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WORST-CASE APPROACH TO STRATEGIC OPTIMAL PORTFOLIO SELECTION UNDER TRANSACTION COSTS AND TRADING LIMITS

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WORST-CASE APPROACH TO STRATEGIC OPTIMAL PORTFOLIO SELECTION UNDER TRANSACTION COSTS AND TRADING LIMITS [2]

We study a worst-case scenario approach to the stochastic dynamic programming problem, presenting a general probability-based framework and some properties of the arising Bellman-Isaacs equation which allow to obtain a closed-form analytic solution. We also adapt the results for a discrete financial market and the problem of strategic portfolio selection in the presence of transaction costs and trading limits with unspecified stochastic process of market parameters. Unlike the classic stochastic programming, the approach is model-free while the solution can be easily found numerically under economically reasonable assumptions. All results hold for a general class of utility functions and several risky assets. For a special case of proportional transaction costs and CRRA utility, we present a numerical scheme which allows to reduce the dimensionality of the Bellman-Isaacs equation by a number of risky assets.

JEL Classification: C61, C63, G11.

Key words: portfolio selection, Bellman-Isaacs equation, stochastic dynamic programming, transaction costs, worst-case optimization.

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Introduction

Stochastic programming approach to optimal portfolio selection is widely used in academic literature since the pioneering works of Merton [Merton, 1969] and Samuelson [Samuelson, 1969] who studied the problem for discrete and continuous time in its simplest form (multi-asset portfolio on a zero-cost market without price impact). Continuous time strategy modeling usually attracts more interest due to the possibility of a closed-form analytic solution of the Hamilton-Jacobi-Bellman equation (or the quasi-variational inequality). The model assumes that m-dimensional price process X is the geometric Brownian motion with SDE

$$dX_t^i = \mu_t^i X_t^i dt + X_t^i \sum_{j=1}^{m} \sigma_t^{ij} dw_t^j, \quad i = \overline{1, m}, \tag{1}$$

where w_t is a Wiener process. Dynamics of the risk-free asset Y is described by SDE

$$dY_t = rY_t dt \tag{2}$$

where r is a risk-free rate. Problem is solved for isoelasic (CRRA) utility, the solution is to keep a constant part of total portfolio wealth in each risky asset (the so called "Merton line" for single asset).

The approach has been extended in various studies: for example, Richard [Richard, 1979] generalizes results to multi-dimensional Markovian price process; Karatzas et al. [Karatzas et al., 1986] solve the problem for HARA utility; Shreve & Xu [Xu and Shreve, 1992] use the dual approach to solve the problem with phase constraints (no short-selling).

Extensive research has been conducted recently for market with transaction costs and price impact. The most notable works in this area are [Davis and Norman, 1990], [Shreve and Soner, 1994], [Ø ksendal and Sulem, 2010] for proportional transaction costs.

The above-mentioned continuous control framework does not allow for a fixed fee per deal. Zakamouline in [Zakamouline, 2002], [Zakamouline, 2005] considers both fixed and proportional costs while maximizing portfolio terminal value over impulse control strategies. Numerical procedure for finding the solution is also presented for CARA-utility. [Vath et al., 2007] presents characteristics of the solution for price-dependent costs function and permanent price impact. Impulse solution for general-shaped concave costs function is studied in [Ma et al., 2013]. Overall, there are very few studies considering both transaction costs and trading limits (phase constraints).

The work by Bertsimas & Lo [Bertsimas and Lo, 1998] drew attention to the problem of optimal liquidation, i. e. optimal selection problem with a boundary condition. It has been researched in a series of works by Almgren & Chriss [Almgren and Chriss, 1999], [Almgren and Chriss, 2001], [Almgren, 2003], [Lorenz and Almgren, 2011], [Almgren, 2012] for discrete time, which consider various models of price impact and the Markowitz approach to define the optimal criterion through the risk-aversion of the portfolio manager. Further extension of the framework can be found, for example, in [Andreev et al., 2011] for a cubic polinomial costs function with stochastic coefficients. Still, most of the research in this area is centered around continuous time market where prices follow the geometric Brownian motion. See, for example, [Schied and Schöneborn, 2009], [Schöneborn, 2011], [Predoiu et al., 2011], [Fruth et al., 2013], [Obizhaeva and Wang, 2013].

In this paper we consider a discrete time market and present the worst-case approach to optimal selection problem. The developed framework does not require specification of the stochastic dynamics of the system. Instead, basic properties of the parameter conditional distribution must be specified, such as expectation and range, for the considered time period (characteristics can be estimated from statistics or by an expert). Optimal strategy is assumed to maximize the worst-case expected value of the general-shaped terminal utility function. The approach admits transaction costs and phase constraints while being oriented for practical use as a decision support system (DSS) during an investment management process. Similar approach, in game-theory terms, was studied in [Deng et al., 2005] for one-period problem and Markowitz optimal criterion without transaction costs. The worst-case framework has also been recently presented by Chigodaev [Chigodaev, 2016] for the option super-hedging problem on a zero-cost market with one risky asset, based on the ideas of Prof. Smirnov introduced at the course of lectures at the Moscow State University. Our framework closely relates to the stochastic differential games theory, but instead we provide a probabilistic interpretation through a set of probability models of the market.

The first chapter describes the framework and states the results for the general dynamic system and expected utility optimization problem. We obtain the sufficient conditions to simplify the arising Bellman-Isaacs equation and study properties of the value function. The second chapter concentrates on the market and the portfolio selection problem: we provide sufficient conditions for the closed-form solution of the Bellman-Isaacs equation for markets with and without transaction costs. The third chapter describes numerical procedure for finding the solution for the particular case of linear costs, which allows reducing dimensionality of the problem. The fourth chapter presents results for the modeled data, the fifth chapter concludes.

1 General framework and dynamic programming principle

To provide some insight, consider a filtered probability space satisfying the usual conditions, which represents a probability model of the system (i. e. the financial market). The usual assumption when solving the optimal control problem is that the dynamics of all the system parameters can be expressed as a trajectory of a known stochastic process Θ_t, $t \in T$, on a measurable space. In this paper we try to solve the problem under weaker assumptions that Θ_t is not explicitly known, though its values belong to a known compact set K_t. We will also assume that the expected values E_t, $t \in T$, of the system parameters are known as well. The importance of specifying E_t will be demonstrated later for a particular optimization problem. As for application in finance, certain expectations about the market dynamics are necessary for any reasonable investment strategy and are provided by the analyst of the company.

To understand the reasoning behind the proposed framework, imagine that Θ_t belongs to a known set of candidate processes and all information concerning the system is contained in Θ_t. The induced probability space can be associated with each of the candidates Θ, let $\mathbb{F}^\Theta$ be the natural filtration and $\mathbb{P}(\Theta^{-1}(A))$ be the induced probability measure. When all the candidate processes are assumed to have the same range of values, $\mathbb{F}^\Theta$ is common for all of them. We plan to define the optimization criterion in the mean sense, so it is easier to describe the framework in terms of a collection of probability measures rather than the candidate processes themselves.

For any space T, consider a collection of measurable spaces $\big\{(K_t, \mathcal{B}_t)\big\}_{t\in T}$, each with a Hausdorff topology, where for all t, K_t is a convex compact set in $\mathbb{R}^l$. Consider a space of $\mathbb{R}^l$-valued functions $\mathbf{X}^T = \big\{\omega(t) \colon T \to \mathbb{R}^l \colon \omega(s) \in K_s \;\; \forall s \in T\big\}$ and a cylindrical σ-algebra $\mathcal{F}$ on $\mathbf{X}^T$, let the sample space $\Omega = \mathbf{X}^T$. Let $\{Q_S\}$ be a collection of inner regular and compatible measures Q_S (in terms of the Kolmogorov existence theorem [Tao, 2011, p. 195]) on $K_S = \prod\limits_{t\in S} K_t$ for each finite subset $S \subseteq T$. According to the Kolmogorov theorem, there is a unique measure Q on Ω such that $Q_S = (\pi_S^T)_* Q$ — a pushforward measure induced by a canonical projection map π for every finite $S \subset T$. Hence the probability space $(\Omega, \mathcal{F}, Q)$ is well-defined for each collection $\{Q_S\}$. Let $\mathbb{Q}$ be the class of all such measures Q. In this paper we consider a discrete time system and a finite-horizon problem, so

$$T = \{t_0, \ldots, t_N\}.$$

(To avoid trivial cases in the proofs, assume $N \geq 2$, though the main results hold for $N = 1$ as well.) Our main idea is to find a solution to the expected functional maximization problem in the worst possible case among all $Q \in \mathbb{Q}$ that produce the given expected values of parameters.

Let $\mathcal{S}_n$ be the state of the system at t_n, and $\mathcal{S}_n \mid H_n$ be the state where the control at t_n is H_n[3]. Optimal strategy will later be defined as an admissible strategy H^{X^*} which provides supremum to the functional $\inf\limits_{Q \in \mathbb{Q}} \mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N^X)$. The proposed optimal control problem is usually solved via the dynamic programming approach. The main difficulty arises for finding an extreme measure in $\mathbb{Q}$. Any numerical implementation would require either a feasible parametrization of the measure set or some analytic results. Moreover, the state $\mathcal{S}$ must be passed as an argument of the value function, so a complex system dynamics would require keeping all the history in the state variable which will greatly increase computational difficulty. Throughout the paper we assume that the process of parameters is Markov, so the system state can be reduced to the current state.

For a probability model $(\Omega, \mathcal{F}, Q)$ consider a family of σ-algebras $\mathcal{F}_t^s \subseteq \mathcal{F}$, $s, t \in T$, $s < t$, such that

$$\mathcal{F}_t^s \subset \mathcal{F}_v^u \qquad \forall\, u \leq s < t \leq v.$$

Definition 1. The *parameter* process is a family of functions

$$\Theta_s(t) = \Theta_s(t, \omega) = \omega(t), \quad s, t \in T, \quad t > s.$$

Note that for all $s < t$, $\Theta_s(t, \cdot) : \Omega \to K_t$ is measurable with respect to $\mathcal{F}_t^s$. For each $C \in \mathcal{F}$ we let $\Theta_s(\cdot, \omega) \in C$ if there is such $\bar{\omega}(t) \in \Omega$ that $\omega(t) = \bar{\omega(t)}$ for all $t \geq s$ and

[3]Formally, $\mathcal{S}_n = \mathcal{S}_n(\omega, H(\omega))$, hence $\mathcal{S}_n \mid H_n^* = \mathcal{S}_n(\omega, H_n^*(\omega))$.Note that, unconditionally, $\mathcal{S}_n$ might depend on all the parameter values and the control history.

$\bar{\omega(t)} \in C$ (see [Skorokhod, 1989, p. 150]). Define a subset $\mathbb{Q}^E$ of $\mathbb{Q}$ as

$$\mathbb{Q}^E = \left\{ Q \in \mathbb{Q} \colon \mathbb{E}_Q^{\Theta_s} \Theta_t = E_t \quad \forall s, t \in T, \quad s < t \right\}.$$

For each $Q \in \mathbb{Q}^E$ define the Markov kernel (the transition probability function)

$$P(s, x, t, A) = Q\big\{\Theta_s(t) \in A \mid \Theta_s(s) = x\big\} = P_{s,x}\big\{\Theta_t \in A\big\} \tag{3}$$

on $T \times K_s \times T \times \mathcal{B}_t$, which satisfy the following conditions:

I) for all $s, t \in T$, $s \leq t$, and all $A \in \mathcal{B}_t$, $P(s, \cdot, t, A)$ is measurable on K_s;

II) for all $s, t \in T$, $s \leq t$, and all $x \in K_s$, $P(s, x, t, \cdot)$ is a probability measure on $\mathcal{B}_t$;

III) the Chapman-Kolmogorov equation holds:

$$P(s, x, t, A) = \int P(s, x, u, dz) P(u, z, t, A) \qquad s < u < t, \quad x \in K_s, \quad A \in \mathcal{B}_t. \tag{4}$$

By definition, $P_{s,x}\big\{\Theta_s(s) = x\big\} = 1$. The space of Markov kernels induced by $\mathbb{Q}^E$ will be denoted $\mathcal{P}(\mathbb{Q}^E)$.

Let $\mathcal{P}^E$ be the space of functions $P(s, x, t, A) \colon T \times K_s \times T \times \mathcal{B}_t \to [0; 1]$ for which I)-III) are true,

$$\int z P(s, x, t, dz) = E_t \qquad s < t, \quad x \in K_s, \tag{5}$$

and

$$P(s, x, s, \{x\}) \equiv 1. \tag{6}$$

We would make an extensive use of measures on subsets S of T, so the following notation is used as the most convenient and descriptive:

$$\mathbb{Q}_S^E = \big\{ (\pi_S^T)_* Q, \quad Q \in \mathbb{Q}^E \big\}, \qquad \mathbb{Q}_{\geq n}^E \overset{df}{=} \mathbb{Q}_{[t_n, t_N]}^E, \qquad \mathbb{Q}_n^E \overset{df}{=} \mathbb{Q}_{\{t_n\}},$$

$$\mathcal{P}^E(\mathbb{Q}_S^E) = \Big\{ Q_S\big\{\Theta_s(t) \in A \mid \Theta_s(s) = x\big\}, \quad Q_S \in \mathbb{Q}_S \Big\},$$

$\mathcal{P}_S^E$ and $\mathcal{P}_{\geq n}^E$ are defined by analogy. For any state $\mathcal{S}_n$ define a set of conditional measures

$$\mathbb{Q}_S^E \mid \mathcal{S}_n = \big\{ Q\{\cdot \mid \mathcal{S}_n\}, \quad Q \in \mathbb{Q}_S^E \big\}.$$

Lemma 1. *For any $S \subseteq T$, $\mathcal{P}_S^E = \mathcal{P}(\mathbb{Q}_S^E)$.*

Proof. $\mathcal{P}(\mathbb{Q}_S^E) \subseteq \mathcal{P}_S^E$ by definition, so we only need to prove that for any $P \in \mathcal{P}_S^E$ there is a $Q \in \mathbb{Q}_S^E$, generating P in accordance with (3). Indeed, it can be shown [Skorokhod, 1989] that one can construct a family of measures

$$P_{s,x}\big\{\Theta_s(s_1) \in A_1, \dots, \Theta_s(s_n) \in A_n\big\} \overset{P_{s,x}\text{-a.s.}}{=} \\ = \int_{A_1} P(s, x, s_1, dx_1) \int_{A_2} P(s_1, x_1, s_2, dx_2) \dots \int_{A_n} P(s_{n-1}, x_{n-1}, s_n, dx_n), \\ s \le s_1 \le \dots \le s_n, \quad s, s_1, \dots, s_n \in S. \tag{7}$$

For $\tau = \inf S$, let $\mu_\tau(A)$ be any probability measure on $(K_\tau, \mathcal{B}_\tau)$. Then we can construct Q as

$$Q\big\{\omega(s_1) \in A_1, \dots, \omega(s_n) \in A_n\big\} = \int \mu_\tau(dx) P_{\tau,x}\big\{\Theta_\tau(s_1) \in A_1, \dots, \Theta_\tau(s_n) \in A_n\big\}. \tag{8}$$

It can be easily seen that for $s \le s_1$, $s \in S$,

$$Q\big\{\omega(s_1) \in A_1, \dots, \omega(s_n) \in A_n \mid \omega(s) = x\big\} \overset{Q\text{-a.s.}}{=} P_{s,x}\big\{\Theta_s(s_1) \in A_1, \dots, \Theta_s(s_n) \in A_n\big\}. \tag{9}$$

Thus, $\mathbb{E}_Q^{\Theta_s} \Theta_t = \int z P(s, \Theta_s, t, dz) = E_t$ and $Q \in \mathbb{Q}_S^E$. Therefore, $\mathcal{P}_S^E \subseteq \mathcal{P}(\mathbb{Q}_S^E)$. $\square$

Lemma 1 shows that by generating Markov kernels from the set of considered measures, we obtain all possible Markov kernels with the required properties.

