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FORMATION OF COALITION STRUCTURES AS A NON-COOPERATIVE GAME

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Formation of coalition structures as a non-cooperative game

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Abstract

The paper defines a family of nested non-cooperative simultaneous finite games to study coalition structure formation with intra and inter-coalition externalities. The novelties of the paper are: a definition of every games embeds a *coalition structure formation mechanism*. E every game has two outcomes - an allocation of players over coalitions and a payoff profile for every player.

The family is parametrized by a maximum coalition size in every coalition structure (a partition) in a game. For every partition a player

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has a partition-specific set of strategies. The mechanism portions a set of strategies of the game (a Cartesian product) into partition-specific strategy domains, what makes every partition to be itself a non-cooperative game with partition-specific payoffs for every player. Payoffs are assigned separately for every partition and are independent from the mechanism.

Every game in the family has an equilibrium in mixed strategies. The equilibrium can generate more than one coalition and encompasses intra and inter group externalities, what makes it different from the Shapley value. Presence of individual payoff allocation makes it different from a strong Nash, coalition-proof equilibrium, and some other equilibrium concepts. The paper demonstrate some applications of the proposed toolkit: for non-cooperative fundamentals of cooperation in a coalition, Bayesian game, stochastic games and construction of a non-cooperative criterium of coalition structure stability.

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1 Introduction

The research topic of this paper was inspired by John Nash's "Equilibrium Points in n-person games "(1950). This remarkably short, but highly influential note of only 5 paragraphs established an equilibrium concept and the proof of its existence which did not require an explicit specification of a final coalition structure for a set of players. Prior to Nash's paper, the generalization of the concept of equilibrium provided by von Neumann for the case of two-players zero-sum game was done by portioning the players into two groups and regarding several players as a single player, without an explicit individual payoff. However up to now, non of these approaches resulted into expected progress in studying intra- and inter group externalities between players.

Cooperative game theory traditionally disregards issues of the strategic interactions between individual players, between players of a coalition and from players in other coalitions. Non-cooperative game theory does overlook the structural aspects of player's partition into groups or into coalition structures.[1] However, such strategic interactions of agents located to different groups or from the same group, currently out of consideration in a traditional game-theoretic framework, could be of importance in a wide range of applications. Further, we present here an example named as a "dinner game" to illustrate importance of these effects.

To illustrate some problems with the existing approachs consider the two examples - a voluntarily division of a group of participants into paintball teams and a voluntarily division of a class into studying groups.

In both examples every player makes a decision from self-interest considerations. Players make individual choices about a team/group and what to do in the team/group including free-riding. There is no need that at the same time every member of any team/group does the same as other members of the same team/group. Thus both examples contain intra-coalition externalities inside a team or a group.

There is no need to explain inter-coalition externalities in the paintball game. For studying groups the inter-coalition externalities can appear from a simultaneous access of everybody to a limited WiFi connection. In every example players are allocated in more than one group. Individual payoff depends on actions of everybody, disregarding a group allocation. Thus we have two examples of games, where players make voluntarily participation choices and are exposed for two types of externalities. There is more than one coalition in every game. So these examples go beyond legacies of any theory above.

In this paper "coalition structure", or "partition"[2] for short, is a collection

[1]Coalition structure terminology was used by Aumann and Dreze (1976).
[2]Existing literature uses both terms.

of non-overlapping subsets from a set of players, which in a union make the original set. A group, or a coalition, is an element of a coalition structure or of a partition.[3]

A partition induces two types of externalities on a player's payoff. The first, through actions of players of the same coalition. These effects will be addressed as *intra*-coalition externalities. The second, from all other players, who are outside coalition, and belong to different coalitions. This effects will be addressed as *inter*-group externalities.

Nash (1953) suggested that cooperation should be studied within a group and in terms of non-cooperative fundamentals. This suggestion is known now as the Nash Program. Cooperative behavior was understood as an activity inside a group with positive externalities between players. Nash did not write explicitly about multi-coalition framework or coalition structures. Coalition structures allow us to study inter-coalition externalities, along with intra-coalition ones, and to separate cooperation in payoffs and cooperation in allocation of players.

The best analogy for the difference between Nash Program and the current research is the difference between a partial and a general strategic equilibrium analysis[4] in economics. The former isolates a market ignoring cross-market interactions, the latter explicitly studies cross market interactions from individual strategic actions of self-interested traders.

The research agenda of the paper is: how to construct coalition structures from actions of self-interested agents. Moreover, the paper offers a generalization of a non-cooperative game from Nash (1950) to address the problem of coalition structure formation absent in Nash (1953). The contributions of this paper are: a construction of a *family* of non-cooperative games and a parametrization of all constructed games by a number of deviators. Every game has an equilibrium in mixed strategies.

[3]the same.

[4]meaning, strategic market games of Shapley and Shubik

Relation of the current approach to existing literature is in Discussion section of the paper. It explains problems with existing approaches and how to overcome then using the suggested model.

Every constructed non-cooperative game of the family has two outcomes: an allocation of players over coalitions and individual payoffs for every player. Thus in every game every player is exposed for intra and inter-coalition externalities. An equilibrium (in mixed strategies) exists for every game in the family and is based only on individual motivation of every player. These are the most important differences from existing equilibrium concepts, including the strong Nash, coalition-proof equilibria, cores, kernels, nucleolus, bargaining sets, etc.

The paper has the following structure: Section 2 presents an example, on why studying inter-coalition externalities requires including coalition structures into individual strategy sets, Section 3 presents a general model of the game, Section 4 presents an example of the game. The last section discusses the approach used and the results of the paper.

