_j$ such that $u_i + u_j < 0$. Then by (23) $\varphi(u) = \varphi(v^1) = \varphi(v^2)$ that contradicts the assumption.

If $\varphi(v^1)_i > \varphi(v^2)_i$, then consider a game αv^1, where $\alpha > 0$ is sufficiently small such that $\varphi(\alpha v^1)_i = \alpha\varphi(v^1)_i < \varphi(v^2)_i$. Then, as in the previous case, we obtain $k(v^2) = k(\alpha v^1)$, and $k(\alpha v^1) = k(v^1)$ by positive homogeneity of φ.

Thus, we have obtained that k is the unique parameter corresponding to the $\varphi = \varphi_k$ solution for games v with $\varphi(v) \neq (0,0)$.

Consider now games v with zero solution value $\varphi(v) = (0,0)$.

Equality (23) implies that $\varphi(v) = (0,0)$ for all games v whose vector $\bar{v}$ is situated on the ray $x_j = kx_i, x_i \leq 0$ going out of zero point. Consider a game v with $v_j < kv_i$. Assume that $\varphi_i(v) < 0$, then the line, connecting $\bar{v}$ and $\varphi(v)$ intersects the line $x_j = kx_i$, and, by self consistency of φ, we should obtain $\varphi(v) = 0$ that contradicts the assumption $\varphi_i(v) < 0$. Assume now that $\varphi_i(v) > 0$. Then the ray going out from the point $\bar{v}$ and passing through $\varphi(v)$ intersects the diagonal in a point $\bar{w} = (w,w), w < 0$. Since, by anonymity of φ, $\varphi(w) = (0,0)$, by (23) we obtain $\varphi(v) = (0,0)$, that again contradicts the assumption.

Hence, $\varphi(v) = (0,0)$ for all games v with $v_i < v_j$ and $v_j \leq kv_i$, and we have proved that $\varphi = \varphi_k$ for some $k \in (-1,1]$ and for all games from $\mathcal{G}^{0sa}$.

Let now the value φ be individually rational. Then $k = \dfrac{\varphi_j(v) - v_j}{\varphi_i(v) - v_i} > 0$, and the proof is over.

□

Proof of Theorem 2. The proof is similar to that of Theorem 1.

Let ψ be an arbitrary solution for the class $\mathcal{G}_2^{sb}$ satisfying all the axioms stated in the Theorem.

By weak covariance of the solution ψ it suffices to prove the theorem only for the class $\mathcal{G}_2^{0sb} \cap \mathcal{G}_2^{sb}$ of games with zero total gain $v(\{i,j\}) = 0$. Let v be such a game. Then $v_i + v_j > 0$. Without loss of generality we may assume that $v_j > v_i$. If $\psi(v) = (0,0)$ for all $v \in \mathcal{G}_2^0$, then $\psi = \psi_{-1}$.

Let now $\psi(v) \not\equiv (0,0)$. Consider a game with $\psi(v) \neq (0,0)$. Evidently, as well as for superadditive games, for any subadditive game v and self covariant value ψ $\psi(w) = \psi(v)$ for all games w satisfying equality (23), or, that is the same,

$$w_j - v_j = (w_i - v_i)\frac{\psi_j(v) - v_j}{\psi_i(v) - v_i}. \tag{24}$$

Denote $l = \frac{\psi_j(v) - v_j}{\psi_i(v) - v_i}$. Then $w_j = lw_i + v_j - lv_i$. Let us show that $l \geq 1$ or $l \leq -1$. Suppose that $0 \leq l < 1$. Then the ray (24) intersects the diagonal in the point $\bar{w} = (w,w)$, where $v_i < w < \psi_i(v)$, so $\bar{w} = \alpha\bar{v} + (1-\alpha)\psi(v)$ for some $\alpha \in (0,1)$. By (23) we obtain $(0,0) = \psi(w) = \psi(v)$, that is a contradiction.

Suppose that $-1 < l < 0$. Then $\psi_j(v) > v_j,\ \psi_i(v) < v_i$, hence, the number w satisfying the equalities and the inequality

$$-1 < l = \frac{w - v_j}{w - v_i} = \frac{\psi_j(v) - v_i}{\psi_i(v) - v_i} < 0,$$

satisfies the inequality $v_i < w$. Thus, the point $\bar{v}$ is placed on the same line between $\psi(v)$ and $\bar{w} = (w,w)$, hence $\bar{v} = \alpha\psi(v) + (1-\alpha)\bar{w}$ for some $\alpha \in (0,1)$, or

$$(1-\alpha)\bar{w} = \bar{v} - \alpha\psi(v).$$

Therefore, by self-consistency and positive homogeneity of ψ we obtain $\psi(v) = \psi(w) = (0,0)$, that is again a contradiction.