Corollary 1. *For any $S \subseteq T$, $\mathbb{Q}_S^E$ is isomorphic to $\mathbb{Q}_{\{\tau\}}^E \otimes \mathcal{P}_S^E$ where $\tau = \inf S$.*

Proof. Consider the map $f \colon \mathbb{Q}_{\{\tau\}}^E \otimes \mathcal{P}_S^E \to \mathbb{Q}_S^E$ given by (8), and the map $g \colon \mathbb{Q}_S^E \to \mathbb{Q}_{\{\tau\}}^E \otimes \mathcal{P}_S^E$ where $g(Q) = (\mu_\tau, P)$,

$$\mu_\tau(A) = Q\left\{\omega(\tau) \in A\right\},$$

$$P(s, x, t, A) = Q\big\{\omega(t) \in A \mid \omega(s) = x\big\}.$$

Since

$$
\begin{aligned}
&\big[f(g(Q))\big]\big\{\omega(s_1) \in A_1, \dots, \omega(s_n) \in A_n\big\} = \\
&\quad = \int \mu_\tau(dx) \int\limits_{A_1} P(\tau, x, s_1, dx_1) \int\limits_{A_2} P(s_1, x_1, s_2, dx_2) \dots \int\limits_{A_n} P(s_{n-1}, x_{n-1}, s_n, dx_n) = \\
&\quad = \int Q\big\{d\omega(\tau)\big\} \int\limits_{A_1} Q\big\{d\omega(s_1) \mid \mathcal{S}_\tau\big\} \int\limits_{A_2} Q\big\{d\omega(s_2) \mid \mathcal{S}_{s_1}\big\} \dots \int\limits_{A_n} Q\big\{d\omega(s_n) \mid \mathcal{S}_{s_{n-1}}\big\} = \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = Q\big\{\omega(s_1) \in A_1, \dots, \omega(s_n) \in A_n\big\}
\end{aligned}
$$

for every measurable $A_1, \dots, A_n$, then $g = f^{-1}$ and f is a bijective map. $\square$

Corollary 2. *For any $S \subseteq T$ such that $[t_{n+1}, t_N] \cap S \neq \emptyset$ where $n = \overline{0, N-1}$, $\mathbb{Q}_S^E \mid \mathcal{S}_n$ is isomorphic to $\mathcal{P}^E_{\{t_n\} \cup ([t_{n+1}, t_N] \cap S)}$.*

Proof. The proof will be provided for $t_{n+1} \leq \inf S$, in other cases it can be conducted by analogy. Consider the map $f\colon \mathcal{P}^E_{\{t_n\} \,\cup\, ([t_{n+1}, t_N] \cap S)} \to \mathbb{Q}_S^E \mid \mathcal{S}_n$, given by (9), and the map $g\colon \mathbb{Q}_S^E \mid \mathcal{S}_n \to \mathcal{P}^E_{\{t_n\} \cup ([t_{n+1}, t_N] \cap S)}$ defined as $P(s, x, t, A) = Q\big\{\omega(t) \in A \mid \omega(s) = x\big\}$. Since

$$
\begin{aligned}
&\big[f(g(Q))\big]\big\{\omega(s_1) \in A_1, \dots, \omega(s_n) \in A_n \mid \mathcal{S}_n\big\} = \\
&\qquad = \int\limits_{A_1} P(t_n, \Theta_n, s_1, dx_1) \int\limits_{A_2} P(s_1, x_1, s_2, dx_2) \dots \int\limits_{A_n} P(s_{n-1}, x_{n-1}, s_n, dx_n) = \\
&\qquad = \int\limits_{A_1} Q\big\{d\omega(s_1) \mid \mathcal{S}_n\big\} \int\limits_{A_2} Q\big\{d\omega(s_2) \mid \mathcal{S}_{s_1}\big\} \dots \int\limits_{A_n} Q\big\{d\omega(s_n) \mid \mathcal{S}_{s_{n-1}}\big\} = \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = Q\big\{\omega(s_1) \in A_1, \dots, \omega(s_n) \in A_n\big\}
\end{aligned}
$$

for every measurable $A_1, \dots, A_n$, then $g = f^{-1}$ and we have constructed a bijective map. $\square$

1.1 The Bellman-Isaacs equation

To derive the Bellman-Isaacs equation, we, in layman's terms, need to be able to decompose the infimum over $\mathbb{Q}^E_{\geq n}$ into infima over $\mathbb{Q}^E_n$ and $\mathbb{Q}^E_{\geq n+1}$.

Lemma 2. *For any $u \in S \subseteq T$ such that both $[t_0; u] \cap S$ and $[u; t_N] \cap S$ contain at least two points, $\mathcal{P}^E_S$ is isomorphic to $\mathcal{P}^E_{[t_0;u] \cap S} \otimes \mathcal{P}^E_{[u;t_N] \cap S}$.*

Proof. To construct a bijective map, we show that for any $P_1^* \in \mathcal{P}^E_{[t_0;u] \cap S}$ and $P_2^* \in \mathcal{P}^E_{[u;t_N] \cap S}$, there is $P \in \mathcal{P}_S^E$:

$$P(s,x,t,A) \equiv \begin{cases} P_1^*(s,x,t,A), & s,t \in [t_0,u] \cap S, \\ P_2^*(s,x,t,A), & s,t \in [u,t_N] \cap S \end{cases}$$

Firstly, note that $P(u,x,u,A)$ is defined correctly since

$$P(u,x,u,A) = P_1^*(u,x,u,A) = P_2^*(u,x,u,A) = \mathbb{I}\{x \in A\}.$$

For $s < u < t$ define

$$P(s,x,t,A) = \int P_1^*(s,x,u,dz) P_2^*(u,z,t,A).$$

The Markov kernel composition belongs to $\mathcal{P}_S^E$ which can be shown by verifying properties I)-III) and (5) for $s < u < t$. II) and III) are obvious, (5) is implied by the Fubini-Tonelli theorem and by property (5) of P_1^*:

$$E_t = E_t \int\limits_{K_u} P_1^*(s,x,u,dz) = \int\limits_{K_u} E_t P_1^*(s,x,u,dz) \overset{(5)}{=} \int\limits_{K_u} \int\limits_{K_t} y P_2^*(u,z,t,dy) P_1^*(s,x,u,dz) =$$
$$= \int\limits_{K_t} y \int\limits_{K_u} P_1^*(s,x,u,dz) P_2^*(u,z,t,dy) = \int\limits_{K_t} y P(s,x,t,dy).$$

To prove property I), we use the fact that a function is measurable if and only if it is a limit of a uniformly converging sequence of simple functions [Kolmogorov and Fomin, 1999]. Since $P_1^*(s,x,u,A)$ and $P_2^*(u,x,t,A)$ are x-measurable,

$$P_1^*(s,x,u,A) = \lim_{N \to +\infty} \sum_{j=1}^{+\infty} c_{N,j}(s,u,A) \mathbb{I}\big\{x \in C_{N,j}(s,u,A)\big\}, \quad \bigcup_j C_{N,j}(s,u,A) = K_s,$$

$$P_2^*(u,x,t,A) = \lim_{M \to +\infty} \sum_{i=1}^{+\infty} b_{M,i}(u,t,A) \mathbb{I}\big\{x \in B_{M,i}(u,t,A)\big\}, \quad \bigcup_i B_{M,i}(u,t,A) = K_u,$$

where all $C_{N,j}$ and $B_{M,i}$ are measurable sets. Then

$$P(s,x,t,A) = \lim_{M \to +\infty} \sum_{i=1}^{+\infty} b_{M,i}(u,t,A) P_1^*\left(s,x,u,B_{M,i}(u,t,A)\right) =$$

$$= \lim_{M\to+\infty} \sum_{i=1}^{+\infty} b_{M,i}(u,t,A) \lim_{N\to+\infty} \sum_{j=1}^{+\infty} c_{N,j}\left(s,u,B_{M,i}(u,t,A)\right) \mathbb{I}\big\{x \in C_{N,j}\left(s,u,B_{M,i}(u,t,A)\right)\big\} =$$

$$= \lim_{M\to+\infty} \lim_{N\to+\infty} \sum_{j=1}^{+\infty} \sum_{i=1}^{+\infty} b_{M,i}(u,t,A) c_{N,j}\left(s,u,B_{M,i}(u,t,A)\right) \mathbb{I}\big\{x \in C_{N,j}\left(s,u,B_{M,i}(u,t,A)\right)\big\}.$$

Since the values of products $b_{M,i}(u,t,A)c_{N,j}$ can coincide, let $\{a_l\}_{l=1}^{+\infty}$ be a set of various values among all i and j, so that

$$a_k = b_{M,i_{k,1}}(u,t,A)c_{N,j_{k,1}} = \ldots = b_{M,i_{k,l_k}}(u,t,A)c_{N,j_{k,l_k}}.$$

Then

$$P(s,x,t,A) = \lim_{\substack{M\to+\infty \\ N\to+\infty}} \sum_{k=1}^{+\infty} a_k \mathbb{I}\left\{x \in \bigcup_{p=1}^{l_k} C_{N,j_{k,p}}\left(s,u,B_{M,i_{k,p}}(u,t,A)\right)\right\} = \lim_{n\to+\infty} f_n(x;s,u,t,A),$$

where f_n is a simple function since all the sets $\big\{f_n(x) = a_k\big\}$ are measurable as unions of measurable sets. Thus, property I) for P holds true. The provided map $f\colon \mathcal{P}^E_{[t_0;u]\cap S} \otimes \mathcal{P}^E_{[u;t_N]\cap S} \to \mathcal{P}^E_S$ can be obviously reversed and proven bijective which proves the main statement. $\square$

Statement 1. *For any $n = \overline{1,N}$, admissible state $\mathcal{S}_{n-1}$, measurable f and $\xi \in m(\mathcal{F}_k)$, $k \geq n$,*

$$\inf_{\xi\in m(\mathcal{F}_k)} \mathbb{E}^{\mathcal{S}_{n-1}} f(\xi) = \mathbb{E}^{\mathcal{S}_{n-1}} \inf_{\xi\in m(\mathcal{F}_k)} f(\xi), \tag{10}$$

granted the infima exist and the conditional expectations exist and finite.

Proof. 1) Let ξ^* belong to the a. s. infimum of f: $f(\xi^*) \overset{\text{a. s.}}{=} \inf\limits_{\xi\in m(\mathcal{F}_k)} f(\xi)$ then

$$\mathbb{E}^{\mathcal{S}_{n-1}} f(\xi^*) \leq \mathbb{E}^{\mathcal{S}_{n-1}} f(\xi),$$

therefore

$$\mathbb{E}^{\mathcal{S}_{n-1}} \inf_{\xi\in m(\mathcal{F}_k)} f(\xi) \leq \inf_{\xi\in m(\mathcal{F}_k)} \mathbb{E}^{\mathcal{S}_{n-1}} f(\xi).$$

On the other hand, since $\xi^* \in m(\mathcal{F}_k)$,

$$\inf_{\xi\in m(\mathcal{F}_k)} \mathbb{E}^{\mathcal{S}_{n-1}} f(\xi) \leq \mathbb{E}^{\mathcal{S}_{n-1}} f(\xi^*) = \mathbb{E}^{\mathcal{S}_{n-1}} \inf_{\xi\in m(\mathcal{F}_k)} f(\xi),$$

hence the equality holds. $\square$

Remark: Statement 1 obviously holds for supremum instead of infimum (*mutatis mutandis*), so we will refer to the Statement in both cases.

Lemma 3. *Let $N \geq 2$, then for each $\xi \in m(\mathcal{F}_N)$ and each $n = \overline{1, N-1}$,*

$$\inf_{Q \in \mathbb{Q}^E_{\geq n}} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q} \xi = \inf_{Q_n \in \mathbb{Q}^E_n} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q_n} \inf_{Q_{\geq n+1} \in \mathbb{Q}^E_{\geq n+1}} \mathbb{E}^{\mathcal{S}_n}_{Q_{\geq n+1}} \xi.$$

Proof. Note that $\mathbb{Q}_{\geq n} \mid \mathcal{S}_{n-1}$ is isomorphic to $\mathcal{P}^E_{\{t_{n-1}, t_N\}}$ by virtue of Corollary 2 for $S = [t_n; t_N]$; $\mathcal{P}^E_{\{t_{n-1}, t_N\}}$ is isomorphic to $\mathcal{P}^E_{\{t_{n-1}, t_n\}} \otimes \mathcal{P}^E_{\geq n}$ by Lemma 2 for $S = [t_{n-1}; t_N]$ and $u = t_n$. Corollary 2, when applied for $S = \{t_n\}$, yields that $\mathcal{P}^E_{\{t_{n-1}, t_n\}}$ is isomorphic to $\mathbb{Q}_n \mid \mathcal{S}_{n-1}$; when applied for $S = [t_{n+1}; t_N]$, yields that $\mathcal{P}^E_{\geq n}$ is isomorphic to $\mathbb{Q}_{\geq n+1} \mid \mathcal{S}_n$. Thus, by transitivity, $\mathbb{Q}_{\geq n} \mid \mathcal{S}_{n-1}$ is isomorphic to $\mathbb{Q}_n \mid \mathcal{S}_{n-1} \otimes \mathbb{Q}_{\geq n+1} \mid \mathcal{S}_n$. Therefore

$$\inf_{Q \in \mathbb{Q}^E_{\geq n}} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q} \xi = \inf_{Q \in \mathbb{Q}^E_{\geq n} | \mathcal{S}_{n-1}} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q} \xi = \inf_{\mathbb{Q}_n | \mathcal{S}_{n-1}} \inf_{\mathbb{Q}_{\geq n+1} | \mathcal{S}_n} \mathbb{E}^{\mathcal{S}_{n-1}} \xi =$$

$$= \inf_{\mathbb{Q}_n | \mathcal{S}_{n-1}} \inf_{\mathbb{Q}_{\geq n+1} | \mathcal{S}_n} \mathbb{E}^{\mathcal{S}_{n-1}} \mathbb{E}^{\mathcal{S}_n} \xi \overset{(10)}{=} \inf_{Q_n \in \mathbb{Q}_n | \mathcal{S}_{n-1}} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q_n} \inf_{Q_{\geq n+1} \in \mathbb{Q}_{\geq n+1} | \mathcal{S}_n} \mathbb{E}^{\mathcal{S}_n}_{Q_{\geq n+1}} \xi.$$

$\square$

Lemma 3 can be interpreted as an equivalent of the semigroup property in proof of the dynamic programming principle.

Consider a utility function $J(S_N)$. J might depend solely on the portfolio structure — for example, in most classic approaches to portfolio optimization, the utility depends on the terminal market value of portfolio. J might also depend on the state of the whole system — consider the previous example when, instead of market value, the utility is based on liquidation value of the portfolio, hence co-depends on the transaction cost parameters.

By *strategy* we mean a sequence $H = \{H_n\}_{n=1}^N$ such that $H_n \in \mathbb{R}^{m+1}$ for each $n = \overline{1, N}$. As before, let $H_{\leq k} = \{H_n\}_{n=1}^k$, $H_{\geq k} = \{H_n\}_{n=k}^N$. We assume that $\mathcal{S}_0$ is observable initial state of the system (which includes H_0).

Definition 2. For a given collection of sets $D_n \in m(\mathcal{F}_{n-1})$, $D_n \subseteq \mathbb{R}^{m+1}$, $D_n \neq \emptyset$, *admissible strategy* is a strategy H such that for all $n = \overline{1, N}$

1. $H_n \in D_n$;

2. $H_n \in m(\mathcal{F}_{n-1})$;

3. $$Q\left\{H_n \in A_n \mid \mathcal{F}_{n-1}\right\} \overset{\text{a.-s.}}{=} Q\left\{H_n \in A_n \mid \mathcal{S}_{n-1}\right\} \qquad \forall A_n \in \mathcal{B}(D_n),\ \forall Q \in \mathbb{Q}^E$$

   (Markov control policy).

A set of all admissible strategies is denoted $\mathcal{A}$; for any $S \subseteq T$,

$$\mathcal{A}_S \mid \mathcal{S}_{n-1} = \left\{\{H_k\}_{S\cap[t_n;t_N]} \colon H \in \mathcal{A},\ H_{n-1} \in \mathcal{S}_{n-1}\right\},$$

while $\mathcal{A} \mid \mathcal{S}_{n-1} = \mathcal{A}_{[t_n;t_N]} \mid \mathcal{S}_{n-1}$.

Definition 3. *Optimal strategy* is a strategy $H^* \in \mathcal{A}$ such that

$$\inf_{Q\in\mathbb{Q}^E} \mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N^*) = \sup_{H\in\mathcal{A}} \inf_{Q\in\mathbb{Q}^E} \mathbb{E}_Q^{\mathcal{S}_0} J(\mathcal{S}_N \mid H_N). \tag{11}$$

Let $\mathbb{S}_n$ be the set of all system states at t_n for all admissible $H_{\leq n}$; $\mathbb{S}_n \mid H_{\leq n}$ be the set of all system states at t_n for the strategy $H_{\leq n}$; let $\mathbb{S}_n \mid H_n$ be a set of all system states at t_n when the control value is H_n. When necessary, for $\mathcal{S} \in \mathbb{S}_n \mid H_{\leq n}$ we will also use the notation $\mathcal{S} \mid H_{\leq n}$ for clarification.