2 A dinner game

The example below demonstrates two ideas. First, a coalition structure, i.e. an allocation of players over coalitions, can be considered as a strategy. This is different from Hart and Kurz (1978). Second, a payoff of all players in a coalition, i.e. and a value of the coalition, may depend on the whole coalition structure due to appearance of inter-coalition externalities. This is different from assumptions of cooperative game theory.

Assume there is an in-house office cafeteria, a live experiment setting for an observer to gain valuable insights into the inner (social) workings. There is no pre-defined sitting arrangements, and the visitors choose the table at which to sit simultaneously from personal preferences. Such personal preferences are related to (i) interactions in different groups of co-workers, and (ii) individual preferences over placement for others (i.e Mike would like to sit with Kate, Kate would like to sit with Mike, but Kate does not want Jane to sit with Jim, and Kate can not prevent it). The resulting sitting arrangement (placement structure) comes from individual choices of everybody.[5]

We assume that a visitor can exhibit a wide variety of social activity while dining, depending on the table co-sitters (table-mates). Thus, a realized activity of a visitor is an outcome of a sitting pattern, and one can say that the visitor produces table-specific (intra-coalition) externalities for his table-mates. The very same activity of the visitor produces extra-coalition externalities for each visitor at any other table. Hence, the dining satisfaction of any visitor at any table depends both on activities of his table-mates and on activities of all visitors other than that sitting at the same table.

For simplicity we assume that everybody independently and simultaneously enters into the cafeteria, makes all the decisions also independently, but concerns about possible decisions of others.

For simplicity the model skips an adjustment period, when players find out choices of others and make allocation arrangements. The adjustment period is substituted by an instant application of sitting (placement) arrangements rules. A rule deals only with a set of strategies, but not with dinner satisfaction from them.

Let everybody knows the simplest rule, a table can be formed only from the mutual agreement of everybody at this table, disregarding of what happens at other tables. If there is a conflict, no one has a bargaining power, and a visitor with a conflict sits alone. For example, if Mike would like to sit only with Kate, whatever is outside, Kate would like to sit with both

[5] The basic assumption is that an individual action of every self-interest agent, located at one coalition can have a non-negligible echo in enjoyment for every other, located at some different coalitions. Construction of the example disregards possible negligibility of these effects. The presence of negligible echoes leaves a space for a co-existence of non-cooperative and cooperative game theories, what is similar for the co-existence of quantum and statistical approach to multi-particle systems.

Mike and Jim, whatever is outside, and Jim only with Jane, without Mike, whatever is outside, then everybody sits alone, as Kate can not sit at two different tables. The same arrangement occurs if Jim chooses to sit with both Mike and Jane. But if Mike would like to sit only with Kate, whatever is outside, *and* Kate would like to sit only with Mike, but Jim with Kate (or with Jim and Kate), then Mike and Kate will sit together and Jim alone. However any outside sitting arrangement has an impact on the final dining satisfaction of Mike, Jane and Jim. A change in the sitting arrangement rule make a different game.

The example embeds possible sitting arrangements (i.e. an allocation of visitors over coalitions) into an individual strategy set. This approach is different from one suggested by Hart and Kurz (1986), where a strategy is a coalition, i.e. in term of the example, a table. In this way Hart and Kurz loose possible existence of inter-table externalities, which may have an effect of coalition formation.

2.1 A game

Consider a game of 4 agents: A is a President; B is a senior vice-president; C_1, C_2 are two other vice-presidents. Agents differ by attractiveness for others to eat together the corporate dinner, but they are have equal bargaining power in a choice of table-mates.[6] Every agent may sit only at one table. As a coalition we take players at *one* table. A coalition structure is an allocation of all four, A,B,C and D, over tables. Naturally empty tables are not taken into account.

Individual set of strategies of every agent is a set of all coalition structures presented in Table 1. In other words, a strategy is a coalition structure, or every coalition structure has only one strategy. All players have identical set of strategies. The players differ by preferences over coalition structures. All individual preferences are known to everybody.

[6]Non-equal bargaining powers will make complicate rules of sitting arrangements.

Table 1: Strategies in the game is a set of all possible coalition structures, or allocations of players over non-overlapping coalitions, or allocations of players over different tables. Parameter K is a maximum coalition size in every coalition structure.

line number	K	coalition structure (strategy)
0	$K = 1$	$\{\{A\}, \{B\}, \{C_1\}, \{C_2\}\}$
1	$K = 2$	$\{\{A, B\}, \{C_1\}, \{C_2\}\}$
2		$\{\{A, B\}, \{C_1, C_2\}\}$
3		$\{\{A, C_1\}, \{B, C_2\}\}$
4		$\{\{A, C_1\}, \{B\}, \{C_2\}\}$
5		$\{\{A, C_2\}, \{B, C_1\}\}$
6		$\{\{A, C_2\}, \{B\}, \{C_1\}\}$
7		$\{\{A\}, \{C_2\}, \{B, C_1\}\}$
8		$\{\{A\}, \{B\}, \{C_1, C_2\}\}$
9		$\{\{A\}, \{C_1\}, \{B, C_2\}\}$
10	$K = 3$	$\{\{A, B, C_1\}, \{C_2\}\}$
11		$\{\{A, B, C_2\}, \{C_1\}\}$
12		$\{\{A, C_1, C_2\}, \{B\}\}$
13		$\{\{B, C_1, C_2\}, \{A\}\}$
14	$K = 4$	$\{A, B, C_1, C_2\}$

In Table 1 strategies (or coalition structures) are grouped by a maximum coalition size of a coalition structure, K. Parameter K restricts a set of strategies, i.e. parametrizes the set of all possible coalition structures by a maximum number of agents at a table. All agents have the same set of strategies for a fixed K. For four agents the variable K has four values, $K = \{1, 2, 3, 4\}$. An increase in K adds new available coalition structures.