Thus, $l \geq 1$ or $l \leq -1$, and the value $\psi(v) = \psi_l(v)$ for some l and for all games v such that $\psi(v) \neq (0,0)$.

Suppose that $l > 1$ and consider a game v such that $l = \frac{v_j}{v_i}$. Then $\psi(v) = (0,0)$, and, by positive homogeneity of ψ $\psi(u) = (0,0)$ for every game $u = (u_i, u_j)$ satisfying $u_j = lu_i$. Now consider a game v with $v_i < v_j < lv_i$. Assume that $\varphi_i(v) < 0$, then the line, connecting $\bar{v}$ and $\varphi(v)$ intersects the line $x_j = lx_i$, and, by self consistency of ψ we obtain $\psi(v) = 0$ that contradicts the assumption. If $\varphi_i(v) > 0$, then the line, connecting v and $\varphi(v)$, would intersects the diagonal in a point $\bar{w}$, and by the same reason we obtain the contradiction $0 = \psi(w) = \psi(v)$.

Analogously the proof of the equality $\psi(v) = (0,0)$ for games with $0 > \frac{v_j}{v_i} > l$ for negative values of l is fullfiled.

Thus, we have proved that $\psi = \psi_l$ for $l \in [-\infty, -1] \cup [1, \infty]$. □

Proof of Proposition 4 The equal share solution and the solution $\phi_{ind}(N,v) = \{v_i\}_{i \in N}$ for additive games are efficient, single-valued, anonymous, weak and self-covariant, and consistent.

Consider a solution ϕ_{kl} for the class of non-additive two-person games satisfying the properties given in Theorem 3. We will find under what parameters k, l the solutions ϕ_{kl} have no consistent extensions to the class of games $\mathcal{G}_{3,0}$ with three players such that $(N,v) \in \mathcal{G}_{3,0}$ if $|N| = 3$, $v_i > 0, v(\{i,j\} = v_i + v_j$ for all $i,j \in N$, . Let for simplicity $N = \{1,2,3\}$. Without loss of generality we may suppose that $v_1 \leq v_2 \leq v_3$. For any $(N,v) \in \mathcal{G}_{3,0}$ denote $\phi(N,v) = (x_1, x_2, x_3)$. Consider all possible cases of relations between the values v_i, x_i, $i = 1,2,3$ and x_1, x_2, x_3. In the sequel we use notation $e_i = v_i - x_i$, $i = 1,2,3$.

By consistency of the solution ϕ for every reduced game $(\{i,j\}, v^x_{ij})$ its characteristic function is defined as follows:

$$v^x_{ij}(\{i\}) = \begin{cases} v_i, & \text{if } e_t \leq 0, \\ v_i + e_t & \text{if } e_t > 0,\ t \in \{1,2,3\}, t \neq i,j. \end{cases} \tag{25}$$

From the definition of the solutions ϕ_{kl} for two-person games it follows that these solutions save inequalities between the individual characteristic functions values: given a game $(\{i,j\}, v)$

$$v_i > v_j \implies \phi_i(\{i,j\}, v) > \phi_j(\{i,j\}, v). \tag{26}$$

Thus, from (25),(26) the inequalities $x_1 \leq x_2 \leq x_3$ imply equalities

$$k \text{ or } l = \begin{cases} \frac{e_j}{e_i} & \text{if } e_t \leq 0, \\ \frac{e_j + e_t}{e_i + e_t} & \text{otherwise }, \end{cases} \tag{27}$$

where $v_j \geq v_i$, $i,j,t = 1,2,3$, and k or l are chosen depending on super- or subadditivity of the reduced game $(\{i,j\}, v^x_{ij})$.

Given the inequalities $v_1 \leq v_2 \leq v_3$, and $x_1 \leq x_2 \leq x_3$, let us consider all cases of inequalities between v_i and x_i, $i = 1,2,3$, and between $v_1 + v_2 + v_3$ and $v(N)$.

1. $v_1 + v_2 + v_3 > v(N)$.

1-1. $e_1 < 0, e_2, e_3 > 0$.

The reduced game on the set $(\{2,3\})$ is subadditive, and we obtain $l = \frac{e_3}{e_2}$.

a) $e_1 + e_2 + 2e_3 > 0$, $e_1 + e_3 + 2e_2 > 0$.

Since $e_3 \geq e_2$, the right-hand side inequality $e_1 + e_3 + 2e_2 > 0$ implies the left-hand side one $e_1 + e_2 + 2e_3 > 0$.