Consider the following dynamic programming equation for the value function $V_n(\mathcal{S})$:

$$V_{n-1}(\mathcal{S}) = \sup_{H_n\in D_n(\mathcal{S})} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}} V_n(\mathcal{S}_n \mid H_n), \quad \mathcal{S} \in \mathbb{S}_{n-1}, \qquad n = \overline{1,N}, \tag{12}$$

$$V_N(\mathcal{S}) = J(\mathcal{S}), \quad \mathcal{S} \in \mathbb{S}_N. \tag{13}$$

Statement 2. *If $n = \overline{1,N}$, $\mathcal{S} \in \mathbb{S}_{n-1}$, $\xi \in m(\mathcal{F}_N)$ and function $f(\xi)$ is continuous and bounded, then $\mathbb{E}_Q^{\mathcal{S}}\ f(\xi)$ is continuous in Q over $\mathbb{Q}_{\geq n}$.*

Proof. By definition, $\mathbb{Q}_{\geq n+1} \mid \mathcal{S}$ consists of Borel measures defined on $\prod\limits_{i=n+1}^{N} K_i$ which can be supplied with a metric d. Consider the Prokhorov metric d_P on $\mathbb{Q}_{\geq n+1} \mid \mathcal{S}$. By its properties,

$$d_P(Q_i, Q) \longrightarrow 0 \qquad \Longrightarrow \qquad Q_i \Rightarrow Q, \qquad Q,\ Q_i \in \mathbb{Q}_{\geq n+1} \mid \mathcal{S},\ i \geq 1.$$

Given the continuity and boundedness of f, weak convergence of measures proves the statement. $\square$

Under specific conditions, the value of $V_n(\mathcal{S})$ is the optimal value of the worst-case expected utility when the initial moment is t_n and the initial state is $\mathcal{S}$. To prove this, a version of the minimax theorem is required. Since the initial version for concave-convex function over compact sets, the result has been generalized for more general cases. Here we use the theorems for two generalizations of convexity which require the compactness of only one set.

Definition 4. A function $f(x, y)$ on $M \times N$ is *quasi-concave* in M if $\{x\colon f(x, y) \geq c\}$ is a convex set for any $y \in N$ and real c.

Definition 5. A function $f(x, y)$ on $M \times N$ is *quasi-convex* in N if $\{y\colon f(x, y) \leq c\}$ is a convex set for any $x \in M$ and real c.

A function $f(x, y)$ on $M \times N$ is *quasi-concave-convex* if it is both quasi-concave in M and quasi-convex in N.

Theorem 1 ([Sion, 1958])**.** *Let M and N be convex spaces, one of which is compact, and $f(\mu, \nu)$ be a function on $M \times N$, quasi-concave-convex, upper semi-continuous in μ for each $\nu \in N$ and lower semi-continuous in ν for each $\mu \in M$. Then*

$$\sup_{\mu \in M} \inf_{\nu \in N} f(\mu, \nu) = \inf_{\nu \in N} \sup_{\mu \in M} f(\mu, \nu).$$

Lemma 4. *Let*

$$V_{n-1}(\mathcal{S}) = \sup_{H_{\geq n} \in \mathcal{A} | \mathcal{S}} \inf_{Q_{\geq n} \in \mathbb{Q}_{\geq n}} \mathbb{E}^{\mathcal{S}}_{Q_{\geq n}} J(\mathcal{S}_N \mid H_N), \quad \mathcal{S} \in \mathbb{S}_{n-1}, \quad n = \overline{1, N}, \tag{14}$$

If for some $n' \in [1; N-1]$

1. *$D_k(\mathcal{S}_{k-1})$ is convex for each $\mathcal{S}_{k-1} \in \mathbb{S}_{k-1}$, $k \geq n' + 1$;*
2. *$\mathbb{E}^{\mathcal{S}}_{Q} \inf\limits_{Q' \in \mathbb{Q}_{\geq n'+1}} \mathbb{E}^{\mathcal{S}_{n'} | H_{n'}}_{Q'} J(\mathcal{S}_N \mid H_N)$ is quasi-concave and upper semi-continuous in $H_{\geq n'+1} \in \mathcal{A} \mid (\mathcal{S}_{n'} \mid H_{n'})$ for each $\mathcal{S} \in \mathbb{S}_{n'-1}$, $H_{n'} \in D_{n'}(\mathcal{S})$ and $Q \in \mathbb{Q}^{E}_{n'}$;*
3. *$\inf\limits_{Q' \in \mathbb{Q}_{\geq n'+1}} \mathbb{E}^{\mathcal{S}_{n'}}_{Q'} J(\mathcal{S}_N \mid H_N)$ is continuous and bounded on $\mathbb{S}_{n'}$ for each $H_{\geq n'+1} \in \mathcal{A} \mid \mathcal{S}_{n'}$,*

then V_{n-1} satisfies (12) *for $n = n'$.*

Proof. We have

$$V_{n'-1}(\mathcal{S}) = \sup_{H_{\geq n'} \in \mathcal{A}|\mathcal{S}} \inf_{Q_{\geq n'} \in \mathbb{Q}_{\geq n'}} \mathbb{E}^{\mathcal{S}}_{Q_{\geq n'}} J(\mathcal{S}_N \mid H_N) \overset{\text{Lemma 3}}{=}$$

$$= \sup_{H_{n'} \in D_{n'}(\mathcal{S})} \sup_{H_{\geq n'+1} \in \mathcal{A}|(\mathcal{S}_{n'}|H_{n'})} \inf_{Q_{n'} \in \mathbb{Q}_{n'}} \mathbb{E}^{\mathcal{S}}_{Q_{n'}} \inf_{Q_{\geq n'+1} \in \mathbb{Q}_{\geq n'+1}} \mathbb{E}^{\mathcal{S}_{n'}|H_{n'}}_{Q_{\geq n'+1}} J(\mathcal{S}_N \mid H_N).$$

$\mathbb{E}^{\mathcal{S}}_{Q_{n'}} \inf_{Q_{\geq n'+1} \in \mathbb{Q}_{\geq n'+1}} \mathbb{E}^{\mathcal{S}_{n'}|H_{n'}}_{Q_{\geq n'+1}} J(\mathcal{S}_N \mid H_N)$ is linear, hence quasi-convex, in $Q_{n'}$. The assumptions yield that it is also quasi-concave and upper semi-continuous in $H_{\geq n'+1}$, and also continuous in $Q_{n'}$ which follows from Statement 2. Since $K_{n'}$ is compact, by the Prokhorov theorem, $\mathbb{Q}_{n'}$ is compact relative to the Prokhorov metric. Convexity of $\mathbb{Q}_{n'}$ follows from convexity of $K_{n'}$ while convexity of $\mathcal{A} \mid (\mathcal{S}_{n'} \mid H_{n'})$ follows from convexity of D_k, $k \geq n'+1$. Therefore, Theorem 1 applies and

$$V_{n'-1}(\mathcal{S}) =$$
$$\sup_{H_{n'} \in D_{n'}(\mathcal{S})} \inf_{Q_{n'} \in \mathbb{Q}_{n'}} \sup_{H_{\geq n'+1} \in \mathcal{A}|(\mathcal{S}_{n'}|H_{n'})} \mathbb{E}^{\mathcal{S}}_{Q_{n'}} \inf_{Q_{\geq n'+1} \in \mathbb{Q}_{\geq n'+1}} \mathbb{E}^{\mathcal{S}_{n'}|H_{n'}}_{Q_{\geq n'+1}} J(\mathcal{S}_N \mid H_N) \overset{\text{Statement 1}}{=}$$

$$= \sup_{H_{n'} \in D_{n'}(\mathcal{S})} \inf_{Q_{n'} \in \mathbb{Q}_{n'}} \mathbb{E}^{\mathcal{S}}_{Q_{n'}} \sup_{H_{\geq n'+1} \in \mathcal{A}|(\mathcal{S}_{n'}|H_{n'})} \inf_{Q_{\geq n'+1} \in \mathbb{Q}_{\geq n'+1}} \mathbb{E}^{\mathcal{S}_{n'}|H_{n'}}_{Q_{\geq n'+1}} J(\mathcal{S}_N \mid H_N) =$$

$$= \sup_{H_{n'} \in D_{n'}(\mathcal{S})} \inf_{Q_{n'} \in \mathbb{Q}_{n'}} \mathbb{E}^{\mathcal{S}}_{Q_{n'}} V_{n'}(\mathcal{S}_{n'} \mid H_{n'}).$$

□

Definition 6. A function $f(x,y)$ on $M \times N$ is *concavelike* in M if for every $x_1,\ x_2 \in M$ and $0 \leq t \leq 1$, there is a $x \in M$ such that

$$tf(x_1,y) + (1-t)f(x_2,y) \leq f(x,y) \qquad \forall y \in N.$$

Definition 7. A function $f(x,y)$ on $M \times N$ is *convexlike* in N if for every $y_1,\ y_2 \in N$ and $0 \leq t \leq 1$, there is a $y \in N$ such that

$$tf(x_1,y) + (1-t)f(x_2,y) \geq f(x,y) \qquad \forall x \in M.$$

A function $f(x,y)$ on $M \times N$ is *concave-convexlike* if it is both concavelike in M and convexlike in N.

Theorem 2 ([Sion, 1958])**.** *Let M be any space, N compact and $f(\mu,\nu)$ a function on*

$M \times N$ that is concave-convexlike. If $f(\mu,\nu)$ is lower semi-continuous in ν for each $\mu \in M$, then

$$\sup_{\mu\in M}\inf_{\nu\in N} f(\mu,\nu) = \inf_{\nu\in N}\sup_{\mu\in M} f(\mu,\nu).$$

Lemma 5. *Let $V_n(\mathcal{S})$ be defined by (14). If for some $n' \in [1; N-1]$*

1. *$\mathbb{E}_Q^{\mathcal{S}} \inf_{Q'\in\mathbb{Q}_{\geq n'+1}} \mathbb{E}_{Q'}^{\mathcal{S}_{n'}|H_{n'}} J(\mathcal{S}_N \mid H_N)$ is concavelike in $H_{\geq n'+1} \in \mathcal{A} \mid (\mathcal{S}_{n'} \mid H_{n'})$ for each $\mathcal{S} \in \mathbb{S}_{n'-1}$, $H_{n'} \in D_{n'}(\mathcal{S})$ and $Q \in \mathbb{Q}_{n'}^E$;*

2. *$\inf_{Q'\in\mathbb{Q}_{\geq n'+1}} \mathbb{E}_{Q'}^{\mathcal{S}_{n'}} J(\mathcal{S}_N \mid H_N)$ is continuous and bounded on $\mathbb{S}_{n'}$ for each $H_{\geq n'+1} \in \mathcal{A} \mid \mathcal{S}_{n'}$,*

then V_n satisfies (12) for $n = n'$.

Proof. The proof repeats Lemma 4, the only difference is using Theorem 2 instead of Theorem 1 to prove that $\sup_{H_{\geq n'+1}\in\mathcal{A}|(\mathcal{S}_{n'}|H_{n'})} \inf_{Q_{n'}\in\mathbb{Q}_{n'}} = \inf_{Q_{n'}\in\mathbb{Q}_{n'}} \sup_{H_{\geq n'+1}\in\mathcal{A}|(\mathcal{S}_{n'}|H_{n'})}$, which is provided by the assumptions of the Lemma. □

Statement 3. *Let f and g be finite functions on $X \times Y$, such that $\inf_Y f(x,y)$ and $\inf_Y g(x,y)$ are attained for each $x \in X$. If for every $y \in Y$ $f(x,y) < g(x,y)$ on X then*

$$\inf_Y f(x,y) < \inf_Y g(x,y) \qquad \forall x \in X.$$

Proof. To prove by contradiction, assume that there is $x' \in X$ such that

$$f(x',y_1) = \inf_Y f(x',y) \geq \inf_Y g(x',y) = g(x',y_2).$$

Then $f(x',y_1) \geq g(x',y_2) > f(x',y_2)$ which contradicts the definition of y_1 as infimum. □

Theorem 3 (Dynamic programming principle in weak form)**.** *Let H^* be an optimal strategy and $V_n(\mathcal{S})$ be defined by (12),(13). Let the assumptions of either Lemma 4 or Lemma 5 hold for every $n = \overline{1, N-1}$. Then for any $n = \overline{1, N}$:*

1. *if $n = 1$, then*

$$H_n^* \in \operatorname*{Arg\,max}_{H_n\in D_n(\mathcal{S}_0)} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_0} V_n(\mathcal{S}_n \mid H_n); \tag{15}$$

2. if $n > 1$ then there is such $\mathcal{S} \in \mathbb{S}_{n-1} \mid H^*_{\leq n-1}$ that

$$H^*_n \in \operatorname*{Arg\,max}_{H_n \in D_n(\mathcal{S})} \inf_{Q_n \in \mathbb{Q}^E_n} \mathbb{E}^{\mathcal{S}}_{Q_n} V_n(\mathcal{S}_n \mid H_n). \tag{16}$$

Proof. The proposed assumptions yield that V_n can be defined by (14). Assume that for some $n' \in [1; N]$ the statement is not true. If $n' = 1$ then there is such $\hat{H}_{n'} \in D_{n'}(\mathcal{S}_0)$, $\hat{H}_{n'} \neq H^*_{n'}$, that

$$\inf_{Q_{n'} \in \mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{n'}} V_{n'}(\mathcal{S}_{n'} \mid H^*_{n'}) < \inf_{Q_{n'} \in \mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{n'}} V_{n'}(\mathcal{S}_{n'} \mid \hat{H}_{n'}); \tag{17}$$

if $n' > 1$, then for any $\mathcal{S} \in \mathbb{S}_{n-1} \mid H^*_{\leq n-1}$ there is such $\hat{H}_{n'} \in D_{n'}(\mathcal{S})$, $\hat{H}_{n'} \neq H^*_{n'}$, that

$$\inf_{Q_{n'} \in \mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}}_{Q_{n'}} V_{n'}(\mathcal{S}_{n'} \mid H^*_{n'}) < \inf_{Q_{n'} \in \mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}}_{Q_{n'}} V_{n'}(\mathcal{S}_{n'} \mid \hat{H}_{n'}). \tag{18}$$

1) Let $n' < N$. For the subclass of strategies $\mathcal{A}^*_{\leq n'} = \{H \in \mathcal{A} \colon H_k = H^*_k,\ k \leq n'\}$, we have

$$H^* \in \mathcal{A}^*_{\leq n'} \subseteq \mathcal{A},$$

therefore

$$\inf_{Q \in \mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_Q J(\mathcal{S}_N \mid H^*_N) = \sup_{H \in \mathcal{A}^*_{\leq n'}} \inf_{Q \in \mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_Q J(\mathcal{S}_N \mid H_N) \overset{\text{Lemma 3}}{=}$$

$$= \sup_{H_{\geq n'+1} \in \mathcal{A} | \left(\mathcal{S}_{n'} | H^*_{n'}\right)} \inf_{Q_{\leq n'} \in \mathbb{Q}^E_{\leq n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\leq n'}} \inf_{Q_{\geq n'+1} \in \mathbb{Q}^E_{\geq n'+1}} \mathbb{E}^{\mathcal{S}^*_{n'}}_{Q_{\geq n'+1}} J(\mathcal{S}_N \mid H_N),$$

where $\mathcal{S}^*_k \in \mathbb{S}_k \mid H^*_{\leq k}$, $k \geq 1$, and $\mathcal{S}^*_0 = \mathcal{S}_0$. Using the same reasoning for $\mathbb{Q}^E_{\leq n'}$ as in Lemmas 4 and 5, it is easy to see that $\sup\limits_{H_{\geq n'+1}}$ and $\inf\limits_{Q_{\leq n'}}$ can be switched by virtue of the provided minimax theorems. Therefore,

$$\inf_{Q \in \mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_Q J(\mathcal{S}_N \mid H^*_N) =$$

$$= \inf_{Q_{\leq n'} \in \mathbb{Q}^E_{\leq n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\leq n'}} \sup_{H_{\geq n'+1} \in \mathcal{A} | \left(\mathcal{S}_{n'} | H^*_{n'}\right)} \inf_{Q_{\geq n'+1} \in \mathbb{Q}^E_{\geq n'+1}} \mathbb{E}^{\mathcal{S}^*_{n'}}_{Q_{\geq n'+1}} J(\mathcal{S}_N \mid H_N) \overset{(14)}{=}$$

$$= \inf_{Q_{\leq n'} \in \mathbb{Q}^E_{\leq n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\leq n'}} V_{n'}(\mathcal{S}^*_{n'}).$$