In a game for a fixed K an agent chooses one coalition structure, an element from his strategy set for this game. A set of strategies of a game with a fixed K is a direct (Cartesian) product of individual strategy sets of all four agents for this game. In a game with a fixed K a choice of all agents is a point in this Cartesian product. A realization of a final partition (a coalition structure) depends on choices of all agents. For simplicity we take an exogenous rule[7] for an allocation of agents into a final coalition structure, as follows.

Rule 1 If two agents choose different coalition structures, but in these coalition structures they are in the same coalition, then the agents obtain this coalition disregard choices of others.

Rule 1′ If three (four) agents choose different coalition structures, but in these coalition structures some choose each other, then these agents obtain this coalition disregard choices of others. The rest from these three (four) stay alone.

Rule 2 If an agent chooses a coalition structure, but in this coalition structure he is in a coalition with other agents, who do not choose the same coalition in their coalition structures, then the agent obtains a singleton coalition, i.e. eats alone, disregard choices of others.

[7]Formal properties of such rules deserve special research. It is important for social welfare maximization for a family of non-cooperative games in the next Section. The rule is chosen for simplicity. Different rule will induce a different final coalition structure.

Rule 3 If an agents chooses a coalition structure, where he is alone (eats alone), then he stays alone disregard choices of others.

Some example are presented below. They serve for an illustrative purpose only, but not to demonstrate an equilibrium of the game or a social welfare maximization.

Example 1. Let agent A choose a coalition structure (a strategy) $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 1 in Table 2); agent B choose $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 1 in Table 2); agent C_1 choose $\{\{A, C_1\}, \{B\}, \{C_2\}\}$ (line 4 in Table 2); and agent C_2 choose $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 6 in Table 2). The final coalition structure or a final partition is constructed as:

1. For agents A, B we apply Rule 1 and obtain a coalition $\{A, B\}$ in a final coalition structure.

2. For agent C_1 we apply Rule 2 and obtain a coalition $\{C_1\}$ in a final coalition structure.

3. For agent C_2 we apply Rule 3 and obtain a coalition $\{C_2\}$ in a final coalition structure.

The final coalition structure is $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 1 in Table 2).

Example 2. Let agent A choose a coalition structure (a strategy) $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 1 in Table 2); agent B choose $\{\{A, B, C_1\}, \{C_2\}\}$ (line 10 in Table 2); agent C_1 choose $\{\{A, C_1\}, \{B\}, \{C_2\}\}$ (line 4 in Table 2); and agent C_2 choose $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 6 in Table 2). The final coalition structure or a final partition is constructed as:

1. For agents A, B, C_1 we apply Rule $1'$ and obtain coalitions $\{A, B\}$, $\{C_1\}$ in a final coalition structure, as B does not choose C_1.

2. For agent C_2 we apply Rule 3 and obtain a coalition $\{C_2\}$ in a final coalition structure.

The final coalition structure is $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 1 in Table 2).[8]

Example 3 differs from Example 1 in a strategy for C_1. Let agent A choose a coalition structure (a strategy) $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 1 in Table 2); agent B choose $\{\{A, B, C_1\}, \{C_2\}\}$ (line 10 in Table 2); agent C_1 choose $\{\{A, C_2\}, \{B, C_1\}\}$ (line 5 in Table 2); and agent C_2 choose $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 6 in Table 2). The final coalition structure or a final partition is constructed as:

1. For agents A, B, C_1 we apply Rule $1'$ and obtain coalitions $\{A\}, \{B\}, \{C_1\}$ in a final coalition structure, as B can not belong to two different coalitions, what is different from Example 2.

2. For agent C_2 we apply Rule 3 and obtain a coalition $\{C_2\}$ in a final coalition structure.

The final coalition structure is $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ (line 1 in Table 2).

The used rule structures a set of strategies into domains. A domain of strategies corresponds to only one coalition structure.

Table 2 contains additional information for the game, payoffs (column 4) and values of coalitions (column 5). Payoffs have the following interpretation. Every agent (besides A) would like to have a dinner together with A, but A only with B. The next ranked alternatives are to have a dinner with B or either with C_1 or C_2 or alone. At the same time every agent wants players outside his table to eat individually, due to possible dissipation of rumors, information exchange or collusion. For every agent a dinner with two or three others is inferior in comparison to any coalition structure with any two-agent coalition. The game is non-cooperative by construction, hence no one can enforce others to form or not to form coalitions.

The one period game is run for a fixed K, is simultaneous, and goes as follows. Agents simultaneously announce individually chosen coalition

[8]Choice of every agent we taken only for an illustrative purpose for the rule, but not to illustrate an equilibrium of the game.

structures for K. A profile of their choices is a point in a set of strategies of all agents. This point is assigned a final coalition structure following the rules above, and a payoff for every agent is assigned according to column 4 from Table 2.

An increase in K adds more feasible strategies for every agent. However payoff (column 4 Table 2) for neither agent increases after $K = 2$, hence every agent will choose only strategies with $K = 2$. Thus we discuss the game for $K = 4$ with all available coalition structures.

We can easily see that agents A and B would like to choose a coalition structure number 1 (line 1 in Table 2), where they obtain the highest payoffs. But this leads to the conflict with the first best choices of agents C_1 and C_2, who would like to choose coalition structures with A (lines 4 and 6). More of that, these choices of C_1 and C_2 are mutually inconsistent and can not be realized. The A or B make C_1 and C_2 be worse off. Thus, the best agents C_1 and C_2 can do is to choose a coalition structure with each other[9], i.e. to choose a coalition structure $\{\{A\},\{B\},\{C_1, C_2\}\}$ (line 8 in Table 2) or a $\{\{A, B\},\{C_1, C_2\}\}$ (line 2 in Table 2). Line 8 is preferable for C_1 and C_2, but they can not prohibit players A and B be to form a coalition. So the line 8 can not be realized, but only line 2. Thus the rule of unilateral agreement results in a coalition structure $\{\{A, B\},\{C_1, C_2\}\}$ from individual action of every agent given coalition structure formation rule.