Then both reduced games on $\{1,2\}$, and on $\{1,3\}$ are subadditive, and we obtain the equalities

$$l = \frac{e_2 + e_3}{e_1 + e_3} = \frac{e_2 + e_3}{e_1 + e_2} = \frac{e_3}{e_2}.$$

These equalities imply $e_2 = e_3$, and the only possibility for l is $l = 1$.

b) $e_1 + e_3 + 2e_2 < 0$.

In this case the reduced game $(\{1,3\}, v^x_{13})$ is superadditive. Therefore,

$$k = \frac{e_3 + e_2}{e_1 + e_2}.$$

Since $k \in [-1, 1]$, the last equality implies $e_3 \leq e_1$ that contradicts the inequalities $e_1 < 0, e_3 > 0$.

The subcases a) and b) exhaust the case 1-1.

1-2. $v_1 + v_2 + v_3 > v(N), \;\; e_1, e_2, e_3 > 0$.

In this case all reduced games on two-person games $(\{i,j\}, v^x_{i,j})$ are subadditive$<$ hence,

$$l = \frac{e_2 + e_3}{e_1 + e_2} = \frac{e_3 + e_2}{e_1 + e_2} = \frac{e_3 + e_1}{e_2 + e_1},$$

implying $e_1 = e_2 = e_3, \;\; l = 1$.

1-3. $v_1 + v_2 + v_3 > v(N), \;\; e_1 < 0, e_2 < 0, e_3 > 0$.

a) $e_1 + e_3 > 0$, $e_2 + e_3 > 0$. As in the previous case we obtain that all two-person reduced games are subadditive, and

$$l = \frac{e_3}{e_1} = \frac{e_3}{e_2} = \frac{e_2 + e_3}{e_1 + e_3} = 1.$$

b) $e_1 + e_3 > 0$, $e_2 + e_3 < 0$. The reduced games $(\{1,3\}, v^x_{1,3})$ and $(\{1,2\}, v^x_{1,2})$ are subadditive, hence,

$$l = \frac{e_3}{e_1} = \frac{e_2 + e_3}{e_1 + e_3}.$$

However, this equality is impossible, since $e_2 + e_3 < e_3, e_1 + e_3 > e_1$.

The impossibility of the case $e_2 + e_3 > 0, e_1 + e_3 < 0$ is shown similar to the case b). Since $e_1 + e_2 < 0$, the case 1-3 has been considered completely, giving the unique possibility $l = 1$.

All the cases $e_i, e_j < 0, e_t > 0, i, j, t = 1, 2, 3$ are considered analogously to the case 1-3. Thus, the case 1 has been considered completely.

2. $v_1 + v_2 + v_3 < v(N)$.

2-1. $e_1, e_2, e_3 < 0$. In this case all two-person reduced games are superadditive, hence, we obtain the following values for k :

$$k = \frac{e_2}{e_1} = \frac{e_3}{e_1} = \frac{e_3}{e_2},$$

implying $k = 1$.

2-2. $e_1 < 0, e_2 < 0, e_3 > 0$.

a) $e_1 + e_2 + 2e_3 < 0$. Then the reduced game $(\{1,2\}, v^x_{1,2})$ is superadditive, and $k = \frac{e_2 + e_3}{e_1 + e_3}$.

a-1) $e_1 + e_3 > 0$. Then $e_2 + e_3 < 0$, and the reduced game $(\{1,3\}, v^x_{1,3})$ is subadditive$<$ and the reduced game $(\{2,3\}, v^x_{2,3})$ is superadditive. Hence,

$$l = \frac{e_3}{e_1}, \;\; k = \frac{e_3}{e_2}.$$

Equalizing two expressions for k, we obtain $k = \frac{e_3}{e_2} = \frac{e_2+e_3}{e_1+e_3}$. However, since $e_3 > e_2 + e_3$, $e_2 < 0 < e_1 + e_3$, the last equality is impossible.

a-2) $e_2 + e_3 > 0$. Then $e_1 + e_3 < 0$, and similar to case a-1) we obtain

$$l = \frac{e_3}{e_2}, \;\; k = \frac{e_3}{e_1}. \tag{28}$$

Equalizing the expressions for k we obtain

$$k = \frac{e_3}{e_1} = \frac{e_2 + e_3}{e_1 + e_3}, \tag{29}$$

implying $e_3^2 = e_1 e_2$. Hence, $k = \sqrt{\frac{e_2}{e_1}}$. Since $|l| \geq |k|$, from (28) it follows $e_2 \geq e_1$, and, since $|k| \leq 1$, equality (29) may hold only if $e_1 = e_2 = e_3$ that is impossible.

a-3) $e_1 + e_3 < 0$, $e_2 + e_3 < 0$. As in the previous cases we obtain $l = \frac{e_2+e_3}{e_1+e_3}$. The reduced games $(\{1,3\}, v^x_{1,3}), (\{2,3\}, v^x_{2,3})$ are superadditive, hence,

$$k = \frac{e_3}{e_1} = \frac{e_2}{e_1}$$

implying $e_1 = e_2$, and $k = l = 1$.