If $n' > 1$ then

$$\inf_{Q_{\le n'}\in\mathbb{Q}^E_{\le n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le n'}} V_{n'}(\mathcal{S}^*_{n'}) = \inf_{Q_{\le n'-1}\in\mathbb{Q}^E_{\le n'-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le n'-1}} \inf_{Q_{n'}\in\mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}^*_{n'-1}}_{Q_{n'}} V_{n'}(\mathcal{S}^*_{n'}) \overset{(18),\,\text{Stat.3}}{<}$$

$$< \inf_{Q_{\le n'-1}\in\mathbb{Q}^E_{\le n'-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le n'-1}} \inf_{Q_{n'}\in\mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}^*_{n'-1}}_{Q_{n'}} V_{n'}(\mathcal{S}_{n'} \mid \hat{H}_{n'}) = \inf_{Q_{\le n'}\in\mathbb{Q}^E_{\le n'}} \mathbb{E}^{\mathcal{S}^*_{n'-1}}_{Q_{\le n'}} V_{n'}(\mathcal{S}_{n'} \mid \hat{H}_{n'});$$

if $n' = 1$ then

$$\inf_{Q_{\le n'}\in\mathbb{Q}^E_{\le n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le n'}} V_{n'}(\mathcal{S}^*_{n'}) = \inf_{Q_{n'}\in\mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le n'}} V_{n'}(\mathcal{S}^*_{n'}) \overset{(17),\,\text{Stat.3}}{<}$$

$$< \inf_{Q_{n'}\in\mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le n'}} V_{n'}(\mathcal{S}_{n'} \mid \hat{H}_{n'}) = \inf_{Q_{\le n'}\in\mathbb{Q}^E_{\le n'}} \mathbb{E}^{\mathcal{S}^*_{n'-1}}_{Q_{\le n'}} V_{n'}(\mathcal{S}_{n'} \mid \hat{H}_{n'}).$$

In any case,

$$\inf_{Q\in\mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_{Q} J(\mathcal{S}_N \mid H^*_N) < \inf_{Q_{\le n'}\in\mathbb{Q}^E_{\le n'}} \mathbb{E}^{\mathcal{S}^*_{n'-1}}_{Q_{\le n'}} V_{n'}(\mathcal{S}_{n'} \mid \hat{H}_{n'}) =$$

$$= \inf_{Q_{\le n'}\in\mathbb{Q}^E_{\le n'-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le n'-1}} \inf_{Q_{n'}\in\mathbb{Q}^E_{n'}} \mathbb{E}^{\mathcal{S}^*_{n'-1}}_{Q_{n'}} \sup_{H_{\ge n'+1}\in\mathcal{A}|\left(\mathcal{S}_{n'}|\hat{H}_{n'}\right)} \inf_{Q_{\ge n'+1}\in\mathbb{Q}_{\ge n'+1}} \mathbb{E}^{\mathcal{S}_{n'}|\hat{H}_{n'}}_{Q_{\ge n'+1}} J(\mathcal{S}_N \mid H_N) =$$

$$= \inf_{Q\in\mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_{Q} J(\mathcal{S}_N \mid \hat{H}_N)$$

for some strategy $\hat{H}$ such that $\begin{cases} \hat{H}_k = H^*_k,\ k < n', \\ \hat{H}_{n'} \ne H^*_{n'} \end{cases}$. Therefore, $\hat{H} \ne H^*$ which contradicts the optimality of H^*.

2) For $n' = N$ we have, by analogy,

$$\inf_{Q\in\mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_{Q} J(\mathcal{S}_N \mid H^*_N) \overset{(13)}{=} \inf_{Q\in\mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_{Q} V_N(\mathcal{S}^*_N) = \inf_{Q_{\le N-1}\in\mathbb{Q}^E_{\le N-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le N-1}} \inf_{Q_N\in\mathbb{Q}^E_N} \mathbb{E}^{\mathcal{S}^*_{N-1}}_{Q_N} V_N(\mathcal{S}^*_N) \overset{(18),\,\text{Stat.3}}{<}$$

$$< \inf_{Q_{\le N-1}\in\mathbb{Q}^E_{\le N-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\le N-1}} \inf_{Q_N\in\mathbb{Q}^E_N} \mathbb{E}^{\mathcal{S}^*_{N-1}}_{Q_N} V_N(\mathcal{S}_N \mid \hat{H}_N) = \inf_{Q_{\le N}\in\mathbb{Q}^E_{\le N}} \mathbb{E}^{\mathcal{S}^*_{N-1}}_{Q_{\le N}} V_N(\mathcal{S}_N \mid \hat{H}_N) =$$

$$= \inf_{Q\in\mathbb{Q}^E} \mathbb{E}^{\mathcal{S}_0}_{Q} J(\mathcal{S}_N \mid \hat{H}_N)$$

for some strategy $\hat{H}$ such that $\hat{H}_N \ne H^*_N$ which contradicts the optimality of H^*. □

Theorem 4 (Verification theorem)**.** *Let $V_n(\mathcal{S})$ be a solution to* (12)*. Then for all $n = \overline{1, N}$ and $\mathcal{S} \in \mathbb{S}_{n-1}$*

1.

$$V_{n-1}(\mathcal{S}) \ge \inf_{Q\in\mathbb{Q}^E_{\ge n}} \mathbb{E}^{\mathcal{S}}_{Q} J(\mathcal{S}_N \mid H_N), \qquad \forall H_{\ge n} \in \mathcal{A} \mid \mathcal{S}; \tag{19}$$

2. if there is a $H^* \in \mathcal{A}$ such that

$$H_n^* \in \operatorname*{Arg\,max}_{H_n \in D_n(\mathcal{S})} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}} V_n(\mathcal{S}_n \mid H_n), \qquad \mathcal{S} \in \mathbb{S}_{n-1}, \tag{20}$$

then $V_{n-1}(\mathcal{S}) = \inf_{Q \in \mathbb{Q}_{\geq n}^E} \mathbb{E}_Q^{\mathcal{S}} J(\mathcal{S}_N \mid H_N^*)$.

Proof. The first part of the theorem is proven by induction. For $n = N$, (19) follows directly from (12). Let (19) hold for V_n. Then

$$V_{n-1}(\mathcal{S}) \overset{(12)}{=} \sup_{H_n \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{S} V_n(\mathcal{S}_n \mid H_n) \geq \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{S} V_n(\mathcal{S}_n \mid H_n) \overset{(19) \text{ for } V_n}{\geq}$$

$$\geq \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{S} \inf_{Q \in \mathbb{Q}_{\geq n+1}^E} \mathbb{E}_Q^{S_n | H_n} J(\mathcal{S}_N \mid H_N) \overset{\text{Lemma 3}}{=} \inf_{Q \in \mathbb{Q}_{\geq n}^E} \mathbb{E}_Q^{S} J(\mathcal{S}_N \mid H_N). \tag{21}$$

Therefore, (19) holds for V_{n-1}.

The second statement is proven by analogy: due to (20), all the inequalities in (21) turn to equalities which proves the statement. □

The dynamic programming principle specifies the necessary conditions for the optimal strategy to satisfy the Bellman-Isaacs equation, while the Verification theorem provides the sufficient conditions. Unfortunately, the result of Theorem 3 is rather weak for $n > 1$ in general case[4]. H_n^* is guaranteed to attain maximum in the dynamic programming equation only for *some* of the system states in $\mathbb{S}_{n-1} \mid H_{\leq n-1}^*$, therefore one cannot be sure that for the given state the equation is satisfied. Weakness of the proposition stems from the necessity of its strong negation to justify the inequality

$$\inf_{Q_{\leq n'-1} \in \mathbb{Q}_{\leq n'-1}^E} \mathbb{E}_{Q_{\leq n'-1}}^{\mathcal{S}_0} \mathcal{V}_{n'}(\mathcal{S}_{n'-1}^*, H_{n'}^*) < \inf_{Q_{\leq n'-1} \in \mathbb{Q}_{\leq n'-1}^E} \mathbb{E}_{Q_{\leq n'-1}}^{\mathcal{S}_0} \mathcal{V}_{n'}(\mathcal{S}_{n'-1}^*, \hat{H}_{n'}) \tag{22}$$

in proof by contradiction, where

$$\mathcal{V}_n(\mathcal{S}, H) = \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}} V_n(\mathcal{S}_n \mid H), \quad \mathcal{S} \in \mathbb{S}_{n-1}, \quad H \in \mathcal{A}_n \mid \mathcal{S}. \tag{23}$$

The result could be strengthened if we assume that the outer infima on both sides are

[4]Note that for $n = 1$ $H_n = H_1 \in m(\mathcal{F}_0)$, hence not a random variable. So the condition (15) is satisfied in a classic sense.

attained at the same measure $Q^* \in \mathbb{Q}^E_{\leq n'-1} \mid S_0$. Then the inequality is equivalent to

$$\mathbb{E}^{\mathcal{S}_0}_{Q^*_{\leq n'-1}} \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, H^*_{n'}) < \mathbb{E}^{\mathcal{S}_0}_{Q^*_{\leq n'-1}} \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, \hat{H}_{n'})$$

which is satisfied when

$$Q^*_{\leq n'-1} \left\{ \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, H^*_{n'}) < \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, \hat{H}_{n'}) \right\} = 1.$$

From this we can assume that H^*_n dominates (in terms of (18)) such $\hat{H}_n$ for which the infimum over $\mathbb{Q}^E_{\leq n'-1} \mid S_0$ can be attained at $Q^*_{n'-1}$.

For $n = \overline{1, N}$ consider the set-valued map $R_{n-1}\colon\ \mathcal{A}_{\leq n-1} \times \mathcal{A}_n \mid \mathcal{S}_{n-1} \to 2^{\mathbb{Q}^E_{\leq n-1}}$ such that

$$R_{n-1}(H'_{\leq n-1}, H_n) = \operatorname*{Arg\,min}_{Q_{\leq n-1} \in \mathbb{Q}^E_{\leq n-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\leq n-1}} \mathcal{V}_n(\mathcal{S} \mid H'_{\leq n-1}, H_n).$$

This allows us to formulate the dynamic programming principle in semi-strong form:

Theorem 5 (Dynamic programming principle in semi-strong form)**.** *Let H^* be an optimal strategy and $V_n(\mathcal{S})$ be defined by* (12),(13)*. Let the assumptions of either Lemma 4 or Lemma 5 hold for every $n = \overline{1, N-1}$. Then for any $n = \overline{1, N}$:*

1. *if $n = 1$, then H^*_n satisfies* (15)*;*

2. *if $n > 1$ then for any $\hat{H}_n$ such that $R_{n-1}(H^*_{\leq n-1}, \hat{H}_n) \cap R_{n-1}(H^*_{\leq n-1}, H^*_n) \neq \emptyset$, and any $Q^* \in R_{n-1}(H^*_{\leq n-1}, \hat{H}_n) \cap R_{n-1}(H^*_{\leq n-1}, H^*_n)$*

$$Q^* \left\{ \mathcal{V}_n(\mathcal{S}^*_{n-1}, H^*_n) \geq \mathcal{V}_n(\mathcal{S}^*_{n-1}, \hat{H}_n) \right\} > 0.$$

Proof. The proof repeats Theorem 3, inequality (22) holds based on the assumptions for $n' > 1$: if there are $\hat{H}_{n'}$ and $Q^* \in R_{n'-1}(H^*_{\leq n'-1}, \hat{H}_{n'}) \cap R_{n'-1}(H^*_{\leq n'-1}, H^*_{n'})$ such that

$$Q^* \left\{ \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, H^*_{n'}) \geq \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, \hat{H}_{n'}) \right\} = 0 \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad Q^* \left\{ \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, H^*_{n'}) < \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, \hat{H}_{n'}) \right\} = 1,$$

then

$$\inf_{Q_{\leq n'-1}\in\mathbb{Q}^E_{\leq n'-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\leq n'-1}} \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, H^*_{n'}) = \mathbb{E}^{\mathcal{S}_0}_{Q^*} \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, H^*_{n'}) <$$

$$< \mathbb{E}^{\mathcal{S}_0}_{Q^*} \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, \hat{H}_{n'}) = \inf_{Q_{\leq n'-1}\in\mathbb{Q}^E_{\leq n'-1}} \mathbb{E}^{\mathcal{S}_0}_{Q_{\leq n'-1}} \mathcal{V}_{n'}(\mathcal{S}^*_{n'-1}, \hat{H}_{n'}).$$

As in Theorem 3, we can use (22) to reach a contradiction and prove the main statement. □

The necessary conditions for optimality might be hard to verify in practice so by optimal strategy we will mean an extreme point in the Bellman-Isaacs equation according to (20) bearing in mind that there can be other optimal strategies. Nevertheless, such strategy is reasonably constructed thus can be considered a feasible solution of the problem. Finding the solution of (12)-(13) is another topic since it implies finding a minimum over a set of probability measures. The problem is numerically difficult in general case but has a closed-form solution for value functions with specific properties:

Theorem 6. *Let $f(x)$ be a finite real-valued concave function on a convex polyhedron $K \subset \mathbb{R}^l$, $\mathbf{E} \in K$, $\mathbb{Q}$ — a set of probability measures on K. Then the solution of the problem*

$$\begin{cases} \int\limits_K f(x)dQ(x) \longrightarrow \inf\limits_{Q\in\mathbb{Q}}, \\ \int\limits_K x_i dQ(x) = E_i,\ i = \overline{1,l}. \end{cases} \tag{24}$$

includes an atomic measure with mass concentrated in $m_K + 1$ extreme points of K, $m_K = \dim(K)$, such that E belongs to their convex combination while mass at each point equals the corresponding barycentric coordinate of E.

Similar result can be obtained via the theory of generalized Tchebycheff inequalities, see [Karlin and Studden, 1966, chapter XII]. The problem of finding the extreme measure is also studied in a more recent work [Goovaerts et al., 2011] for a number of measure classes. In this paper we provide a constructive proof of the Theorem for measures on bounded polyhedrons and obtain a closed-form analytic solution. The proof is based on the fact that a bounded polyhedron always has such facet that all its points lie above the facet-defined hyperplane. To illustrate, below is the proof for $K = [a; b]$, $a < b$: consider a linear function $l(x)$, such that $l(a) = f(a)$ and $l(b) = f(b)$:

$$l(x) = \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b).$$

Since f is concave, $f(x) \geq l(x)$ on $[a,b]$. Therefore

$$\int\limits_{[a,b]} f(x)dQ(x) \geq \int\limits_{[a,b]} l(x)dQ(x) = \frac{b-E}{b-a}f(a) + \frac{E-a}{b-a}f(b) = \int\limits_{[a,b]} f(x)dQ^*(x),$$

where

$$Q^*(x) = p\delta(x-a) + (1-p)\delta(x-b), \tag{25}$$

$p = \frac{b-E}{b-a}$. $p \in [0,1]$ since $E \in [a,b]$, which proves the statement. The proof for the general case can be found in Appendix 1. The result could be used for (12)-(13) to find infimum over a set of measures more effectively since the set of extreme measure candidates would be known and finite.

For a bounded polyhedron K, let $m(K) = \dim(K)$ and let $\mathcal{G}(K,E)$ be a set of combinations of its $m(K)+1$ extreme points such that their convex combination contains a point $E \in K$. For a combination $G = (G_0, \ldots, G_{m(K)}) \in \mathcal{G}(K,E)$, let $p^i(G,E)$ be the barycentric coordinate of E in the convex combination corresponding to the point G_i, $i = \overline{0, m(K)}$.

Lemma 6. *For some $1 \leq n' \leq N$ let $K_{n'}$ be a bounded polyhedron and let the value function $V_{n'}(S_{n'})$ be concave in $\Theta_{n'}$ on $\mathbb{S}_{n'}$. Let $\mathcal{S}_{n'} \mid (H^*, \Theta^*)$ be a system state at $t_{n'}$ where the control $H_{t_{n'}} = H^*$ and the parameter value $\Theta_{t_{n'}} = \Theta^*$. Then*

$$V_{n'-1}(\mathcal{S}) = \sup_{H_{n'} \in D_{n'}(\mathcal{S})} \min_{G_{n'} \in \mathcal{G}(K_{n'}, E_{n'})} \sum_{i=0}^{m(K_{n'})} p^i(G_{n'}, E_{n'}) V_{n'}\left(S_{n'} \mid (H_{n'}, G_{i,n'})\right). \tag{26}$$

Throughout the rest of the paper we consider only polyhedral sets K_n, $n = \overline{1,N}$, which are automatically bounded due to compactness. If concavity in unknown parameters Θ holds for all t_n then the Bellman-Isaacs equation can be simplified to (26) for all $n = \overline{1,N}$ which reduces the problem of finding V_{n-1} at each $\mathcal{S}$ to solving a finite number of $(m+1)$ - dimensional optimization problems. If there is only one parameter ($l = 1$), the problem is further simplified since there is only one combination of extreme points for each n and the extreme measure is known exactly, see (25).

The remaining part of the research considers application of the provided results to the optimal portfolio selection problem in discrete-time financial market. We consider a general class of price models without requiring a full specification of the dynamics and provide sufficient and economically reasonable conditions to simplify the Bellman-Isaacs equation.

2 Optimal portfolio selection problem

Let $H_n^X \in \mathbb{R}^m$ be the vector of volumes of the risky assets at t_n, and $H_n^Y \in \mathbb{R}$ be the volume of the risk-free asset at t_n, while the asset prices are $X_n \in \mathbb{R}^m$ and $Y_n \in \mathbb{R}$ respectively. Let K_n be a bounded convex polyhedron for $n = \overline{1, N}$.