Agents A and B will obtain an equilibrium coalition $\{A, B\}$, but this will make them worse off in comparison to their first best choice, where C_1 and C_2 eat separately. Agents A and B can not enforce C_1 and C_2 to eat separately.

So in equilibrium agents A and B can choose either strategies 1 or 2, agents C_1, C_2 can choose either strategy 2 or 8, and the final coalition structure will be strategy 2. Equilibrium mixed strategies for both agents are probability spaces over two points in individual strategy sets.

[9]In sociology this behavior is referred as a cooperation: agents C_1 and C_2 group together against inferior outcomes for every, agent A will never choose them. This problem will be addressed further in this paper.

An outcome of the game is an equilibrium allocation of agents over two coalitions $\{\{A, B\}, \{C_1, C_2\}\}$ with an equilibrium payoff profile, $(8, 8, 5, 5)$. Every agent obtains a second-best payoff in comparison to a desirable due to strategic actions of some other agents.

The reasons, why coalition structures should be incorporated into a strategy set, should be clear now. First, there are inter-coalition externalities, which can be different in different coalition structures. Compare payoffs for A and B in coalition structures $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ and $\{\{A\}, \{B\}, \{C_1, C_2\}\}$.

Second, necessity to navigate in a set of coalition structures. If an agent, say A, chooses only a coalition, $\{A, B\}$ for A, then A can not discriminate between two different strategies (or coalition structures), $\{\{A, B\}, \{C_1\}, \{C_2\}\}$ and $\{\{A\}, \{B\}, \{C_1, C_2\}\}$, where A obtains different payoffs. The same can be said about every agent in the game.

The fifth column in Table 2 is a list of values for coalitions in coalition structures if to calculate values using cooperative game theory. We can see that the same coalition may have different value in different coalition structure.[10] Shapley value (1953) can not be applied, as it assumes superadditivity of payoffs, what is absent in the game. The result is also different from partition approach (Yi, 1999), as, for example, there is no initial allocation of players over coalitions.

Application of core analysis (Aumann, 1960) is also inappropriate here. Additionally to coalition payoffs, which depend on coalition structures, there are other differences. The example allows to describe a deviation of many players, from different coalitions, and a variety of interactions of these players. There is any well-constructed reasoning for these cases in the cooperative game theory.

The final (equilibrium) coalition structure contains two coalitions in an equilibrium. The same arguments, especially absence of two types of ex-

[10] Thus it is not clear how to transform the game in a coalition from as the same coalition has different total value in different coalition structures.

Table 2: Strategies and payoffs in the corporate dinner game

K	num	Best final partitions	Non-cooperative payoff profile $(U_A, U_B, U_{C_1}, U_{C_2})$	Values of coalitions as in cooperative game theory
$K=1$	0	$\{\{A\},\{B\},\{C_1\},\{C_2\}\}$	(1,1,1,1)	1_A, 1_B, 1_{C_1}, 1_{C_2}
$K=2$	1	$\{\{A,B\},\{C_1\},\{C_2\}\}$	(10,10,3,3)	20_{AB}, 3_{C_1}, 3_{C_2}
	2*	$\underline{\{\{A,B\},\{C_1,C_2\}\}}$	(8,8,5,5)	16_{AB}, $10_{C_1,C_2}$
	3	$\{\{A,C_1\},\{B,C_2\}\}$	(3,3,5,5)	8_{AC_1}, 8_{BC_2}
	4	$\{\{A,C_1\},\{B\},\{C_2\}\}$	(3,3,10,3)	13_{AC_1}, 3_B, 3_{C_2}
	5	$\{\{A,C_2\},\{B,C_1\}\}$	(3,3,5,5)	8_{AC_2}, 8_{BC_1}
	6	$\{\{A,C_2\},\{B\},\{C_1\}\}$	(3,3,3,10)	13_{AC_2}, 3_B, 3_{C_2}
	7	$\{\{A\},\{C_2\},\{B,C_1\}\}$	(3,5,8,3)	3_A, 3_{C_2}, 13_{BC_1}
	8	$\underline{\{\{A\},\{B\},\{C_1,C_2\}\}}$	(5,5,8,8)	5_A,5_B, $16_{C_1,C_2}$
	9	$\{\{A\},\{C_1\},\{B,C_2\}\}$	(3,5,3,8)	3_A, 3_{C_1}, 13_{BC_2}
$K=3$	10	$\{\{A,B,C_1\},\{C_2\}\}$	(0,0,0,0)	0_{ABC_1}, 0_{C_2}
	11	$\{\{A,B,C_2\},\{C_1\}\}$	(0,0,0,0)	0_{ABC_2}, 0_{C_1}
	12	$\{\{A,C_1,C_2\},\{B\}\}$	(0,0,0,0)	$0_{AC_1C_2}$, 0_B
	13	$\{\{B,C_1,C_2\},\{A\}\}$	(0,0,0,0)	$0_{BC_1C_2}$, 0_A
$K=4$	14	$\{A,B,C_1,C_2\}$	(0,0,0,0)	$0_{A,B,C_1,C_2}$

ternalities, are the differences from the strong Nash equilibrium (Aumann, 1959), and the coalition-proof equilibrium (Bernheim, Peleg, and Whinston, 1977).

The game above is constructed in a way that all coalition structures are parametrized by a size of a maximum coalition size, K. An increase in K adds new coalition structures, new strategies and new payoffs. Thus we may say that an increase in K results in a new game. Such games we suggest to name as nested games, formally introduced further in this paper. In the dinner game above an increase of K above $K = 2$ does not make agents change their equilibrium strategies. This is the way to construct non-cooperative coalition structure stability criterium presented in this paper.