The subcase 2-2a) has been considered completely.

b)$e_1 + e_2 + 2e_3 > 0$. Then the reduced game $(\{1,2\}, v^x_{1,2})$ is subadditive, and $l = \frac{e_2+e_3}{e_1+e_3}$.

b-1) $e_1 + e_3 > 0$. Since $e_1 + e_2 < 0$, the reduced game $(\{1,2\}, v^x_{1,2})$ is subadditive, $l = \frac{e_3}{e_1}$. Similar to case a-2), equalizing two expressions for l we obtain that the equality $l = \frac{e_3}{e_1} = \frac{e_2+e_3}{e_1+e_3}$ is impossible.

b-2) $e_1 + e_3 < 0$ that implies $e_2 + e_3 > 0$. Analogously to the previous cases we obtain

$$k = \frac{e_3}{e_1}, \;\; l = \frac{e_3}{e_2} = \frac{e_2 + e_3}{e_1 + e_3}.$$

Since $|l| \geq |k|$, from these equalities we obtain that $e_1 < e_2$.

From the last equality it follows $e_3^2 + e_1 e_3 = e_2^2 + e_2 e_3$. Hence, $e_3^2 > e_2^2 > e_1^2$, implying that, in view of $|k| \leq 1$ the only possibility is $e_3 = -e_1$, i.e., $k = -1$, $e_1 + e_3 = 0 \;\; \to \;\; e_2 = 0$. Therefore, $e_1 + e_2 + e_3 = 0$ that contradicts the conditions of case 2.

b-3). $e_1 + e_3 < 0, e_2 + e_3 < 0$. Then from the superadditive reduced games $(\{1,3\}, v^x_{1,3})$, $(\{2,3\}, v^x_{2,3})$ we obtain

$$k = \frac{e_3}{e_1} = \frac{e_3}{e_2}.$$

The last equality implies $e_1 = e_2$, hence, $k = 1$.

If the reduced game $(\{1,2\}, v^x_{1,2})$ is superadditive, then we obtain the expression for k once more. If this game is subadditive, then $l = \frac{e_2+e_3}{e_1+e_3} = 1$.

Thus, we have finished studying the case 2-2b), and all possibilities have been considered. We have shown that for three-person games considered in cases 1–2, a single-valued consistent solution ϕ that coincides with $\varphi_{k,l}$ on two-person reduced game w.r.t. $x = \phi(\{1,2,3\}, v)$, and $v_i - x_i \neq 0$, for $i = 1,2,3$, has the unique possibility for parameters $k, l : k, l = 1$.

If some of excesses equal zero, then the parameters k, l may take values $k = 0$ and $l = \infty$. Let us show that the second equality $l = \infty$ is impossible. Consider the example of subadditive three-person game from (Arín, Iñarra 2001):

$$N = \{1,2,3\},\ v_1 = v_2 = 1, v_3 = 0,\ v(\{1,3\}) = 1.4,\ v(\{2,3\}) = 1.3, v(\{1,2\}) = v(N) = 2.2.$$

Let $y = \varphi_{k,l}(N,v)$. Apply the usual notation $s_{ij}(y) = \max\{v(\{i\}) - y_i, v(\{i,k\}) - (y_i + y_k)\}$, $i, j, k \in \{1,2,3\}$.

By reducing the game on two-person player sets w.r.t. y we obtain that $k($ or $)l = \frac{s_{ij}(y)}{s_{ji}(y)}$, if $y_i > y_j$.

First, we show that $s_{ij}(y) \geq 0$ for all $i, j \in N$. Consider all pairs i, j.

1) Suppose that $s_{12}(y) \leq 0$. Since $k = 0$ or 1, then it should be $s_{12}(y) = s_{21}(y) \leq 0$, that implies inequalities $y_1 \geq 1, y_1 + y_3 \geq 1.4,\ \ y_2 \geq 1, y_2 + y_3 \geq 1.3$, and the efficiency equality $y_1 + y_2 + y_3 = 2.2$ These inequality and equality are inconsistent.