Definition 8. *Portfolio* at time $t_n, n = \overline{0, N}$, is a vector $H_n = (H_n^X, H_n^Y)^T$.

Definition 9. *Market value* of the portfolio H_n is

$$W_n = H_n^{X^T} X_n + H_n^Y Y_n. \tag{27}$$

Let $W_n^X = H_n^{X^T} X_n$ and $W_n^Y = H_n^Y Y_n$ be market values of the risky and the risk-free positions respectively. In presence of transaction costs, difference arises between the market and the liquidation value (i. e. the real value of a portfolio when liquidated on the market). Henceforth, by "portfolio value" we shall mean the market value.

$H_0 \in \mathbb{R}^{m+1}$ is a given initial portfolio. External capital and asset movements in the framework are considered zero, so the following budget equation holds at every n:

$$\Delta H_n^{X^T} X_{n-1} + \Delta H_n^Y Y_{n-1} = -C_{n-1}(\Delta H_n^X, \mathcal{S}_{n-1}) \tag{28}$$

$$\Leftrightarrow$$

$$H_n^Y = Y_{n-1}^{-1} \left(W_{n-1} - H_n^{X^T} X_{n-1} - C_{n-1}(\Delta H_n^X, \mathcal{S}_{n-1}) \right), \tag{29}$$

where $C_{n-1}(\Delta H, \mathcal{S})$ is the value of transaction costs for the deal of volume ΔH at t_{n-1} given the state $\mathcal{S}$[5].

Consider trading limits at each t_n in the form of phase constraints $H_n \in D_n \subseteq \mathbb{R}^{m+1}$, $n = \overline{1, N}$, where $D_n \in m(\mathcal{F}_{n-1})$. Since H_n means the portfolio which must be hold throughout the n-th investment period, the structure of H_n is determined at t_{n-1} based on the available information, which explains the $\mathcal{F}_{n-1}$-measurability of H_n and its domain D_n.

Budget equation demonstrates that H_n^Y can always be expressed through H_n^X via (29), so in the rest of the paper we consider H^X as a strategy and formulate phase constraints in terms of H^X only. An admissible strategy is then defined according to Definition 2.

[5]Throughout the paper, $\Delta H > 0$ means buying $|\Delta H|$ of the asset and $\Delta H < 0$ means selling $|\Delta H|$ respectively. For a vector ΔH this applies element-wise.

2.1 Price dynamics

The general framework admits a frivolous choice of the market parameter vector. For example, parameters can describe price returns, risk-free rate, market liquidity or credit quality of the emitent. In each case an aimed study is required to find the optimal solution. In this study we consider only the parameters that afflict price dynamics assuming that the remaining parameters are known or can be provided with a reliable point estimate. Most of the financial literature considers a multiplicative price dynamics:

$$\Delta X_n = s_n X_{n-1},$$

e. g. the model of Cox,Ross and Rubinstein or the Black-Scholes market. To illustrate the approach, one could consider $\Theta_n = s_n$, but to have a better understanding of how it can be used in practice, we treat s_n as a sum of two components which stand for the expected value of price return and the deviation from it:

$$\Delta X_n = \mu_n X_{n-1} \Delta t_n + \sigma_n X_{n-1} \sqrt{\Delta t_n},\ n = \overline{1, N}, \tag{30}$$

where $\sigma_n \in \mathbb{R}^{m \times m}$ are diagonal matrices with random elements $\sigma_n^1, \ldots, \sigma_n^m$ on the main diagonal, $\mu_n \in \mathbb{R}^{m \times m}$ are matrices with non-negative non-diagonal elements, see [Yaozhong, 2000] for details. μ_n is assumed known for all n and can be interpreted as a forecast of the returns made by the company analyst for the forthcoming investment periods. We consider

$$\Theta_n = (\sigma_n^1, \ldots, \sigma_n^m)^T \quad \forall n = \overline{1, N},$$

the vector of parameters represent deviation from the forecast which is *a priori* unknown, though its range can be estimated by an expert to whom we refer as "risk-manager". Construction of (30) resembles GBM model but does not assume normality or symmetry of the returns distribution, thus avoiding the most criticized assumptions of the model (see [Cont, 2001]). Rather, (30) represents a general class of price processes with multiplicative dynamics.

One could consider an additive model and conduct a similar research but additive dynamics is less common in financial literature and in practice due to the lack of positivity of prices. It is also possible to abstain from imposing any *a priori* assumptions about price, but the optimal strategy can turn out to be ineffective due to the complete lack of information about price. In the multiplicative model, range of Θ can sometimes be estimated based on the market trading policy. For example, MOEX stops the continuous auction for the particular security on the Main Board and performs two discrete auctions per trading session if the price of the security deviates by more than 20% from the previous close[6]. If $\mu_n = 0$ this allows to consider $K_n = \bigotimes_{i=1}^{m}[-0.2; 0.2]$ for all n as reasonable bounds for the parameters (in general case, K_n will depend on μ_n). Of course, such a rough estimate can lead to ineffective portfolio management strategy since it does not rely on any additional information about the risky prices.

Note that μ_n and K_n must be chosen in such a way that s_n is always positive. Otherwise, the price of some assets is allowed to become non-positive with non-zero probability, while negative prices are not economically reasonable and zero prices are not considered in the current framework (all issuers are assumed default-free).

Risk-free dynamics is given, as usual, by

$$\Delta Y_n = r_n Y_{n-1} \Delta t_n,\ n = \overline{1, N}, \tag{31}$$

where $r_n \geq 0$ is the risk-free rate known for every n. Since we don not consider risk-free rate as an unknown parameter, a flat rate structure would suffice to demonstrate the results. Risk-free asset can be a cash account in a secure bank or even a separate fixed income investment portfolio with no credit and liquidity risk.

To simplify the problem, we also assume that the optimal portfolio structure for the next period depends solely on the market prices, the portfolio structure and the value of the risk-free position at the end of the previous period (or at t_0), thus

$$\mathcal{S}_n = (X_n, H_n^X, W_n^Y).$$

For ease of notation we will write D_n instead of $D_n(X, H^X, W^Y)$ though dependency on the state is still implied unless mentioned otherwise. Then the Bellman-Isaacs equation can be represented as

[6] http://moex.com/a775 [as of 20.09.2016]

$$V_{n-1}(X,H^X,W^Y) = \sup_{H_n^X \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n\Big((1+\mu_n \Delta t_n + \Theta_n \sqrt{\Delta t_n})X, H_n^X,$$
$$\big(W^Y - (H_n^X - H^X)^T X - C_{n-1}(H_n^X - H^X, S_{n-1})\big)(1+r_n \Delta t_n)\Big),\ n = \overline{1,N}, \quad (32)$$

$$V_N(X,H^X,W^Y) = J(X,H^X,W^Y),$$

which, in presence of Θ-concavity of V_n, can be rewritten as

$$V_{n-1}(X,H^X,W^Y) =$$
$$= \sup_{Z \in D_n} \min_{G \in \mathcal{G}_n(K_n,E_n)} \sum_{i=0}^{m(K_n)} p^i(G_n,E_n) V_n\Big((1+\mu_n \Delta t_n + \operatorname{diag}(G_{i,n})\sqrt{\Delta t_n})X, Z,$$
$$\big(W^Y - (Z - H^X)^T X - C_{n-1}(H_n^X - H^X, S_{n-1})\big)(1+r_n \Delta t_n)\Big),\ n = \overline{1,N}, \quad (33)$$

$$V_N(X,H^X,W^Y) = J(X,H^X,W^Y),$$

where $\operatorname{diag}(G_{i,n})$ is a diagonal matrix with elements of vector $G_{i,n}$ on the main diagonal.

In this chapter we research the specific properties of the value function. First, we study the market with zero transaction costs ($C_n(\Delta H_n^X, \mathcal{S}_{n-1}) \equiv 0$, $n = \overline{1,N}$) and provide sufficient conditions for Θ-concavity of the value function. Then we present the analogous result for the general market, which requires stricter conditions. We also assume that the optimal strategy and the value function are finite over the considered regions. This assumption is reasonable for practical use when infinite values point to poor specification of parameter estimates or optimal criteria. In general, boundedness depends on the trading limits, the terminal utility and the stochastic parameter range, and is beyond the scope of the research.

2.2 Zero-cost market

For some region $\mathbb{S}^* \subset \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ consider a set-valued function $D(X,H^X,W^Y) : \mathbb{S}^* \to 2^{\mathbb{R}^m}$.

Assumption 1. *$D(X,H^X,W^Y)$ is such that*

1. *For every* $(X, H^X, W^Y) \in \mathbb{S}^*$

$$Z \in D(X, H^X, W^Y) \iff A^T Z \in D(A^{-1}X, A^T H^X, W^Y) \tag{34}$$

for all invertible matrices A;

2. *For every* $(X, H^X, W^Y) \in \mathbb{S}^*$

$$Z \in D(X, H^X, W^Y) \iff Z \in D(X, 0, W^Y + {H^X}^T X); \tag{35}$$

3. *For every* $\alpha \in [0,1]$ *and every* $(X, H_1^X, W_1^Y), (X, H_2^X, W_2^Y) \in \mathbb{S}^*$,

$$Z_1 \in D(X, H_1^X, W_1^Y),\ Z_2 \in D(X, H_2^X, W_2^Y) \quad \Rightarrow$$

$$\Rightarrow\ \alpha Z_1 + (1-\alpha) Z_2 \ \in\ D(X, \alpha H_1^X + (1-\alpha) H_2^X, \alpha W_1^Y + (1-\alpha) W_2^Y). \tag{36}$$

Assumption 1′. *$D(X, H^X, W^Y)$ is such that (35), (36) hold and for every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$Z \in D(X, H^X, W^Y) \iff AZ \in D(A^{-1}X, AH^X, W^Y) \tag{37}$$

$$\forall A = \operatorname{diag}(a^1, \dots, a^m) > 0.$$

As an example of the function which satisfies Assumption 1, consider

$$D(X, H^X, W^Y) = \left\{ Z \in \mathbb{R}^m \colon -\beta^X W \le Z^T X \le (1+\beta^Y) W \right\}, \tag{38}$$

$$W = W^Y + {H^X}^T X.$$

Now consider a function $V(X, H^X, W^Y) : \mathbb{S}^* \to \mathbb{R}$.

Assumption 2. *$V(X, H^X, W^Y)$ is such that for every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$V(AX, H^X, W^Y) = V(X, A^T H^X, W^Y) \quad \forall A. \tag{39}$$

Assumption $2'$**.** *$V(X, H^X, W^Y)$ is such that for every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$V(AX, H^X, W^Y) = V(X, AH^X, W^Y) \quad \forall A = \mathrm{diag}(a^1, \dots, a^m) > 0. \tag{40}$$

Assumption 3. *$V(X, H^X, W^Y)$ is such that for every $(X, H^X, W^Y) \in \mathbb{S}^*$*

$$V(X, H^X, W^Y) = V(X, 0, W^Y + {H^X}^T X). \tag{41}$$

Assuming that the utility function J and the constraint sets D_n, $n = \overline{1, N}$, satisfy some of the above-mentioned assumptions, we prove that some properties of J are inherited by the value functions V_n across all n, including concavity in W^Y which leads to concavity in parameters.

Theorem 7. *For a zero-cost market, let the following assumptions hold:*

1. *$J(X, H^X, W^Y)$ satisfies Assumptions 2 and 3 for $\mathbb{S}^* = \mathbb{S}_N$.*

2. *$J(X, H^X, W^Y)$ is concave in W^Y for each X, H^X such that $(X, H^X, W^Y) \in \mathbb{S}_N$.*

3. *$D_n(X, H^X, W^Y)$ satisfies Assumption 1 for $\mathbb{S}^* = \mathbb{S}_n$ for every $n = \overline{1, N}$.*

4. *X_n and Y_n are defined by* (30) *and* (31) *respectively.*

Then the value function satisfies (33)*.*

Theorem 8. *For a zero-cost market, let the following assumptions hold:*

1. *$J(X, H^X, W^Y)$ satisfies Assumptions $2'$ and 3 for $\mathbb{S}^* = \mathbb{S}_N$.*

2. *$J(X, H^X, W^Y)$ is concave in W^Y for each X, H^X such that $(X, H^X, W^Y) \in \mathbb{S}_N$.*

3. *$D_n(X, H^X, W^Y)$ satisfies Assumption $1'$ for $\mathbb{S}^* = \mathbb{S}_n$ for every $n = \overline{1, N}$.*

4. *X_n and Y_n are defined by* (30) *and* (31) *respectively.*

5. *μ_n is diagonal for every $n = \overline{1, N}$.*

Then the value function satisfies (33).

It can be easily noticed that all the previous results of this and other sections can be obtained for a class of measures with known support but without fixed expectation values. A natural question arises about the necessity of specifying the expectation at all. Below we illustrate the importance of it in the context of the worst-case portfolio selection problem. We provide sufficient conditions under which lack of the expectation constraint leads to risk-free strategy being always optimal, which makes the investment process degenerative.

Theorem 9. *For a zero-cost market, consider the Bellman-Isaacs equation* (12)-(13) *for* $\mathbb{Q}_n = \bigcup_{E \in K_n} \mathbb{Q}_n^E$, $n = \overline{1, N}$, *and let the following assumptions hold:*

1. $J(X, H^X, W^Y)$ *satisfies Assumptions 2 and 3 for* $\mathbb{S}^* = \mathbb{S}_N$.
2. $J(X, H^X, W^Y)$ *is concave in* W^Y *for each* X, H^X *such that* $(X, H^X, W^Y) \in \mathbb{S}_N$.
3. $D_n(X, H^X, W^Y)$ *satisfies Assumption* $1'$ *for* $\mathbb{S}^* = \mathbb{S}_n$ *for every* $n = \overline{1, N}$.
4. $0 \in D_n(X, H^X, W^Y)$ *for every* $n = \overline{1, N}$.
5. X_n *and* Y_n *are defined by* (30) *and* (31) *respectively.*
6. μ_n *is diagonal and the main diagonal of* $(r_n I - \mu_n)\sqrt{\Delta t_n}$ *is a vector in* K_n *for every* $n = \overline{1, N}$.

Then $H^{X^*} = 0$ *is an optimal strategy.*

2.3 Market with transaction costs

For the general market, consider the costs function $C_{n-1}(\Delta H^X, \mathcal{S}_{n-1}) \stackrel{\text{df}}{=} C_{n-1}(\Delta H^X, X_{n-1})$, $n = \overline{1, N}$. While being quite narrow, this simple class of functions includes the most commonly used proportional costs model.

Consider a function $C(\Delta H, X) : \mathbb{R}^m \times \mathbb{R}_+^m \to \mathbb{R}_+$.

Assumption 4. $C(\Delta H, X)$ *is such that*

1. $C(\Delta H, X)$ *is non-negative, non-decreasing in* $|\Delta H|$ *and convex in* ΔH *on* $\mathbb{R}^m$ *for any* $X \in \mathbb{R}_+^m$;

2. *for every* $X \in \mathbb{R}^m_+$, $\Delta H \in \mathbb{R}^m$ *and* $A = \operatorname{diag}(a^1, \ldots, a^m) > 0$

$$C(A\Delta H, X) = C(\Delta H, AX); \tag{42}$$

With this assumption it is quite easy to prove the analog to Theorem 8:

Theorem 10. *Let the following assumptions hold:*

1. $J(X, H^X, W^Y)$ *is non-decreasing in* W^Y *for each* X, H^X *such that* $(X, H^X, W^Y) \in \mathbb{S}_N$.
2. $J(X, H^X, W^Y)$ *satisfies Assumption* $2'$ *for* $\mathbb{S}^* = \mathbb{S}_N$.
3. $J(X, H^X, W^Y)$ *is jointly concave in* H^X, W^Y *for each* X *such that* $(X, H^X, W^Y) \in \mathbb{S}_N$.
4. $D_n(X, H^X, W^Y)$ *satisfies Assumption* $1'$ *for* $\mathbb{S}^* = \mathbb{S}_n$ *for every* $n = \overline{1, N}$.
5. $C_{n-1}(\Delta H^X, X)$ *satisfies Assumption 4 for every* $n = \overline{1, N}$.
6. X_n *and* Y_n *are defined by* (30) *and* (31) *correspondingly.*
7. μ_n *is diagonal for every* $n = \overline{1, N}$.

Then the value function satisfies (33).

The key difference between Theorems 8 and 10 are the assumption of joint concavity in H^X, W^Y and monotonicity in W^Y. However, these properties are not restrictive since inherent to a wide range of classic utility functions of the form

$$J(X, H^X, W^Y) = J(W^Y + {H^X}^T X - C_N(H^X, X)), \tag{43}$$

i.e. non-decreasing concave functions of terminal liquidation value of the portfolio, including CARA and CRRA utilities.