2.2 Conclusion from the example

The dinner game is constructed from the following assumptions.

1. A set of allocation of players over coalitions is an individual strategy set, and a player makes a choice over it.

2. To reconcile possibly contradicting choices of players we use exogenously given Nash umpire, Nash (1953), or a rule for allocation of all players over coalitions structures. It takes choices of all players over coalition structures (a strategy profile of a game) and assigns a final coalition structure with transportation of chosen strategies inside this coalition structure. This rule does not solve a social welfare maximization problems, but only eliminates possible conflict between players, like in Nash (1953).

3. Construct a set of all strategies of a game in a standard way (with a Cartesian product). A choice of all agents, a strategy profile, is a point in this Cartesian product. The Cartesian product is structured or divided by rule of allocation of players into non-overlapping subsets.

Each subset corresponds to only one allocation of players over coalitions. Every such allocation becomes a non-cooperative game itself.

The same in another way: for every point in the Cartesian product the central planner assigns an allocation of visitors over tables (coalitions).

The approach is similar to the idea of Arrow to construct state-contingent payoffs, where an agent has an action for every contingent outcome.

4. Every point in the Cartesian product is assigned a vector of payoffs. At the same time every point is assigned an allocation of agents over coalitions. Thus an allocation of players and a profile of payoffs are constructed independently[11] over the same appropriately constructed set of strategies.

The game demonstrates importance of interaction of players for construction of coalition structures. The theory of this type of games is in the next section.

3 Formal setup of the model

Nash (1950, 1951) suggested a non-cooperative game which consists of a set of players N, with a general element i, sets of individual finite strategies $S_i, i \in N$, and payoffs, defined as a mapping from a set of all strategies into a set of payoff profiles of all players, $\Big(U_i(s)\Big)_{i \in N}$, such that $S = \times_{i \in N} S_i \mapsto \Big(U_i(s)\Big)_{i \in N} \subset \mathbb{R}^{\#N}$, where $\Big(U_i(s)\Big)_{i \in N} < \infty, \forall s \in S$.

The suggested game modifies the mapping by preliminary portioning S into coalition structure specific domains and assigning payoffs for every point in these domains. The division of S is done with a coalition structure formation mechanism, defined further. The coalition structures are parametrized by K, a maximum coalition size.

[11]Independently is only in a formally syntactic sense, not in semantic or pragmatic!

Let there is a set of agents N, with a general element i, a size of N is $\#N$, a finite integer, $2 \leq \#N < \infty$.

There is an index K, which will serve as a parameter, $K \in \{1, \ldots, \#N\}$. This parameter has two interpretations. Let for N agents there is a coalition with a maximum size K. Then no more than K agents are required to dissolve it. The reverse is also true: we need no more than K agents to form this maximum size K coalition. Closeness of a construction of the object under investigation requires these two simultaneous interpretations for K be equivalent.

Every value of K from the set $\{1, \ldots, \#N\}$ induces a family of coalition structures (or a family of partitions) $\mathcal{P}(K)$ over the set of all players N:

$$\mathcal{P}(K) = \Big\{ \{P = \{g_j \colon g_j \subset N; \#g \leq K; \cup_j g_j = N; \forall j_1 \neq j_2 \Rightarrow g_{j_1} \cap g_{j_2} = \emptyset\} \Big\}.$$

An element P of $\mathcal{P}(K)$ is a partition of players into coalitions, $P = \{g_j\}$, where g_j is a coalition. Every coalition g_j has a size (a number of members) no bigger than K, but can be less. The condition $j_1 \neq j_2 \Rightarrow g_{j_1} \cap g_{j_2} = \emptyset$ is interpreted as that an agent can participate only in one coalition.

If we increase value of the parameter K by one, then we need to add partitions for a difference between the families, i.e. for the $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. This makes families of partitions for different K be nested: $\mathcal{P}(K = 1) \subset \ldots \subset \mathcal{P}(K) \subset \ldots \subset \mathcal{P}(K = N)$. The bigger is K, the more coalition structures (or partitions) are involved into consideration. The grand coalition, i.e. a coalition of size N, which includes all players, is present only in the family $\mathcal{P}(K = N)$.

For every partition P an agent i has a finite strategy set $S_i(P)$.[12] In the dinner game a player had only one strategy per a coalition structure. In the example of the next section a player has two strategies per a coalition structure.

[12]Finite strategies are chosen as used in Nash (1950).

A set of strategies of agent i for a family of coalition structures $\mathcal{P}(K)$ is

$$S_i(K) = \Big\{ s_i(K) \colon s_i(K) \in \{S_i(P) \colon P \in \mathcal{P}(K)\} \Big\}$$

with a general element $s_i(K)$. For a given K an agent chooses $s_i(K)$ from $S_i(K)$. *A choice of $s_i(K)$ means a choice of a desirable partition and an action for this partition.*[13] If we increase the parameter K by one, then we need to construct additional strategies only for the newly available coalition structures from $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. This makes strategy sets for different K be nested: $S_i(K=1) \subset \ldots \subset S_i(K) \subset \ldots \subset S_i(K=N)$. All players make choices simultaneously.

The set of strategies of all players for a fixed K is $S(K) = \times_{i \in N} S_i(K)$, a direct (Cartesian) product of individual strategy sets of all players for the given K. A choice of all players $s(K) = \Big(s_i(K), \ldots, s_N(K)\Big)$ is a point in $S(K)$. For simplicity if there is no ambiguity we will write $s = \Big(s_1, \ldots, s_N\Big) \equiv s(K) = \Big(s_i(K), \ldots, s_N(K)\Big)$. It is clear that an increase in K induces nested strategy sets: $S(K=1) \subset \ldots \subset S(K=N)$.