2) Suppose that $s_{13}(y) = s_{31}(y) \leq 0$. Then $y_1 \geq 1, y_1 + y_2 \geq 2.2$. the last inequality implies $y_3 \leq 0$. However, from $s_{31}(y) \leq 0$ it follows $y_3 \geq 0$, that is $y_3 = 0$. The inequality $y_2 + y_3 \geq 1.3$ implies $y_2 \geq 1.3$, and this inequality together with $y_1 \geq 1$ are inconsistent with the efficiency equality $y_1 + y_2 = 2.2$

3) Suppose that $s_{23}(y) = s_{32}(y) \leq 0$. It is not difficult to check that the system of inequalities

$$\begin{array}{rl} y_2 \geq 1, & y_1 + y_2 \geq 2.2, \\ y_3 \geq 0, & y_1 + y_3 \geq 1.4, \\ y_1 + y_2 + y_3 & = 2.2 \end{array} \tag{30}$$

is inconsistent.

Therefore, all two-person reduced games are subadditive, and the solution $\psi_l(N,v)$ should have parameters $l = 1$ or $l = \infty$. The first case is impossible as cases 1)-3) above show. The second possibility means that

$$y_j > y_i \implies s_{ji}(y) \geq 0, \text{ and } s_{ij}(y) = 0.$$

The impossibility of this relation had been shown in (Arín, Iñarra 2002).

Proof of Theorem 4. First, let us show that there are consistent extensions of the solutions 1)–3). In fact, the prenucleolus equals the standard solution for two-person games and satisfies other properties given in the Theorem. Hence, on the class of all two-person games the prenucleolus coincides with the solution ϕ_{11}.

Let us show the existence of consistent extension of the solution ϕ_{01}. Given an arbitrary game (N,v), denote by $C(N,v)$ its core, and by $PC(N,v)$ its positive core. It is non empty for every TU game (Orshan, Sudhölter 2010). On the class of subadditive two-person games the positive core coincides with the prenucleolus, and on the class of two-person superadditive games it coincides with the core.

Consider the solution PC_{lexmin} assigning to every game the payoff vector from the positive core on which the maximum of the lexmin relation is attained. This solution is single-valued, anonymous, and on the class of balanced games it coincides with the the lexmin core solution (Yanovskaya 1997, Arín, Iñarra 2001). Hence, on the class of subadditive two-person games this solution coincides with the prenucleolus, and on the class of superadditive two-person games the solution PC_{lexmin} coincides with the constrained egalitarian solution, thus on the class of all two-person games $PC_{lexmin} = \phi_{01}$.

The proof of consistency of the solution PC_{lexmin} can be fulfilled with the help of a slight modification of the corresponding proof for balanced games in Yanovskaya (1997).

Let ϕ be an arbitrary solution satisfying the conditions of the Theorem. The equal share solution is defined for games with arbitrary set of players, and satisfies all the properties given in the Theorem (see Proposition 1).

If $\phi \neq ES$, then $\phi = \phi_{kl}, k, l \neq -1$ on the class of non-additive two-person games (Theorem 3), then the uniqueness of the values $k = 0$ or $k = 1$, $l = 1$ for the class of non-additive games has been proved in Proposition 4.

Now let us show that for every additive two-person game $(\{i, j\}, v)$ this solution $\phi(\{i, j\}, v) = \phi_{ind}(\{i, j\}, v) = (v_i, v_j)$. For this purpose we should check what a solution – the equal share solution or the solution ϕ_{ind} on the class of additive games – may be consistent with the solutions ϕ_{11} and ϕ_{01} on the class of non-additive two-person games.

First, consider the prenucleolus. On the class of convex games the prenucleolus is the unique single-valued, anonymous, and consistent solution, being equal to the standard solution on the subclass of two-person games (Theorem I^{con}.)

This solution coincides with ϕ_{11} on the class of all two-person games. Since there are convex games with more that 3 players such that at least one reduced game on two-player set and w.r.t. the prenucleolus is additive, we obtain that there are no consistent solutions differing from the standard solution on the class of additive games, since for every additive two-person game $ST(\{i, j\}, v) = (v_i, v_j) = \phi_{ind}(\{i, j\}, v)$.

Now consider the solution $PC_{lexmin} \in PC$. For every additive two-person game $(\{i, j\}, v)$ $PC(\{i, j\}, v) = (v_i, v_j)$. Since the positive core is consistent, we obtain that $\phi = \phi_{ind}$ for every additive two-person game.

Thus, only the solution ϕ_{ind} can be combined with consistent extensions of the solutions ϕ_{11}, ϕ_{01} on the class of all TU games.

□

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