3 Numeric solution of the Bellman-Isaacs equation with linear costs

When costs function is linear in volume, it is possible to restate the problem in terms of $W^X = {H^X}^T X$ instead of X and H^X separately. This leads to the Bellman-Isaacs equation where the value function depends on m less variables compared to the general case, which is useful for numerical purposes. Let $C_{n-1}(\Delta H, X) = \lambda_{n-1}(\Delta H)|\Delta H|X$, $n = \overline{1, N}$, where $\lambda_{n-1}(h) \equiv \begin{cases} \lambda^+_{n-1}, \ h \geq 0, \\ \lambda^-_{n-1}, \ h < 0. \end{cases}$ λ^+_{n-1} and λ^-_{n-1} are proportionality coefficients for transaction costs function for buy and sell deals respectively. Below we present a numeric scheme assuming that the limit book is symmetrical ($\lambda^+_{n-1} = \lambda^-_{n-1}$). The assumption is widely-used in literature and can sometimes be a necessary condition for the arbitrage-free market (see [Gatheral, 2010]). Here, the symmetry is used for the sake of convenience and not required.

Consider the isoelastic utility $J(X, H^X, W^Y) = (W^Y + {H^X}^T X - \lambda_N |{H^X}^T| X)^\gamma/\gamma$ and $m = 1$ (multidimensional case can be researched by analogy). By denoting $\pi^X = \frac{W^X}{W_0}$ and $\pi^Y = \frac{W^Y}{W_0}$, we can work in terms of dimensionless variables and obtain the Bellman-Isaacs equation as

$$V_{n-1}(\pi^X, \pi^Y) = \sup_{h \in D_n(W^X, W^Y)} \Big[p_n V_n\big(\overline{s}_n h, \pi^Y \tilde{r}_n - (h - \pi^X)\tilde{r}_n - \lambda_{n-1}\big|h - \pi^X\big|\tilde{r}_n\big) + \\ + (1 - p_n) V_n\big(\underline{s}_n h, \pi^Y \tilde{r}_n - (h - \pi^X)\tilde{r}_n - \lambda_{n-1}\big|h - \pi^X\big|\tilde{r}_n\big)\Big], \quad n = \overline{1, N}, \quad (44)$$

$$V_N(\pi^X, \pi^Y) = (\pi^Y + \pi^X - \lambda_N |\pi^X|)^\gamma/\gamma, \quad (45)$$

where

$$D_n = \big\{h : -\beta^X_n(\pi^X + \pi^Y) \leq h \leq (1 + \beta^Y_n)(\pi^Y + \pi^X)\big\}, \quad n = \overline{1, N}. \quad (46)$$

Dimensionality of the state space can be reduced up to $m + 1$ for even more general case when $C_n(\Delta H, X)$ is a, possibly non-linear, function of $\Delta H^T X$. However, in this case the value function will depend on (W^X, W^Y) rather than the dimensionless (π^X, π^Y) which will require additional scaling during numeric procedures.

The described method has been implemented for MatLab R2012a. Generally speaking, the framework can be decomposed into several blocks. Actual implementation depends on the specific formalization of the problem. The blocks are:

1. Defining key aspects of the problem which is conducted by both investor and portfolio manager. At this stage, one defines investment horizon, control moments $t_1, \ldots, t_N$, initial market and portfolio state, optimal criteria, admissible set of assets for future investments (stock selection).

2. Preliminary analysis of market data: estimation of initial market parameters and a priori distributions for the Bayesian method.

3. Update procedure for statistically estimated parameters and updates of expert forecasts.

4. Numerical solution of the Bellman-Isaacs equation based on the current estimates.

5. Analysis of performance and the control characteristics. Strategy can be stopped prematurely due to the reset of the strategy which is decided by the portfolio manager.

We consider the case when the value function is concave in parameters Θ, hence the Bellman-Isaacs equation can always be reduced to the simplified form. If phase constraints are compact, maximum is always achieved. For example, in one-dimensional case, constraint set (38) is an interval, while in multidimensional case it can be modified as

$$D(X, H^X, W^Y) = \left\{ Z \in \mathbb{R}^m : \begin{array}{ccccc} -\beta^X W & \leq & Z^T X & \leq & (1+\beta^Y)W, \\ & & |Z|^T X & \leq & (1+\tilde{\beta}^Y)W \end{array} \right\}, \tag{47}$$

where $W = W^Y + H^{X^T} X$, so that D is compact and still satisfies Assumption 1 which can be readily verified. (47) can be interpreted as limits for total size of short positions and limit for the amount invested in risky assets. Without the second constraint, one could infinitely short one risky asset and invest in another without violating the limit.

The value function can be calculated recursively according to (44), however this method becomes too slow with the increase of the number of steps N. Hence, we propose a step-by-step reconstruction of the value function on (π^X, π^Y) grid: first, for t_N, then for t_{N-1} by using known values at t_N, and so on up to t_0. As a byproduct, we obtain reconstructed value function for the whole grid which can be used for future analysis and strategy modeling if market parameters and forecasts are assumed unchanged.

At time t_N value function is known from analytic representation of J. Suppose that V_n is reconstructed for the grid. To find V_{n-1} according to (44), we might need V_n values in points both inside and outside the grid. Thus, either interpolation and extrapolation methods or appropriate parametric form $\hat{V}_n$ of the value function is required. The latter approach has been used during modeling, parametric form was chosen so that it is concave for any values of the calibration coefficients. We find that for isoelastic J, all V_n, $n = \overline{0, N-1}$, can be approximated (even in the presence of costs and constraints) by isoelastic function of the form

$$\hat{V}_n(\pi^X, \pi^Y) = \left(b_n^{X^T} \pi^X + b_n^Y \pi^Y + c_n \right)^\gamma / \gamma, \tag{48}$$

with fitting reduced to simple linear regression. Since V_{n-1} depends solely on V_n, it is possible to calculate values of V_{n-1} on the grid in parallel mode.

4 Modeling results

This chapter presents results of implementing the proposed framework to modeled market. We consider one risky asset and stationary parameters μ_n, E_n, K_n. Economic interpretation allows to divide them into two main groups characterizing price forecast and deviation from it. Hence, during investment process, the groups can be estimated by different departments (analyst/trader and risk-manager). We assume that μ is given by an expert analyst since poorly estimated based on market information. K is estimated and updated via a Bayesian method based on observable data: we assume that the data follows a known stochastic process with unknown parameters (GBM was used since it was the basis for the multiplicative price model). The parameters are estimated and K is found as credible interval of detrended returns. E is assumed zero to keep all the information about price forecast within μ.

Analyst's forecasts of μ are characterized by the forecasting power. Denote the price change over interval $[t_{n-1}, t_n]$ as ΔX_n, $n = \overline{1, N}$. At time t_{n-1}, the forecasted value is modeled as a random variable $\tilde{\mu}_n$ with normal distribution such that

$$\tilde{\mu}_n - \frac{\Delta X_n}{X_{n-1}} = \frac{\Delta X_n}{X_{n-1}} \xi, \quad \xi \sim \mathcal{N}(0, \varepsilon^2). \tag{49}$$

ε^{-1} is a measure of forecast's precision, hence analyst's forecasting power. This method of forecast modeling is extremely rough but allows to define dimensionless measure of forecasting power. Estimates for K are characterized by credible interval for specified level α.

Below we demonstrate the worst-case strategy and discuss the results for one realized scenario of market price. Parameters are the following: $m = 1$; $N = 5$; $C_{n-1}(\Delta H, X) = \lambda|\Delta H|X$, $n = \overline{1, N}$; price follows GBM with drift 0.03 and volatility 0.005; risk-free rate equals 0.02; initial prices are 1; initial capital is 10; $\Delta t_n = 1$. The utility function is isoelastic with $\gamma = 0.6$. Strategy is constrained by $D(X, H^X, W^Y)$, defined in (38), with stationary coefficients $\beta_n^X = \beta_n^Y \equiv 1$. Between each neighboring control moments t_n, 500 price observations are assumed available for Bayesian updates. Figures 1 - 3 demonstrate the realized price trajectory and the worst-case strategy results for high forecasting power $\varepsilon = 1$ when $\lambda = 0$ and $\lambda = 0.05$ (loss of 5% of each deal's value).

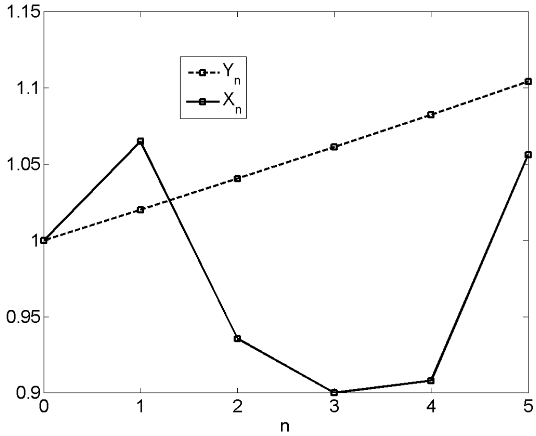

Figure 1: Realized price trajectory.

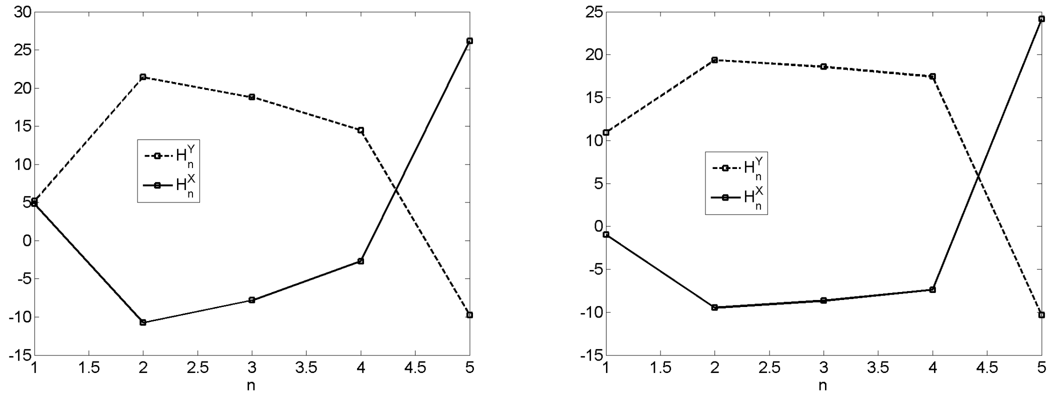

Figure 2: The worst-case optimal strategy: to the left — at costs-free market; to the right — at $\lambda = 0.05$. $\varepsilon = 1$.

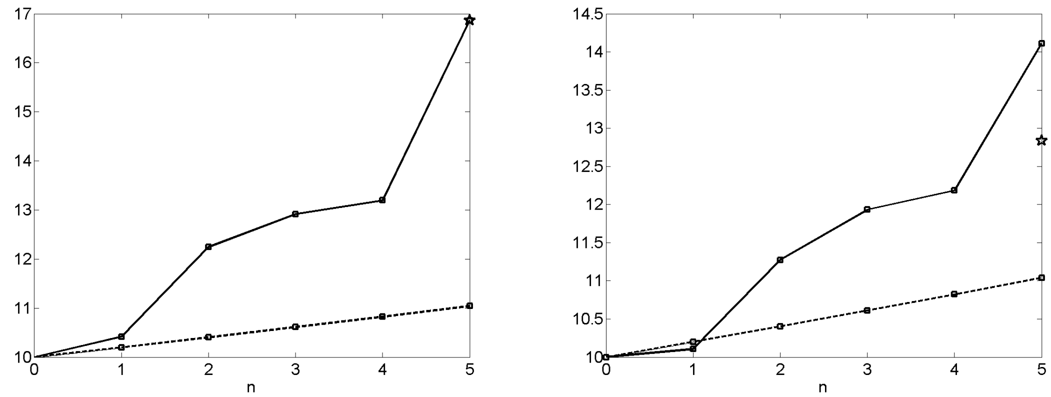

Figure 3: Portfolio market value W_n when using the worst-case optimal strategy: to the left — at costs-free market; to the right — at $\lambda = 0.05$. Dashed line is market value according to risk-free strategy $H^X \equiv 0$. Pentagonal star denotes liquidation value of the portfolio at the end of the strategy. $\varepsilon = 1$.

Since the forecasting power is big enough, all decisions made by DSS were correct in terms of long/short position, hence portfolio value increased at every step. In the presence of costs, transacted volumes are smaller and the total profit becomes less. Further increase in λ shows that at some point costs are so large that risky investments are not worth investing into, even if all the decisions are correct. Figure 4 demonstrates results for the same price scenario and $\lambda = 0.12$.

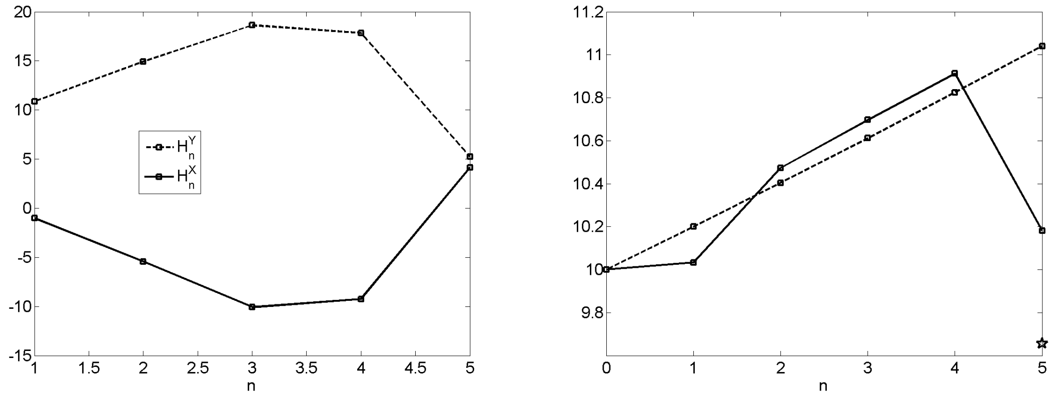

Figure 4: To the left is optimal strategy, to the right is market value of the portfolio for $\lambda = 0.12$. $\varepsilon = 1$. Markup repeats Fig. 2 and 3

For the same values of parameters, we simulated market dynamics and compared the expectation of liquidation value. Based on 100 iterations, we obtained that, for $\lambda = 0.05$, expected optimal portfolio value outperforms risk-free value by 3.05%, and by 28.32% for $\lambda = 0$. This shows that even for $\lambda = 0.05$ the worst-case strategy produces better results than risk-free investment.

5 Conclusion

We present a probabilistic framework for the worst-case approach to the stochastic dynamic programming problem in discrete time with terminal utility maximization criterion. Unlike the classic game-theoretic framework, we assume that the dynamic model of the stochastic system is a black box with some observable characteristics, and formulate a model-free approach to the optimal control of a general Markov stochastic system in a class of Markov strategies on a finite horizon. One of the assumptions of the framework is the boundedness of parameters' range which can be considered a mild restriction if the range is chosen big enough. We also present a closed-form solution to the problem for a specific class of the terminal utility functions. The results are then applied to the strategic portfolio selection problem for discrete time when the stochastic process of asset prices is not specified. The choice of the discrete time is caused by the intended purpose to implement the approach as a decision support system during an investment process rather than an automatic trading system. Besides, even high-frequency trading cannot be continuous due to the latency of the trading platform which, especially in presence of fixed transaction costs per deal, makes discrete dynamics more viable.

The selection problem is solved when only the expected value and range of the future price returns are known. The worst-case framework allows to avoid using a specific price model hence accepting the implied assumptions. For example, the canonical model of geometric Brownian motion assumes normal distribution of returns which has been criticized lately [Cont, 2001]. The proposed approach has been adapted to the general multiplicative model without the assumptions of GBM. Therefore an expert could change forecasts and ranges of possible price returns according to the state of the market, which is crucial during crisis and eventual shocks.

The main results hold for several risky assets in presence of convex transaction costs and trading limits. For proportional costs, we present a simpler numerical scheme for the Bellman-Isaacs equation by reducing the dimensionality of the value function's state space.

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Appendix 1. The extreme measure problem for m-dimensional convex polyhedral support

In this section we consider a finite real-valued function $f(x)$ over a convex hull K of a finite point set in $\mathbb{R}^l$. Let $\mathrm{aff}(K)$ be the affine hull of K, $\dim(\mathrm{aff}(K)) = m_K \leq l$. For the general results in convex geometry, we refer to [Artamonov and Latyshev, 2004]. Let $U(K)$ be the set of extreme points of K. Note that $K = \mathrm{conv}\,(U(K))$.

