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ON MAXIMAL VECTOR SPACES OF FINITE NON-COOPERATIVE GAMES

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On maximal vector spaces of finite non-cooperative games *

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Abstract

We consider finite non-cooperative N person games with fixed numbers m_i, $i = 1, \ldots, N$, of pure strategies of player i. We propose the following question: is it possible to extend the vector space of finite non-cooperative $m_1 \times m_2 \times \ldots \times m_N$-games in mixed strategies such that all games of a broader vector space of non-cooperative N person games on the product of unit $(m_i - 1)$-dimensional simpleces have Nash equilibrium points? We get a necessary and sufficient condition for the negative answer. This condition consists of a relation between the numbers of pure strategies of the players. For two-person games the condition is that the numbers of pure strategies of the both players are equal.

Keywords: Finite non-cooperative N person games; vector space; Nash equilibrium point; maximality.

Subject Classification: C72

1 Introduction

In game theory, as a rule results on equilibrium point existence are proved not for a particular game but rather for classes of games. Often these classes or games turn out to be vector spaces.

For example, Nash theorem [1] states equilibrium point existence for elements of the vector space of finite non-cooperative N person games of any fixed size. When the set of strategy profiles is fixed, a linear combination of games is defined as the game with payoff functions equal to linear combinations of their payoff functions.

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Differential games provide us with another source of objects for which existence of equilibrium points is important for all elements of some vector space of games. There a necessary condition for equilibrium existence in pure strategies is the equality of maxmin and minmax of the left hand side of the Isaacs equation [2]. The coefficients of this equation are the derivatives of the game value of which little is known. Thus one has to consider these coefficients as arbitrary real numbers and check the equality of maxmin and minmax with arbitrary coefficients, and this is equivalent to checking equilibrium existence for all games of the corresponding vector space.

Note that linear combinations of games appear in the theory of games with incomplete information where the payoffs are not commonly known, i.e. players do not know exactly what game is played. In these games the state of knowledge of a player is represented by a linear combination of all the possible games (see Harsanyi [3] and Aumann, Maschler [4]). In classical setting matrix game with incomplete information on both sides is given by payoff matrices A_1, A_2, …, A_K of the same size. Before the game starts a chance move determines the "state of nature" $k \in \{1, 2, \dots, K\}$ and therefore the payoff matrix A_k according to probability distribution $\mathbf{p} = (p_1, p_2, \dots, p_K)$. Thus the matrix A_k is played with probability p_k. Players know the probability vector $\mathbf{p}$ and do not know the result of chance move. In this game players are faced with the matrix game given by payoff matrix

$$A(\mathbf{p}) = \sum_{k=1}^{K} p_k A_k.$$

To describe a set of vector spaces of games it is sufficient to characterize its maximal elements. For two-person zero-sum games maximality may be treated in the sense of partial order by inclusion. In the general case maximality of a vector space of games means that this space can not be extended in any essential way, i.e. in any extension the class of functions a player has to maximize is preserved for each player.

Consider the vector space of 2×2 matrix games, or, equivalently, the four-dimensional vector space of matrices of size 2×2. The subset of such 2×2 matrices having saddle points is not a vector space. Indeed, the sum of two matrices with saddle points does not necessarily have a saddle point, for example one may take matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

But there are vector spaces contained in the set of matrices with saddle points. It is easy to see that if 2×2 matrix A has a saddle point then all matrices αA, where $\alpha \in \mathbb{R}^1$, have saddle points as well. This one-dimensional vector space is not maximal in the set of matrices having saddle points as it is contained in the set of matrices $\alpha A + \beta C$ where $\beta \in \mathbb{R}^1$ and C is the matrix consisting of 1's only, and these matrices have saddle points.

Let us show that there exist four three-dimensional vector spaces of 2×2 matrices with saddle points. Consider two linear independent matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Matrices $\alpha A_1 + \beta A_2$, where $\alpha, \beta \in \mathbb{R}^1$, form two-dimensional vector space of matrices with saddle points. As in the previous example with the help of the constant matrix C it may be expanded to a three-dimensional vector space of matrices with saddle points. The matrices of this vector space have the following form

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \gamma \end{bmatrix},$$

i.e. the elements of the second row are equal. An analogous result is true for the vector space of matrices with equal elements in the first row as well as with equal elements of the first (or second) column. Obviously each of these four three-dimensional vector spaces is maximal in the set of matrices with saddle points as the latter is a strict subset of the four-dimensional space.