The dinner game example has the simplest case: every coalition structure was exactly one strategy for a player. An increase in a number of people at a table increased a number of possible coalition structures and individually feasible strategies.

One may ask a question - is it possible to make a choice in two stages: first, all players choose a partition, then everybody chooses a coalition, Such reformulation of the game leads to the concept of strategic equilibrium, introduced by Mertens (1995), Hillas, Kohlberg (2002). The current paper avoids introduction of too many concepts.

The dinner game demonstrates that we need a mechanism, which resolves possible conflicts between choices of all players. One may think about it

[13] A desirable partition may not realize due to a conflicts in individual choices. A coalition structure formation mechanism resolves conflicts of partition choices between players.

as an enforcement in terms of Nash (1953). For every value of K from the set $\{1, \ldots, \#N\}$ we define a coalition structure formation mechanism (a mechanism or a rule for short) $\mathcal{R}(K)$. For every strategy profile $s = (s_1, \ldots, s_N) \in S(K)$ the mechanism assigns a final coalition structure P, $P \in \mathcal{P}(K)$, and transports the strategy profile s into the partition P. Further we will see that this makes every coalition structure or a partition P to be a standard non-cooperative game, as every P will have a set of players, a non-trivial set of strategies and partition specific payoffs over it's strategy set.

Definition 1. *For every K* ***a coalition structure formation mechanism*** *$\mathcal{R}(K)$ is a set of measurable mappings such that:*

1. *A domain of $\mathcal{R}(K)$ is a set of all strategy profiles of $S(K)$.*
2. *A range of $\mathcal{R}(K)$ is a finite number of subsets $S(P) \subset S(K)$, $P \in \mathcal{P}(K)$. Every $S(P)$ is a strategy set for a coalition structure P.*
3. *$\mathcal{R}(K)$ divides $S(K)$ into coalition structure specific strategy sets, such that the union of all $S(P)$ makes the original set $S(K) = \cup_{P \in \mathcal{P}(K)} S(P)$.*
4. *Two different coalition structures, $\bar{P}$ and $\tilde{P}$, $\bar{P} \neq \tilde{P}$, have different coalition structure strategy sets $S(\bar{P}) \cap S(\tilde{P}) = \emptyset$.*

Formally the same:

$$\mathcal{R}(K) \colon S(K) = \times_{i \in N} S_i(K) \mapsto \colon \begin{cases} S(K) = \cup_{P \in \mathcal{P}(K)} S(P), \\ \forall s = (s_1, \ldots, s_N) \in S(K) \\ \exists P \in \mathcal{P}(K) \colon s \in S(P), \\ \forall \bar{P}, \tilde{P} \in \mathcal{P}(K), \bar{P} \neq \tilde{P} \Rightarrow S(\bar{P}) \cap S(\tilde{P}) = \emptyset. \end{cases} .$$

Hence there are two ways to construct $S(K)$: in terms of initial individual strategies $S(K) = \times_{i \in N} S_i(K)$, and in terms of realized partition strategies

$S(K) = \cup_{P \in \mathcal{P}(K)} S(P)$. Representation of $S(K)$ in terms of coalition structure specific strategy sets may not be a direct product of sets, see an example in the next section.

If K increases we need to add only a mechanism for strategy sets from $S(K+1) \setminus S(K)$. This supports consistency of coalition structure formation mechanisms for different K. The family of mechanisms is nested: $\mathcal{R}(K = 1) \subset \ldots \subset \mathcal{R}(K) \subset \ldots \subset \mathcal{R}(K = N)$. We assume that a mechanism is given from outside. One may observe that the construction of partitions and strategy sets with an increase in K follows the plan of the Dinner game.

Payoffs in the game are defined as state-contingent payoffs (or payoffs of Arrow-Debreu securities) in finance. For every coalition structure P player i has a payoff function $U_i(P) \colon S(P) \to \mathbb{R}_+$, such that every set $U_i(P)$ for every $i \in N$ is bounded, $U_i(P) < \infty$. Payoffs are considered as von Neumann-Morgenstern utilities. All payoffs of i for a game with no more than K deviators make a family: $\mathcal{U}_i(K) = \Big\{ U_i(P) \colon P \in \mathcal{P}(K), U_i(P) < \infty \Big\}$. Every coalition structure has it's own set of strategies and a corresponding set of payoffs. Thus every coalition structure is a non-cooperative game.

An increase in K increases a number of possible partitions and a set of feasible strategies for every player. We need to add only payoffs for the partitions in $\mathcal{P}(K+1) \setminus \mathcal{P}(K)$. Thus we obtain a nested family of payoff functions:

$$\mathcal{U}_i(K = 1) \subset \ldots \subset \mathcal{U}_i(K) \subset \ldots \subset \mathcal{U}_i(K = N).$$

We can easily see that this construction of payoffs allows to obtain both intra and inter coalition (or group) externalities, as payoffs are defined directly over strategy profiles of all players and independent from allocation of players in coalition structures.

Definition 2 (**a simultaneous coalition structure formation game**). *A non-cooperative game for coalition structure formation with a maximum* *coalition size K is*

$$\Gamma(K) = \Big\langle N, \Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}, \Big(S_i(K), \mathcal{U}_i(K)\Big)_{i\in N}\Big\rangle,$$

where $\Big\{K, \mathcal{P}(K), \mathcal{R}(K)\Big\}$ *- coalition structure formation mechanism (a social norm, a social institute),* $\Big(S_i(K), \mathcal{U}_i(K)\Big)_{i\in N}$ *- properties of players in* N*, (individual strategies and payoffs), such that:*

$$\times_{i\in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \Big\{S(P) \colon P \in \mathcal{P}(K)\Big\} \to \Big\{(\mathcal{U}_i(K))_{i\in N}\Big\}.$$

The mechanism of games construction is simple and natural.