Lemma 7. *Let $f(x)$ be concave on K and $E \in K$. Then there is $\tilde{U} \subseteq U(K)$ consisting of $m_K + 1$ points, and affine function $l(x)$, such that*

1. *$E \in \mathrm{conv}(\tilde{U})$;*
2. *$l(A) = f(A)$, $\forall A \in \tilde{U}$;*
3. *$l(x) \leq f(x)$, $\forall x \in K$;*

Proof. Note that when K contains a single point, $m_K = 0$ and the statement is trivial so the rest of the proof assumes $m_K > 0$.

Instead of l-dimensional sets K and $U(K)$, we will work in terms of m_K-dimensional equivalent subsets of $\mathrm{aff}(K)$ due to bijectivity. It can be easily shown that concave $f(x)$ attains minimum on $U(K)$. Define the set

$$U_f(K) = \{B = (A, f(A)),\ A \in U(K) \subset \mathrm{aff}(K)\}.$$

Let $M = \mathrm{conv}(U_f(K))$, then M is a polyhedron and can be defined in terms of a non-singular linear system of

$$g_k(x,h) = \sum_{i=1}^{m_K} a_i^k x_i + a_{m_K+1}^k h + a_0^k, \quad k = \overline{1,r},\ x \in \mathrm{aff}(K),\ h \in \mathbb{R}$$

so that $M = \left\{x, h \colon g_k(x,h) \geq 0,\ k = \overline{1,r}\right\}$. Since f is finite and $U(K)$ is bounded, M is a bounded set, thus $a_{m_K+1}^1, \ldots, a_{m_K+1}^r$ cannot all be zero. By construction, $\dim(M)$ is either m_K or $m_K + 1$. Since $E \in K$, by Carathèodory theorem, there is a combination of $m_K + 1$ extreme points in $U(K)$ such that E is the convex combination of these points. Let $\mathbb{U}_E$ be a set of such combinations.

1) If $\dim(M) = m_K$, then there is a k^* such that $g_{k^*}(x,h) = 0$ for each $(x,h) \in M$. Assume that $a^{k^*}_{m_K+1} = 0$. Then

$$g_{k^*}(x,h) = \sum_{i=1}^{m_K} a_i^{k^*} x_i + a_0^{k^*} = 0$$

defines a m_K-dimensional facet which contains m_K+1 extreme points of $U_f(K)$. Since $\dim(K) = m_K$, the corresponding m_K+1 extreme points of K are in general position, hence $a_0^{k^*} = \ldots = a_{m_K}^{k^*} = 0$ which contradicts the definition of polyhedron. Therefore, $a^{k^*}_{m_K+1} \neq 0$ and we can take

$$l(x) = -\sum_{i=1}^{m_K} \frac{a_i^{k^*}}{a^{k^*}_{m_K+1}} x_i - \frac{a_0^{k^*}}{a^{k^*}_{m_K+1}}$$

and any combination of extreme points from $\mathbb{U}_E$.

2) Let $\dim(M) = m_K + 1$. Let $\Pi(E) = \{\Pi_k\}_{k=1}^{r^*}$ be a set of hyperplanes defined by combinations of the extreme points in $U_f(K)$ such that the combinations of the corresponding points from $U(K)$ belong to $\mathbb{U}_E$. For ease of notation, let each $\Pi_k \in \Pi(E)$ be defined by $g_k(x,h) = 0$. Our next goal is to prove that there is a hyperplane $\Pi_{k^*} \in \Pi(E)$ such that $a^{k^*}_{m_K+1} \neq 0$ and

$$h \geq -\sum_{i=1}^{m_K} \frac{a_i^{k^*}}{a^{k^*}_{m_K+1}} x_i - \frac{a_0^{k^*}}{a^{k^*}_{m_K+1}} \qquad \forall (x,h) \in M. \tag{50}$$

By contradiction, assume that for any $k = \overline{1, r^*}$ either $a^k_{m_K+1} = 0$ or there is $(x^k, h^k) \in M$ such that

$$h^k < -\sum_{i=1}^{n} \frac{a_i^k}{a^k_{m_K+1}} x_i^k - \frac{a_0^k}{a^k_{m_K+1}} \quad \Longleftrightarrow \quad a^k_{m_K+1} g_k(x^k, h^k) < 0.$$

In the latter case, if $a^k_{m_K+1} > 0$ then $g_k(x^k, h^k) < 0$, thus $(x^k, h^k) \notin M$ which contradicts the assumption. Therefore, $a^k_{n+1} \leq 0$ for every $k = \overline{1, r^*}$. This means that if $g_k(E, h) \geq 0$ is satisfied for all $k = \overline{1, r^*}$ for some $h \in \mathbb{R}$, then the inequalities are satisfied for any $(E, h - \Delta)$, $\Delta \geq 0$. If we prove that any point (E, h) satisfying inequalities $g_k(E, h) \geq 0$ for $k = \overline{1, r^*}$, automatically satisfies inequalities for $k = \overline{r^* + 1, r}$, then we will show that M is not bounded which leads to contradiction.

To prove the required statement, note that for any $k = \overline{1, r^*}$ there are extreme points $A_{k,1}, \ldots, A_{k,m_K+1} \in U(K)$ such that $E \in \operatorname{conv}(A_{k,1}, \ldots, A_{k,m_K+1})$. Therefore

$$E = \sum_{i=1}^{m_K+1} \lambda_{k,i} A_{k,i}, \quad \sum_{i=1}^{m_K+1} \lambda_{k,i} = 1, \lambda_{k,i} \geq 0.$$

Then linearity of the polyhedral inequalities implies

$$g_j\left(E, \sum_{i=1}^{m_K+1} \lambda_{k,i} f(A_{k,i})\right) \geq 0 \qquad \forall k = \overline{1, r^*}, \quad \forall j = \overline{r^* + 1, r}. \tag{51}$$

Let

$$\underline{F} = \min_{1 \leq k \leq r^*} \sum_{i=1}^{m_K+1} \lambda_{k,i} f(A_{k,i}) = \sum_{i=1}^{m_K+1} \lambda_{\underline{k},i} f(A_{\underline{k},i}),$$

$$\overline{F} = \max_{1 \leq k \leq r^*} \sum_{i=1}^{m_K+1} \lambda_{k,i} f(A_{k,i}) = \sum_{i=1}^{m_K+1} \lambda_{\overline{k},i} f(A_{\overline{k},i}).$$

Then linearity of the inequalities and (51) imply that

$$g_j(E, h) \geq 0 \qquad \forall h \in [\underline{F}, \overline{F}], \quad \forall j = \overline{r^* + 1, r}.$$

Since M is bounded, the line $\{(E, h),\ h \in \mathbb{R}\}$ intersects M at points $(E, \sum_{i=1}^{m_K+1} \lambda_{k,i} f(A_{k,i}))$ at least for one $k \in [1, r^*]$, thus the set $\{h\colon (E, h) \in M\} \subseteq [\underline{F}, \overline{F}]$. Hence the inequalities $g_k(E, h) \geq 0$ can be satisfied for each $k = \overline{1, r^*}$ only for $h \in [\underline{F}, \overline{F}]$ which means that the rest of the inequalities will be satisfied as well.

We have proven that if for any $k = \overline{1, r^*}$ either $a^k_{m_K+1} = 0$ or there is $(x^k, h^k) \in M$ such that

$$h^k < -\sum_{i=1}^{n} \frac{a_i^k}{a^k_{m_K+1}} x_i^k - \frac{a_0^k}{a^k_{m_K+1}},$$

then M is unbounded which contradicts the definition. Therefore, there are coefficients $a_1^{k^*}, \ldots, a^{k^*}_{m_K+1}$ such that $k^* \in [1, r^*]$ and (50) is true. Then we can consider

$$l(x) = -\sum_{i=1}^{m_K} \frac{a_i^{k^*}}{a^{k^*}_{m_K+1}} x_i - \frac{a_0^{k^*}}{a^{k^*}_{m_K+1}}.$$

By construction, $l(x) \leq f(x)$ on K. Since the corresponding hyperplane Π_{k^*} contains $m_K + 1$ extreme points $(A_1, f(A_1)), (A_{m_K+1}, f(A_{m_K+1}))$, $l(A_i) = f(A_i)$ for $i = \overline{1, m_K + 1}$. By definition of Π_{k^*}, $E \in \text{conv}(A_1, \ldots, A_{m_K+1})$. □

Lemma 7 allows to prove Theorem 6 which states that if $f(x)$ is a finite real-valued concave function on a convex polyhedron $K \subset \mathbb{R}^l$, $E \in K$ and $\mathbb{Q}$ is a set of probability measures on K, then the solution of the problem

$$\begin{cases} \int\limits_K f(x)dQ(x) \longrightarrow \inf\limits_{Q\in\mathbb{Q}}, \\ \int\limits_K x_i dQ(x) = E_i,\ i=\overline{1,l}. \end{cases}$$

includes an atomic measure with mass concentrated in m_K+1 extreme points of K, $m_K = \dim(K)$, such that E belongs to their convex combination while mass at each point equals the corresponding barycentric coordinate of E.

Proof of Theorem 6. Consider any $l(x)$ from Lemma 7. Then $l(x)$ equals $f(x)$ at m_K+1 extreme points $A_0,\ \ldots,\ A_{m_K} \in K$ such that $E=\sum\limits_{i=0}^{m_K} p_i A_i$, $\sum\limits_{i=0}^{m_K} p_i = 1$, $p_i=0$, $i=\overline{0,m_K}$. Since $l(x)$ is affine, we have

$$\int f(x)\,dQ(x) \geq \int l(x)\,dQ(x) = l(E) = \sum_{i=0}^{m_K} p_i l(A_i) = \sum_{i=0}^{m_K} p_i f(A_i) = \int f(x)\,dQ^*(x),$$

where

$$Q^*(x) = \sum_{i=0}^{m_K} p_i \delta_x(\{A_i\}),$$

which proves the statement. □

Below we provide the analytic formulas for $p_0,\ldots,p_{m_K}$: let each extreme point A_i have affine coordinates $\left(x_1^i,\ldots,x_{m_K+1}^i\right)^T$ in $\mathrm{aff}(K)$, $i=\overline{0,m_K}$, and let E have affine coordinates $(E_1,\ldots,E_{m_K})^T$[7]. Then $p_0,\ldots,p_{m_K}$ is the solution of the linear system

$$\begin{bmatrix} x_1^0 - E_1 & \ldots & x_1^{m_K} - E_1 \\ \vdots & \ddots & \vdots \\ x_{m_K}^0 - E_{m_K} & \ldots & x_{m_K}^{m_K} - E_{m_K} \\ 1 & \ldots & 1 \end{bmatrix} p = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Since E is a convex combination of the extreme points, $p \geq 0$. The solution can be found numerically while analytic form is found by using the Cramer's rule and the Laplace expansion:

[7] If $\dim(K)=m$, affine coordinates can be made equal to the absolute coordinates in $\mathbb{R}^l$.

$$p_i = \frac{\Delta_i}{\Delta}, \qquad \Delta = \begin{vmatrix} x_1^0 - E_1 & \dots & x_1^{m_K} - E_1 \\ \vdots & \ddots & \vdots \\ x_{m_K}^0 - E_{m_K} & \dots & x_{m_K}^{m_K} - E_{m_K} \\ 1 & \dots & 1 \end{vmatrix},$$

$$\Delta_i = (-1)^{m_K+i} \begin{vmatrix} x_1^0 - E_1 & \dots & x_1^{i-1} - E_1 & x_1^{i+1} - E_1 & \dots & x_1^{m_K} - E_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{m_K}^0 - E_{m_K} & \dots & x_{m_K}^{i-1} - E_{m_K} & x_{m_K}^{i+1} - E_{m_K} & \dots & x_{m_K}^{m_K} - E_{m_K} \end{vmatrix}.$$

Appendix 2. Properties of the Bellman-Isaacs equation

For the case $m = 1$ let $K_n = [\underline{\sigma}_n; \overline{\sigma}_n]$. The following notations for $n = \overline{1, N}$ will be used throughout this section:

$$
\begin{aligned}
&s_n = I + \mu_n \Delta t_n + \sigma_n \sqrt{\Delta t_n}, \\
&s_n(G_{i,n}) = I + \mu_n \Delta t_n + \operatorname{diag}(G_{i,n}) \sqrt{\Delta t_n}, \\
&\tilde{r}_n = 1 + r_n \Delta t_n,
\end{aligned} \tag{52}
$$

where I denotes the identity matrix of the appropriate size, $G_{i,n}$ is the i-th extreme point in the collection $G_n \in \mathcal{G}_n(K_n, E_n)$. For ease of notation, in this chapter product of two vectors means the scalar product and subtraction of a scalar r from the matrix s means $s - rI$ for the appropriate identity matrix I.

Lemma 8. *For a zero-cost market, assume that $V_{n-1}(X, H^X, W^Y)$ is defined by* (12) *for some $n = \overline{1, N}$ and*

1. *$V_n(X, H^X, W^Y)$ satisfies Assumptions 2 and 3;*
2. *$V_n(X, H^X, W^Y)$ is concave in W^Y for each X, H^X such that $(X, H^X, W^Y) \in \mathbb{S}_n$;*
3. *X_n and Y_n are defined by* (30) *and* (31) *respectively.*

Then (33) *holds for the given n.*

Proof. Using Assumptions 2 and 3, (12) can be transformed as:

$$
V_{n-1}(X, H^X, W^Y) = \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z, W_n^Y) =
$$

$$
= \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(s_n X, Z, W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n) \overset{(41)}{=}
$$

$$
\overset{(41)}{=} \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(s_n X, 0, W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n + Z^T s_n X) \overset{(39)}{=}
$$

$$
\overset{(39)}{=} \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X, 0, W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n + Z^T s_n X).
$$

Since V_n is concave in W^Y, function under the expectation sign is concave in s_n and Theorem

6 applies. By using (39),(41) for backward transformation, we derive

$$V_{n-1}(X, H^X, W^Y) = \sup_{Z \in D_n} \min_{G_n \in \mathcal{G}_n(K_n, E_n)} \sum_{i=0}^{m(K_n)} p^i(G_n, E_n) V_n(X, 0,$$
$$W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n + Z^T s_n(G_{i,n}) X) =$$

$$= \sup_{Z \in D_n} \min_{G_n \in \mathcal{G}_n(K_n, E_n)} \sum_{i=0}^{m(K_n)} p^i(G_n, E_n) V_n(s_n(G_{i,n}) X, Z, W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n),$$

which coincides with (33) after substituting formulas for $s_n(G_{i,n})$ and $\tilde{r}_n$. □

Similar proof can be derived under the weaker assumptions if μ_n, hence s_n, is diagonal:

Corollary 3. *Under the assumptions of Lemma 8, assume that $V_n(X, H^X, W^Y)$ satisfies Assumptions $2'$ and 3 and μ_n is diagonal. Then (33) holds for the given n.*

Assumption 3 can be replaced by joint concavity in H^X and W^Y:

Lemma 9. *For a zero-cost market, assume that $V_{n-1}(X, H^X, W^Y)$ is defined by (12) for some $n = \overline{1, N}$ and*

1. *$V_{n+1}(X, H^X, W^Y)$ satisfies Assumptions 2;*
2. *$V_{n+1}(X, H^X, W^Y)$ is jointly concave in W^Y and H^X for each X such that $(X, H^X, W^Y) \in \mathbb{S}_n$;*
3. *X_n and Y_n are defined by (30) and (31) respectively.*

Then (33) holds for the given n.

Proof. Using Assumption 2, transform (12):

$$V_{n-1}(X, H^X, W^Y) = \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z, W_n^Y) =$$

$$= \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(s_n X, Z, W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n) =$$
$$= \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X, s_n^T Z, W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n).$$

Since V_n is jointly concave in H^X, W^Y, it is concave in s_n and Theorem 6 applies. The rest of the proof follows Lemma 8. □

Corollary 4. *In view of Lemma 9, assume that $V_n(X, H^X, W^Y)$ satisfies only Assumption $2'$ and μ_n is diagonal. Then* (33) *holds for the given n.*

Lemmas 8 and 9 with corollaries provide sufficient conditions to reduce equation (12) to the simpler form where the extreme measure is concentrated in the extreme points of the support. Diagonality of μ_n, together with diagonality of σ_n, is quite constraining and seem to lead to independency of the risky asset prices. However, dependence among $\sigma_n^1, \dots, \sigma_n^m$ is still allowed; besides, μ_n can be estimated according to the model which allows dependent dynamics of parameters. Therefore, dependency can be accounted for during practical use (though "outside" the worst-case framework).

Now we obtain sufficient conditions under which the required properties of V_n are inherited by V_{n-1}.