Sobolev [3] considered the set of all vector spaces of $n \times n$ matrix games with equilibrium points in pure strategies ($n \times n$ matrices with saddle points) and showed that the dimension of maximal vector space belonging to this set does not exceed $(n-1)^2 + 1$, $(n \geq 3)$. For the case of $n = 2$, as we demonstrated above it is $3 = (n-1)^2 + 2$.

Kreps [4] proved that the set of all vector spaces of two-person zero-sum continuous (i.e., with continuous payoff function) games on the unit square with saddle points contains, besides maximal vector spaces of infinite dimension, maximal linear spaces of any finite dimension greater than 3.

In this paper we consider finite non-cooperative N person games with fixed numbers m_i, $i = 1, \ldots, N$, of pure strategies of player i. Finite non-cooperative N person game in mixed strategies is given by an N-tuple of real payoff functions over the product of unit simpleces $S = \prod_1^N S_i$. Dimension of S_i is equal to $m_i - 1$. The payoff function of a player depends linearly on the strategies of any player when the strategies of all other players are fixed. Such games form a vector space of games on S with Nash equilibrium points.

Then we consider non-cooperative N person games on $S = \prod_1^N S_i$ with arbitrary continuous payoff functions over S. We denote $\mathbf{Eq}(S)$ the set of all vector spaces of continuous games on S with equilibrium points in pure strategies. We will never go to mixed extension of these games. Note that the set S_i itself is the set of probability measures on $\{1, \ldots, m_i\}$.

We propose the following question: *is it possible to extend the vector space of finite non-cooperative $m_1 \times m_2 \times \ldots \times m_N$-games in mixed strategies such that all games of a broader vector space of non-cooperative N person games on S have Nash equilibrium points?* So the question is whether the vector space of finite non-cooperative $m_1 \times m_2 \times \ldots \times m_N$-games on S is maximal in $\mathbf{Eq}(S)$.

Our main result is a necessary and sufficient condition for the vector space of finite non-cooperative N person games on S to be maximal in $\mathbf{Eq}(S)$. This condition consists of a relation between the numbers of pure strategies of the players. For two-person games the condition is that the numbers of pure strategies of the both players are equal.

2 Basic notions and preliminary results

Let X_i be the set of strategies of Player i, $i = 1, \ldots, N$, where X_i is a convex compact subset of the non-negative orthant of a finite dimensional Euclidian space. Sometimes the set X_i happens to be a simplex; in this case we denote it by S_i to emphasize this fact. In particular, we do so in the Introduction and in the statement of our main result.

Non-cooperative N person game F on $X = \prod_{i=1}^N X_i$ is given by payoff functions of players $F_i : X \to \mathbb{R}^1$, $i = 1, \ldots, N$.

Strategy profile $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*) \in X$ is the Nash equilibrium point of game F on X if for all $i = 1, \ldots, N$ and all $\mathbf{x}_i \in X_i$ the inequality

$$F_i(\mathbf{x}^*) \geq F_i(\mathbf{x}_1^*, \ldots, \mathbf{x}_{i-1}^*, \mathbf{x}_i, \mathbf{x}_{i+1}^* \ldots, \mathbf{x}_N^*)$$

holds.

A linear combination $\alpha H + \beta F$, $\alpha, \beta \in \mathbb{R}^1$ of non-cooperative games H and F on X is defined as the game on X with payoff functions $\alpha H_i + \beta F_i$, where H_i and F_i are Player i's payoff functions in the games H and F correspondingly.

Let $\mathcal{G}(X)$ be a set of vector spaces of non-cooperative games on X and let $\mathbf{H}(X) \in \mathcal{G}(X)$ be a vector space of non-cooperative games H on X.