The novelty of the paper is an introduction of coalition structure formation mechanism, which portions the set of all strategies into non-cooperative partition-specific games. If we omit the mechanism part of the game and eliminate restriction on coalition sizes then we obtain the traditional non-cooperative game of Nash: $\times_{i\in N} S_i(K) \to \Big(U_i(s)\Big)_{i\in N}$. Construction of a game $\Gamma(K)$ makes every partition P be an individual game.

Another novelty of the paper is an introduction of nested games.

Definition 3 (**family of games**). *A family of games is* ***nested*** *if :*

$$\Gamma = \Gamma(K=1) \subset \ldots \subset \Gamma(K) \subset \ldots \subset \Gamma(K=N).$$

Nested games appear as a result of parametrization of a game by a maximum coalition size (or by a maximum number of deviators, what is equivalent). All games have consistent nesting of components, for the same set of players N.

The suggested games allow to study intra and inter coalition externalities for players and avoid coalition-as-one-unit approach of cooperative game theory, study individual motivations and individual payoff allocations.

Let $\Sigma_i(K)$ be a set of all mixed strategies of i, i.e. a space of probability measures over $S_i(K)$, $\Sigma_i(K) = \left\{\sigma_i(K)\colon \int_{S_i(K)} d\sigma_i(K) = 1\right\}$, with a general element $\sigma_i(K)$, where an integral is Lebegue-Stiltjes integral. Sets of mixed strategies for all other players are defined in the standard way $\Sigma_{-i}(K) = \left\{\left(\sigma_j(K)\right)_{j\neq i}\colon \forall j \neq i \text{ there is } \int_{S_j(K)} d\sigma_j(K) = 1\right\}$.

Expected utility can be defined in terms of strategies the players choose or in terms of final partition-specific strategies. Expected utility of i in terms of individual strategies is:

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \int_{S(K)=\times_{i\in N} S_i(K)} U_i(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K)$$

or in terms of partition-specific strategies is

$$EU_i^{\Gamma(K)}(\sigma_i(K), \sigma_{-i}(K)) = \sum_{P\in\mathcal{P}(K)} \int_{S(P)} U_i(P)(s_i, s_{-i}) d\sigma_i(K) d\sigma_{-i}(K).$$

Expected utilities are constructed in the standard way.

Definition 4 (an equilibrium in a game $\Gamma(K)$). *A mixed strategies profile $\sigma^*(K) = (\sigma_i^*(K))_{i\in N}$ is an equilibrium strategy profile for a game $\Gamma(K)$ if for every $\sigma_i(K) \neq \sigma_i^*(K)$ the following inequality for every player i from N holds true:*

$$EU_i^{\Gamma(K)}\left(\sigma_i^*(K), \sigma_{-i}^*(K)\right) \geq EU_i^{\Gamma(K)}\left(\sigma_i(K), \sigma_{-i}^*(K)\right).$$

Equilibrium in the game $\Gamma(K)$ is defined in a standard way. It's existence is just an expansion of Nash theorem. However this result for non-cooperative games with coalition structure formation is different from the results of cooperative games, where an equilibrium for a deviation of more than one player may not exist. For example, in coalition form games with empty cores. Another outcome of the model is that there is no need to introduce additional properties of games, like transferable/non-transferable utilities, axioms on a system of payoffs, super-additivity, or weights. The equilibrium also captures cases with multi-coalition formation, what is left away from the Shapley value. Equilibrium existence result can be generalized for the whole family of games.

Theorem 1. *The family of games* $\mathcal{G} = \{\Gamma(K), K = 1, 2, \ldots, N\}$ *has an equilibrium in mixed strategies,* $\sigma^*(\mathcal{G}) = (\sigma^*(K = 1), \ldots, \sigma^*(K = N))$, $\Big(\sigma^*(K)\Big)_{i \in N}$.

Technical side off the result is obvious. The theorem expands the classic Nash theorem.

The interesting question is how robust is an equilibrium to an increase in K. For a example, the corporate dinner game above was a game for $K = 2$, where an equilibrium did not change with an increase in values of K from $K = 2$ to $K = 3, 4$.

An equilibrium in the game can also be characterized by equilibrium partitions.

Definition 5 (**equilibrium coalition structures or partitions**). *A set of partitions* $\{P^*\}(K)$, $\{P^*\}(K) \subset \mathcal{P}(K)$, *of a game* $\Gamma(K)$, *is a set of equilibrium partitions, if it is induced by an equilibrium strategy profile* $\sigma^*(K) = (\sigma_i^*(K))_{i \in N}$.

In the same way we can define equilibrium payoffs for the whole family of games. This consideration is important for construction of a non-cooperative stability criterion presented further in this paper.

In the example below we will see that there can be more than one equilibrium partition, and equilibrium partitions may change with an increase in the number of deviators.

4 An example of a game of two players with coalition structure formation

The example serves to demonstrate three points.

The above suggested game can be constructed. It is flexible to changes in rules of coalition structure formation. This allows to ask questions about socially desirable coalition structure formation rules. This is important for social design literature, based on self-interest behavior. This is the new direction of the future work.

The construction of the game suggests the protocol, an order how to explain the new type of the game.

The example demonstrates that: the efficient outcome in the standard Prisoner's dilemma does not deal with cooperation. The same payoff profile $(0;0)$ can be obtained in two different coalition structures $\{1,2\}$ and $\{\{1\},\{2\}\}$. Which of them is a cooperation? Thus the term "cooperation" should be additionally defined.

The example below is based on a generalization of the Prisoner's Dilemma. The example is presented step by step in the same way as was introduced the formal model.