Statement 4. *For a zero-cost market, assume that $V_{n-1}(X, H^X, W^Y)$ is defined by* (12) *and $D_n(X, H^X, W^Y)$ satisfies* (35) *for some $n = \overline{1, N}$. Then $V_{n-1}(X, H^X, W^Y)$ satisfies Assumption 3.*

Proof. The budget equation (29) implies

$$V_{n-1}(X, 0, W^Y + H^{X^T} X) = \sup_{Z \in D_n(X, 0, W^Y + H^{X^T} X)} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z, W^Y \tilde{r}_n + H^{X^T} X \tilde{r}_n - (Z - 0)^T X \tilde{r}_n) =$$

$$= \sup_{Z \in D_n(X, H^X, W^Y)} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z, W^Y \tilde{r}_n - (Z - H^X)^T X \tilde{r}_n) =$$

$$= V_n(X, H^X, W^Y).$$

$\square$

Lemma 10. *For a zero-cost market, assume that $V_{n-1}(X, H^X, W^Y)$ is defined by* (12) *for some $n = \overline{1, N}$ and*

1. *$D_n(X, H^X, W^Y)$ satisfies* (34) *for $\mathbb{S}^* = \mathbb{S}_n$;*

2. *$V_n(X, H^X, W^Y)$ satisfies Assumption 2 for $\mathbb{S}^* = \mathbb{S}_n$;*

3. *X_n and Y_n are defined by* (30) *and* (31) *respectively.*

Then $V_{n-1}(X, H^X, W^Y)$ satisfies Assumption 2 for $\mathbb{S}^* = \mathbb{S}_{n-1}$.

Proof. By letting $Z' = A^T Z$, we obtain

$$V_{n-1}(AX, H^X, W^Y) = \sup_{Z \in D_n(AX, H^X, W^Y)} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(AX_n, Z, W^Y \tilde{r}_n - (Z - H^X)^T AX \tilde{r}_n) =$$

$$= \sup_{Z' \in D_n(X, A^T H^X, W^Y)} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(AX_n, A^{T^{-1}} Z', W^Y \tilde{r}_n - (A^{T^{-1}} Z' - H^X)^T AX \tilde{r}_n) =$$

$$= \sup_{Z' \in D_n(X, A^T H^X, W^Y)} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, A^T A^{T^{-1}} Z', W^Y \tilde{r}_n - (A^T A^{T^{-1}} Z' - A^T H^X)^T X \tilde{r}_n) =$$

$$= \sup_{Z' \in D_n(X, A^T H^X, W^Y)} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z', W^Y \tilde{r}_n - (Z' - A^T H^X)^T X \tilde{r}_n) = V_{n-1}(X, A^T H^X, W^Y).$$

□

Corollary 5. *In view of of Lemma 10, assume that $V_n(X, H^X, W^Y)$ satisfies only Assumption $2'$, $D_n(X, H^X, W^Y)$ satisfies only (37) and μ_n is diagonal. Then $V_{n-1}(X, H^X, W^Y)$ satisfies Assumption $2'$.*

Lemma 11. *For a zero-cost market, assume that $V_{n-1}(X, H^X, W^Y)$ is defined by (12) for some $n = \overline{1, N}$ and*

1. *$D_n(X, H^X, W^Y)$ satisfies (36) for $\mathbb{S}^* = \mathbb{S}_n$;*
2. *X_n and Y_n are defined by (30) and (31) respectively.*

Then

1. *If $V_n(X, H^X, W^Y)$ is jointly concave in H^X, W^Y for each X such that $(X, H^X, W^Y) \in \mathbb{S}_n$, then $V_{n-1}(X, H^X, W^Y)$ is jointly concave in H^X, W^Y for each X such that $(X, H^X, W^Y) \in \mathbb{S}_{n-1}$.*

2. *If $V_n(X, H^X, W^Y)$ is concave in W^Y for each X and H^X such that $(X, H^X, W^Y) \in \mathbb{S}_n$ and satisfies Assumption 3, then $V_{n-1}(X, H^X, W^Y)$ is jointly concave in H^X and W^Y for each X such that $(X, H^X, W^Y) \in \mathbb{S}_{n-1}$.*

3. *If $V_n(X, H^X, W^Y)$ is concave in W^Y for each X and H^X such that $(X, H^X, W^Y) \in \mathbb{S}_n$, then $V_{n-1}(X, H^X, W^Y)$ is concave in W^Y for each X and H^X such that $(X, H^X, W^Y) \in \mathbb{S}_{n-1}$.*

Proof. 1) We begin with the first statement. (36) implies, that for any $\alpha \in [0,1]$ the set of Z that can be represented as $\alpha Z_1 + (1-\alpha)Z_2$ with $Z_1 \in D_n(X, H_1^X, W_1^Y)$ and $Z_2 \in D_n(X, H_2^X, W_2^Y)$, belongs to the set

$$D_n' = D_n(X, \alpha H_1^X + (1-\alpha)H_2^X, \alpha W_1^Y + (1-\alpha)W_2^Y).$$

Therefore,

$$V_{n-1}(X, \alpha H_1^X + (1-\alpha)H_2^X, \alpha W_1^Y + (1-\alpha)W_2^Y) = \sup_{Z \in D_n'} \inf_{Q_n \in \mathbb{Q}_n^E}$$
$$\mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z, \alpha W_1^Y \tilde{r}_n + (1-\alpha)W_2^Y \tilde{r}_n - \big(Z - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X \tilde{r}_n) \geq$$

$$\geq \sup_{\substack{Z = \alpha Z_1 + (1-\alpha)Z_2 \\ Z_1 \in D_n(X, H_1^X, W_1^Y) \\ Z_2 \in D_n(X, H_2^X, W_2^Y)}} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z,$$
$$\alpha W_1^Y \tilde{r}_n + (1-\alpha)W_2^Y \tilde{r}_n - \big(Z - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X \tilde{r}_n) \geq$$

$$\geq \sup_{\substack{Z_1 \in D_n(X, H_1^X, W_1^Y) \\ Z_2 \in D_n(X, H_2^X, W_2^Y)}} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, \alpha Z_1 + (1-\alpha)Z_2,$$
$$\alpha W_1^Y \tilde{r}_n + (1-\alpha)W_2^Y \tilde{r}_n - \big(\alpha Z_1 + (1-\alpha)Z_2 - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X \tilde{r}_n) =$$

$$= \sup_{\substack{Z_1 \in D_n(X, H_1^X, W_1^Y) \\ Z_2 \in D_n(X, H_2^X, W_2^Y)}} \inf_{Q_n \in \mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, \alpha Z_1 + (1-\alpha)Z_2,$$
$$\alpha\big[W_1^Y \tilde{r}_n - (Z_1 - H_1)^T X \tilde{r}_n\big] + (1-\alpha)\big[W_2^Y \tilde{r}_n - (Z_2 - H_2)^T X \tilde{r}_n\big]) \geq$$

$$\geq \sup_{\substack{Z_1 \in D_n(X,H_1^X,W_1^Y)\\ Z_2 \in D_n(X,H_2^X,W_2^Y)}} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} \Big[\alpha V_n(X_n, Z_1, W_1^Y\tilde{r}_n - (Z_1 - H_1)^T X\tilde{r}_n) +$$

$$+ (1-\alpha)V_n(X_n, Z_2, W_2^Y\tilde{r}_n - (Z_2 - H_2)^T X\tilde{r}_n)\Big]. \quad (53)$$

Since

$$\inf\big[\alpha f(x) + (1-\alpha)g(x)\big] \geq \alpha \inf f(x) + (1-\alpha)\inf g(x),$$

we obtain

$$V_{n-1}(X, \alpha H_1^X + (1-\alpha)H_2^X, \alpha W_1^Y + (1-\alpha)W_2^Y) \geq$$

$$\geq \alpha \sup_{Z_1\in D_n(X,H_1^X,W_1^Y)} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z_1, W_1^Y\tilde{r}_n - (Z_1 - H_1)^T X\tilde{r}_n) +$$

$$+ (1-\alpha) \sup_{Z_1\in D_n(X,H_2^X,W_2^Y)} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z_2, W_2^Y\tilde{r}_n - (Z_2 - H_2)^T X\tilde{r}_n) =$$

$$= \alpha V_n(X, H_1^X, W_1^Y) + (1-\alpha)V_n(X, H_2^X, W_2^Y).$$

2) The second statement is proven by analogy. The key difference is using Assumption 3 to obtain

$$V_{n-1}(X, \alpha H_1^X + (1-\alpha)H_2^X, \alpha W_1^Y + (1-\alpha)W_2^Y) = \sup_{Z\in D_n'} \inf_{Q_n\in\mathbb{Q}_n^E}$$
$$\mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, Z, \alpha W_1^Y\tilde{r}_n + (1-\alpha)W_2^Y\tilde{r}_n - \big(Z - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X\tilde{r}_n) =$$

$$= \sup_{Z\in D_n'} \inf_{Q_n\in\mathbb{Q}_n^E} \mathbb{E}_{Q_n}^{\mathcal{S}_{n-1}} V_n(X_n, 0, \alpha W_1^Y\tilde{r}_n + (1-\alpha)W_2^Y\tilde{r}_n -$$
$$- \big(Z - \alpha H_1^X - (1-\alpha)H_2^X\big)^T X\tilde{r}_n + Z^T X_n\tilde{r}_n).$$

Note that Z is missing in the second argument of V_n and appears only in the third argument expression which is linear in Z. Therefore we can obtain (53) by using concavity of V_n only in W^Y. The rest is proved by analogy.

3) The proof of the third statement repeats the proof of the first one when $H_1^X = H_2^X = H^X$. $\square$

Obtained Lemmas lead to Theorems 7 and 8, which provide sufficient conditions for the extreme measure problem in the Bellman-Isaacs equation to have an atomic solution.

Proof of Theorem 7. Properties of the constraint sets D_n, $n = \overline{1,N}$, and Statement 4 imply that V_n satisfies Assumption 3 for every $n = \overline{1,N}$. Hence by Lemma 11, concavity in W^Y holds for every $n = \overline{1,N}$. Lemma 8 concludes the proof. □

Proof of Theorem 8. Proof follows Theorem 7 by using corollaries of the above-mentioned Lemmas. □

Proof of Theorem 9. Lemma 10 implies that $V_n(X, H^X, W^Y)$ satisfies Assumptions 2 and 3 for every n, while Lemma 11 implies concavity in W^Y for every n. Then for any $n = \overline{1,N}$

$$V_{n-1}(X, H^X, W^Y) = \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q_n} V_n\Big(s_n X, Z, \big(W^Y - (Z - H^X)X\big)\tilde{r}_n\Big) =$$

$$= \sup_{Z \in D_n} \inf_{Q_n \in \mathbb{Q}_n} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q_n} V_n\Big(X, 0, \big(W^Y - (Z - H^X)X\big)\tilde{r}_n + s_n Z X\Big).$$

Since $0 \in D_n(X, H^X, W^Y)$,

$$V_{n-1}(X, H^X, W^Y) \geq V_n\big(X, 0, (W^Y + H^X X)\tilde{r}_n\big).$$

Consider the measure $Q_n^* \in \mathbb{Q}_n^{E^*}$, where E_n^* is the main diagonal of $(r_n - \mu_n)\sqrt{\Delta t_n}$, $n = \overline{1,N}$, with mass $p_n^{i^*}$ concentrated at the extreme points $G_{0,n}, \ldots, G_{m(K_n),n}$ of K_n. Then we have

$$\sum_{i=0}^{m(K_n)} p_n^{i^*} \sigma_n^j(G_{i,n}) = E_n^{j^*}, \quad j = \overline{1,m}, \quad \Leftrightarrow \quad \sum_{i=0}^{m(K_n)} p_n^{i^*} \operatorname{diag}(G_{i,n}) = (r_n - \mu_n)\sqrt{\Delta t_n} \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \sum_{i=0}^{m(K_n)} p_n^{i^*} \left(s_n(G_{i,n}) - \tilde{r}_n\right) = 0. \tag{54}$$

Since

$$V_{n-1}(X, H^X, W^Y) \leq \sup_{Z \in D_n} \mathbb{E}^{\mathcal{S}_{n-1}}_{Q_n^*} V_n\Big(X, 0, \big(W^Y - (Z - H^X)X\big)\tilde{r}_n + s_n Z X\Big) =$$

$$= \sup_{Z \in D_n} \sum_{i=0}^{m(K_n)} p_n^{i^*} V_n\Big(X, 0, \big(W^Y - (Z - H^X)X\big)\tilde{r}_n + s_n(G_{i,n}) Z X\Big).$$

Concavity in W^Y implies

$$V_{n-1}(X,H^X,W^Y) \le \sup_{Z\in D_n} V_n\Big(X,0,(W^Y+H^X X)\tilde{r}_n - ZX\tilde{r}_n + \sum_{i=0}^{m(K_n)} p_n^{i*} s_n(G_{i,n})ZX\Big) =$$

$$= \sup_{Z\in D_n} V_n\Big(X,0,(W^Y+H^X X)\tilde{r}_n + \sum_{i=0}^{m(K_n)} p_n^{i*}\big(s_n(G_{i,n})-\tilde{r}_n\big)ZX\Big) = V_n\big(X,0,(W^Y+H^X X)\tilde{r}_n\big).$$

Therefore, $V_{n-1}(X,H^X,W^Y) = V_n\big(X,0,(W^Y+H^X X)\tilde{r}_n\big)$ and the maximum is achieved at $H_n^{X*}=0$ which proves the statement. □

Lemma 12. *Assume that $V_{n-1}(X,H^X,W^Y)$ is defined by (12) for some $n=\overline{1,N}$ and*

1. *$D_n(X,H^X,W^Y)$ satisfies (37) for $\mathbb{S}^*=\mathbb{S}_n$;*
2. *$V_n(X,H^X,W^Y)$ satisfies Assumption 2′ for $\mathbb{S}^*=\mathbb{S}_n$;*
3. *$C_{n-1}(H^X,X)$ satisfies (42);*
4. *X_n and Y_n are defined by (30) and (31) respectively;*
5. *μ_n is diagonal.*

Then $V_{n-1}(X,H^X,W^Y)$ satisfies Assumption 2′ for $\mathbb{S}^*=\mathbb{S}_{n-1}$;

Proof. Proof follows Lemma 10, since the transaction costs function satisfies (42). □

Lemma 13. *Assume that $V_{n-1}(X,H^X,W^Y)$ is defined by (12) for some $n=\overline{1,N}$ and*

1. *$D_n(X,H^X,W^Y)$ satisfies (36) for $\mathbb{S}^*=\mathbb{S}_n$;*
2. *$V_n(X,H^X,W^Y)$ is non-decreasing in W^Y for each X,H^X such that $(X,H^X,W^Y)\in\mathbb{S}_n$;*
3. *$C_{n-1}(H,X)$ is convex in H for each $X\in\mathbb{R}_+^m$;*
4. *X_n and Y_n are defined by (30) and (31) respectively;*
5. *μ_n is diagonal.*

Then

1. *If $V_n(X,H^X,W^Y)$ is jointly concave in H^X,W^Y for each X such that $(X,H^X,W^Y)\in\mathbb{S}_n$, then $V_{n-1}(X,H^X,W^Y)$ is jointly concave in H^X,W^Y for each X such that $(X,H^X,W^Y)\in\mathbb{S}_{n-1}$.*

2. *If* $V_n(X, H^X, W^Y)$ *is concave in* W^Y *for each* X *and* H^X *such that* $(X, H^X, W^Y) \in \mathbb{S}_n$*, then* $V_{n-1}(X, H^X, W^Y)$ *is concave in* W^Y *for each* X *and* H^X *such that* $(X, H^X, W^Y) \in \mathbb{S}_{n-1}$.

Proof. Proof of the first statement follows Lemma 11 since $-C_{n-1}(H, X)$ is concave in H and V_n is non-decreasing in W^Y. Proof of the second statement repeats proof of the first one when $H_1^X = H_2^X = H^X$. $\square$

Lemma 14. *Assume that* $V_{n-1}(X, H^X, W^Y)$ *is defined by* (12) *for some* $n = \overline{1, N}$ *and*

1. $V_n(X, H^X, W^Y)$ *is non-decreasing in* W^Y *for each* X, H^X *such that* $(X, H^X, W^Y) \in \mathbb{S}_n$*;*
2. $V_n(X, H^X, W^Y)$ *satisfies Assumption* $2'$ *for* $\mathbb{S}^* = \mathbb{S}_n$*;*
3. $V_n(X, H^X, W^Y)$ *is jointly concave in* H^X, W^Y *for each* X *such that* $(X, H^X, W^Y) \in \mathbb{S}_n$*;*
4. $C_{n-1}(H, X)$ *satisfies Assumption 4;*
5. X_n *and* Y_n *are defined by* (30) *and* (31) *respectively;*
6. μ_n *is diagonal.*

Then (33) *holds for the given* n.

Proof. Proof follows Lemma 9 since $-C(H, X)$ is concave in H and V_n is non-decreasing in W^Y. $\square$

Proof of Theorem 10. By analogy to Theorem 8, Theorem 10 is derived from the corresponding Lemmas. $\square$

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