Definition of maximality. We say that $\mathbf{H}(X)$ is *maximal in* $\mathcal{G}(X)$ iff an inclusion

$$\mathbf{H}(X) \subset \mathbf{H}'(X) \in \mathcal{G}(X) \tag{1}$$

implies that for any player i, $i = 1, \ldots, N$, the classes of functions maximized by Player i in games in $\mathbf{H}(X)$ and in games in $\mathbf{H}'(X)$ coincide (up to functions not depending on Player i's strategies).

For zero-sum two-person games it means that the inclusion (1) implies $\mathbf{H}(X) = \mathbf{H}'(X)$.

Now consider $\mathbf{Eq}(X)$ (the set of vector spaces of continuous games on X with equilibrium points in pure strategies). As we indicated in Introduction we never go to mixed extension of games on X.

Lemma 1. *Let A be a set such that for some $1 \leq i_0 \leq N$ it holds that $A \subset X_{i_0}$ and $\bar{A} \neq X_{i_0}$ where $\bar{A}$ denotes the closure of the set A. If for any game $H \in \mathbf{H}(X)$ there exists an equilibrium point $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, \mathbf{x}_N^*) \in X$ such that $\mathbf{x}_{i_0}^* \in A$, then the vector space $\mathbf{H}(X)$ is not maximal in $\boldsymbol{Eq}(X)$.*

Fix a non-cooperative game K on X with continuous payoff functions K_i, $i = 1, \ldots, N$, on X. Fix number i, $1 \leq i \leq N$, and two points $\mathbf{a} \in X_i$, $\mathbf{b} \in X_i$, $\mathbf{a} \neq \mathbf{b}$. Denote $\mathbf{x}^i$ a strategy profile of all players but Player i and denote the set of such profiles $X^i = \prod_{j \neq i} X_j$. Let us define function $K_i^{\mathbf{a},\mathbf{b}}$ on $[\mathbf{a}, \mathbf{b}] \times X^i$ as follows: for any $\mathbf{x}^i \in X^i$ function $K_i^{\mathbf{a},\mathbf{b}}$ is linear on $\mathbf{x}_i$ over the interval $[\mathbf{a}, \mathbf{b}]$ and

$$K_i^{\mathbf{a},\mathbf{b}}(\mathbf{a}, \mathbf{x}^i) = K_i(\mathbf{a}, \mathbf{x}^i), \qquad K_i^{\mathbf{a},\mathbf{b}}(\mathbf{b}, \mathbf{x}^i) = K_i(\mathbf{b}, \mathbf{x}^i).$$

Lemma 2. *Let $\bar{\mathbf{x}} \in [\mathbf{a}, \mathbf{b}] \times X^i$ be an equilibrium point of the game $H + \beta K$ where $H \in \mathbf{H}(X)$, K is a continuous game on X and $\beta \in \mathbb{R}^1$. Then the following inequality holds*

$$\beta K_i(\bar{\mathbf{x}}) \geq \beta K_i^{\mathbf{a},\mathbf{b}}(\bar{\mathbf{x}}),$$

where function $K_i^{\mathbf{a},\mathbf{b}}$ is defined above.

Lemma 3. *Assume that for any $\mathbf{x}^* \in IntX$ there exists a game $H \in \mathbf{H}(X)$ with the unique equilibrium point $\mathbf{x}^*$.*

Let K be a continuous game on X. If for any game $H \in \mathbf{H}(X)$ and for any $\beta > 0$ the game $H + \beta K$ has an equilibrium point, then Player i's payoff function K_i, $i = 1, \ldots, N$, is concave as a function of his strategy $\mathbf{x}_i$ (i.e. its subgraph is convex) for any fixed strategy profile of other players $\mathbf{x}^i \in X^i$.

The proofs of Lemmas 1-3 is given in Appendix.

3 Main result

Let us consider finite non-cooperative $m_1 \times m_2 \times \ldots \times m_N$-games of N players in mixed strategies, i.e. games on $S = \prod_{i=1}^N S_i$ (dimension of simplex S_i is equal to $m_i - 1$, $i = 1, \ldots, N$) with linear player payoff function on strategies of any player. Let $\mathcal{H}(S)$ denote the vector space of such games, $\mathcal{H}(S) \in \mathbf{Eq}(S)$.

The proof of the main result is based on the lemmas above and on the following fact (see [5]). *The inequality*

$$\max_{i=1,\ldots,N}(m_i - 1) = (m_{i_0} - 1) \leq \sum_{i \neq i_0}(m_i - 1) \tag{2}$$

for the dimensions of simplices S_i, $i = 1, \ldots, N$, is the necessary and sufficient condition that any completely mixed strategy profile in $S = \prod_{i=1}^N S_i$ is the unique equilibrium point of some finite non-cooperative game of size $m_1 \times m_2 \ldots \times m_N$.

A strategy $\mathbf{s}_i \in S_i$ of Player i is *completely mixed* if all components of vector $\mathbf{s}_i$ are positive, i.e. $\mathbf{s}_i \in IntS_i$. A strategy profile $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_N) \in S = \prod_{i=1}^N S_i$ is *completely mixed* if all strategies $\mathbf{s}_i$, $i = 1, \ldots, N$, are completely mixed, i.e. $\mathbf{s} \in IntS$.

Our main result is as follows:

Theorem 1. *The vector space $\mathcal{H}(S)$ of finite non-cooperative N person games on S is maximal in $\boldsymbol{Eq}(S)$ iff the inequality (2) holds.*

For two-person games this means the equality $m_1 = m_2$.

Proof. Necessity. Let inequality (2) be not satisfied. It means that there exists a number $1 \leq i_0 \leq N$ such that

$$(m_{i_0} - 1) > \sum_{i \neq i_0}(m_i - 1).$$

In this case any finite non-cooperative $m_1 \times \ldots \times m_N$-game $H \in \mathcal{H}(S)$ has an equilibrium point $\mathbf{s}^* = (\mathbf{s}_1^*, \ldots, \mathbf{s}_N^*)$ such that the component $\mathbf{s}_{i_0}$ belongs to the boundary of the simplex S_{i_0} (see [5] or [6]). Hence the condition of Lemma 1 is satisfied and $\mathcal{H}(S)$ is not maximal in $\mathbf{Eq}(S)$.

Sufficiency. Let inequality (2) hold. Then for any completely mixed strategy profile $\mathbf{s} \in IntS$ there exists a game in $\mathcal{H}(S)$ which has a unique equilibrium point $\mathbf{s}$ (see [5]).

Let K be a continuous non-cooperative game on S such that for any game $H \in \mathcal{H}(S)$

and for any $\beta \in \mathbb{R}^1$ the game $H + \beta K$ has an equilibrium point. Then the conditions of Lemma 3 are satisfied both for game K and for game $-K$. Hence payoff functions K_i and $-K_i$ are concave (i.e. the subgraph is convex) on $\mathbf{s}_i$. So we get that functions K_i, $i = 1, \ldots, N$ are linear on $\mathbf{s}_i$. It means that the vector space $\mathcal{H}(S)$ of finite non-cooperative $m_1 \times \ldots \times m_N$-games is maximal in the set $\mathbf{Eq}(S)$ of all vector spaces of continuous games on S with equilibrium points.

□

Remark 1. The result of Theorem 1 remains true if we restrict ourselves to games with infinitely differentiable payoff functions.

Remark 2. Let $\mathcal{H}_0(S)$, $S = S_1 \times S_2$, be the vector space of $m_1 \times m_2$-matrix games in mixed strategies. As in the case of bimatrix games, the equality $m_1 = m_2$ is still the criterion of maximality of $\mathcal{H}_0(S)$ in the set $Eq_0(S)$ of vector spaces of zero-sum continuous (infinitely differentiable) games on S having saddle points.

Remark 3. Compactness is the only property of a simplex used in the proofs of Theorem 1 and of the mentioned result from [5]. Thus, the result of Theorem 1 is true for a vector space $\mathcal{H}(X)$ of games on a product $X = \prod_{i=1}^{N} X_i$ of convex compact subsets $X_i \subset \mathbb{R}_+^{m_i - 1}$ such that the payoff function of a player is linear on the set of strategies of any player.

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Appendix.

Proof of Lemma 1. Let fix i_1, $1 \leq i_1 \leq N$, $i_1 \neq i_0$. Consider a non-cooperative game K on X with the following payoff functions

$$K_i(\mathbf{x}) \equiv 0, \quad \textbf{if } i \neq i_1; \quad K_{i_1}(\mathbf{x}) = \varphi(\mathbf{x}_{i_0}) \cdot \psi(\mathbf{x}_{i_1}), \tag{3}$$

where $\mathbf{x} \in X$, ψ is a continuous non-linear function on X_{i_1}, φ is a continuous function on X_{i_0}, such that $\varphi(\mathbf{x}_{i_0}) = 0$, if $\mathbf{x}_{i_0} \in A$ and φ is not identically equal to zero on X_{i_0}.

Here is the simplest example of the set $A \subset X_{i_0}$ satisfying Lemma's condition and of the function K_{i_1} on X described above. Let $N = 2$ and

$$X_1 = \{x = (x_1, x_2) : x_1 + x_2 = 1,\ x_1, x_2 \geq 0\};$$
$$X_2 = \{y = (y_1, y_2, y_3) : y_1 + y_2 + y_3 = 1,\ y_i \geq 0, i = 1, 2, 3\}.$$

Let A be equal to the boundary of the simplex X_2. It is easy to see that the set A satisfies the lemma's condition. Thus we get $i_0 = 2$ and $i_1 = 1$. We may define function K_1 on $X = X_1 \times X_2$ using formula (3) where

$$\varphi(y) = y_1 \cdot y_2 \cdot y_3 \quad \textbf{and} \quad \psi(x) = (x_1 + x_2)^2.$$

Now come back to the Lemma 1's proof. Note that $\varphi(\mathbf{x}^*_{i_0}) = 0$ where $\mathbf{x}^*$ is an equilibrium point of a game $H \in \mathbf{H}(X)$ such that $\mathbf{x}^*_{i_0} \in A$. By the lemma such equilibrium point exists for any game $H \in \mathbf{H}(X)$.

Consider the vector space $\mathbf{H}'(X) \supset \mathbf{H}(X)$ of non-cooperative games on X of the form $H + \beta K$, where game $H \in \mathbf{H}(X))$, $\beta \in \mathbb{R}^1$ and the non-cooperative game K on X is described above.

It is easy to see that for any $\beta \in \mathbb{R}^1$ the equilibrium point $\mathbf{x}^*$ of a game $H \in \mathbf{H}(X)$ such that $\mathbf{x}^*_{i_0} \in A$ is an equilibrium point of the game $H + \beta K$ on X as well. Thus, $\mathbf{H}'(X) \in \mathbf{Eq}(X)$.

As function ψ is non linear on X_{i_1}, the set of functions maximized by Player i_1 in games in $\mathbf{H}'(X)$ is essentially broader than the set of functions maximized by him in games in $\mathbf{H}(X)$. Hence, the vector space $\mathbf{H}(X)$ is not maximal in $\mathbf{Eq}(X)$.

□

Proof of Lemma 2. As function $H_i(\mathbf{x}_i, \bar{\mathbf{x}}^i) + \beta K_i^{\mathbf{a},\mathbf{b}}(\mathbf{x}_i, \bar{\mathbf{x}}^i)$ depends linearly on $\mathbf{x}_i$ over interval $[\mathbf{a}, \mathbf{b}]$ and $\bar{\mathbf{x}}_i \in (\mathbf{a}, \mathbf{b})$, we get

$$H_i(\bar{\mathbf{x}}) + \beta K_i^{\mathbf{a},\mathbf{b}}(\bar{\mathbf{x}}) \leq \max_{\mathbf{x}_i \in [\mathbf{a},\mathbf{b}]} (H_i(\mathbf{x}_i, \bar{\mathbf{x}}^i) + \beta K_i^{\mathbf{a},\mathbf{b}}(\mathbf{x}_i, \bar{\mathbf{x}}^i)) =$$
$$\max(H_i(\mathbf{a}, \bar{\mathbf{x}}^i) + \beta K_i^{\mathbf{a},\mathbf{b}}(\mathbf{a}, \bar{\mathbf{x}}^i), H_i(\mathbf{b}, \bar{\mathbf{x}}^i) + \beta K_i^{\mathbf{a},\mathbf{b}}(\mathbf{b}, \bar{\mathbf{x}}^i)).$$

But as functions K_i and $K_i^{\mathbf{a},\mathbf{b}}$ coincide at points $\mathbf{a}$ and $\mathbf{b}$,

$$H_i(\bar{\mathbf{x}}) + \beta K_i^{\mathbf{a},\mathbf{b}}(\bar{\mathbf{x}}) \leq \max(H_i(\mathbf{a}, \bar{\mathbf{x}}^i) + \beta K_i(\mathbf{a}, \bar{\mathbf{x}}^i), H_i(\mathbf{b}, \bar{\mathbf{x}}^i) + \beta K_i(\mathbf{b}, \bar{\mathbf{x}}^i)) \leq$$
$$\leq \max_{\mathbf{x}_i \in [\mathbf{a},\mathbf{b}]} (H_i(\mathbf{x}_i, \bar{\mathbf{x}}^i) + \beta K_i(\mathbf{x}_i, \bar{\mathbf{x}}^i)).$$

Since $\bar{\mathbf{x}}$ is an equilibrium point of the game $H + \beta K$, the last maximum is obtained at the point $\bar{\mathbf{x}}_i$. It proves the required inequality.

□

Proof of Lemma 3. The lemma is proved by contradiction. Assume that the hypothesis of the lemma is satisfied. Suppose that for a certain number i there exists a strategy profile $\bar{\mathbf{x}}^i \in X^i$ of all players but Player i such that function $K_i(\mathbf{x}_i, \bar{\mathbf{x}}^i)$ is not concave as a function of $\mathbf{x}_i$. In view of continuity of function K_i on all variables it can believed that $\bar{\mathbf{x}}^i \in IntX^i$. So there exists an interval $(\mathbf{a}, \mathbf{b}) \subset IntX_i$ such that $\bar{\mathbf{x}}_i \in (\mathbf{a}, \mathbf{b})$ and the following inequality holds

$$K_i(\bar{\mathbf{x}}_i, \bar{\mathbf{x}}^i) < K_i^{\mathbf{a},\mathbf{b}}(\bar{\mathbf{x}}_i, \bar{\mathbf{x}}^i),$$

function $K_i^{\mathbf{a},\mathbf{b}}$ over $[\mathbf{a}, \mathbf{b}] \times X^i$ is defined before Lemma 2.

Using continuity of function K_i once again we get that there exists an ε-neighborhood of the point $\bar{\mathbf{x}} = (\bar{\mathbf{x}}_i, \bar{\mathbf{x}}^i)$, $\varepsilon(\bar{\mathbf{x}}) \subset IntX$, such that for all $\mathbf{x} = (\mathbf{x}_i, \mathbf{x}^i) \in \varepsilon(\bar{\mathbf{x}})$ the inequality

$$K_i(\mathbf{x}_i, \mathbf{x}^i) < K_i^{\mathbf{a},\mathbf{b}(\mathbf{x}_i)}(\mathbf{x}_i, \mathbf{x}^i), \tag{4}$$

holds where

$$\mathbf{b}(\mathbf{x}_i) \in IntX_i \quad \textbf{and} \quad \mathbf{x}_i \in (\mathbf{a}, \mathbf{b}(\mathbf{x}_i)).$$

By Lemma's condition there exists game $H \in \mathbf{H}(X)$ with the unique equilibrium point $\bar{\mathbf{x}}$. If positive β sufficiently small then an equilibrium point of game $H + \beta K$, which exists by Lemma's condition, belongs to $\varepsilon(\bar{\mathbf{x}})$. Denote $\mathbf{x}^*$ this point. By inequality (4) we get that for $\beta > 0$

$$\beta K_i(\mathbf{x}^*) < \beta K_i^{\mathbf{a},\mathbf{b}(\mathbf{x}_i^*)}(\mathbf{x}^*).$$

On the other hand as the conditions of Lemma 2 are satisfied the contrary inequality holds as well. Hence our supposition is not true and payoff functions functions K_i, $i = 1, \ldots, N$, are concave in the Player i's strategy.

□

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