There are 2 players, $i = 1, 2$. They can form two types of partitions. If $K = 1$ then there is only one final partition, $\mathcal{P}(K = 1) \equiv P_{separ} = \{\{1\},\{2\}\}$. If $K = 2$ there are two final partitions, which make a family of partitions $\mathcal{P}(K = 2) = \Big\{\{\{1\},\{2\}\},\{1,2\}\Big\}$. Further we will use the notation $P_{joint} = \{1,2\}$. Clearly, partition structures $\mathcal{P}(K = 1)$ and $\mathcal{P}(K = 2)$ are nested.

In every partition a player has two strategies: H(igh) and L(ow). Player i for a game with $K = 1$ has a strategy set $S_i(K = 1) = \{L_{i,P_{separ}}, H_{i,P_{separ}}\}$. Player i in the game with $K = 2$ has the extended strategy set $S_i(K = 2) = \{L_{i,P_{separ}}, H_{i,P_{separ}}, L_{i,P_{joint}}, H_{i,P_{joint}}\}$. Clearly, strategy sets $S_i(K = 1)$ and $S_i(K = 2)$ are nested.

Set of strategies for the game with $K = 1$ is $S(K = 1) = \{L_{1,P_{separ}}, H_{1,P_{separ}}\}\times$

$\{L_{2,P_{separ}}, H_{2,P_{separ}}\}$. These payoffs are in corresponding top-left cells of Table 6. Every cell contains a payoff profile and a *final* coalition structure. The payoffs in these cells are identical to the payoffs of the Prisoner's Dilemma.

Set of strategies for the game with $K = 2$ is

$$S(2) = \{L_{1,P_{separ}}, H_{1,P_{separ}}, L_{1,P_{joint}}, H_{1,P_{joint}}\} \times \{L_{2,P_{separ}}, H_{2,P_{separ}}, L_{2,P_{joint}}, H_{2,P_{joint}}\}.$$

Strategy sets of games with $K = 1$ and $K = 2$ are nested, i.e. $S(K = 1) \subset S(K = 2)$. Additional payoffs of the game for $K = 1$ are replicated from the game with $K = 1$, see Table 6.

For $K = 1$ there are four outcomes, for $K = 2$ there are sixteen outcomes, twelve new in comparison with a game for $K = 1$.

A coalition structure formation mechanisms for different K are $\mathcal{R}(K = 1), \mathcal{R}(K = 2)$ and are defined as :

$$\mathcal{R}(K = 1) \colon S(K = 1) \mapsto \begin{cases} S(\{\{1\}, \{2\}\}), \\ \forall s \in S(K = 1) = \{L_{1,P_{separ}}, H_{1,P_{separ}}\} \times \{L_{2,P_{separ}}, H_{2,P_{separ}}\} \end{cases}$$

and

$$\mathcal{R}(K = 2) \colon S(K = 2) \mapsto \begin{cases} S(\{1, 2\}), & \text{if } s \in \{L_{1,P_{joint}}, H_{1,P_{joint}}\} \times \{L_{2,P_{joint}}, H_{2,P_{joint}}\} \\ S(K = 2) \setminus S(\{1, 2\}\}), & \text{else} \end{cases}$$

The same in words. The grand coalition $P_{joint} = \{1, 2\}$ can be formed only from a unanimous agreement of both players to form this coalition, irrespective which actions they choose to do inside this coalition. It is clear that for $K = 1$ it can not be formed. The grand coalition, or a partition P_{joint}, can be formed only over the strategy set $(L_{1,P_{joint}}, H_{1,P_{joint}}) \times (L_{2,P_{joint}}, H_{2,P_{joint}})$. Otherwise the partition P_{separ} is formed. Thus for $K = 2$ a set of strategies for the partition P_{separ} is not a direct (Cartesian) product, but a union of sets. It is also clear that an increase in K results in nested mechanisms:

$\mathcal{R}(K=1) \subset \mathcal{R}(K=2)$.

From Table 6 we can see that the whole strategy set of the game is partitioned into *coalition structure specific domains*. Every coalition structure (or a partition) is a non-cooperative game with it's own strategy set and payoff profiles. Final partition for a player may not coincide with an individual choice. A set of strategies for the partition P_{separ} is not a product of sets. Finally the games for $K = 1$ and $K = 2$ are nested. For $K = 1$ the equilibrium is marked as *.

Consider the game with a maximum coalition size $K = 2$ as described by the Table 6. If both players choose strategies only for the partition P_{separ}, then the game is the standard Prisoner's Dilemma game. However the same coalition structure can be formed if one of the players does not choose any strategy for $P_{joint} = \{1, 2\}$. Thus for the *final* partition $P_{separ} = \{\{1\}, \{2\}\}$ there are three equilibria, and every equilibrium is inefficient. They are marked with the upper index **, although the grand coalition can not be formed. And there is one equilibrium for the grand coalition $\{1, 2\}$.

The partition $P_{joint} = \{1, 2\}$ can be formed only if both players choose it. Within this partition there is one inefficient equilibrium, marked also as **.

Compare efficient payoff profiles for the partitions P_{separ} and P_{joint}. They have equal payoff profile $(0; 0)$, but it are realized in different final coalition structures. From observing only the payoff profile $(0; 0)$ we can not make a conclusion, which coalition structure is formed either P_{separ} or P_{joint}. Another interpretation is that cooperation takes place into different coalition structures: in one players are separate, in another together. Which of them is cooperation?

Using the same game we can demonstrate appearance of intra- and inter-coalitions externalities. If partition $P_{joint} = \{1, 2\}$ is formed, then an individual payoff of a player depends on a strategy of another in the *same* coalition (presence of intra-coalition or intra-group externality). If partition $P_{separ} = \{\{1\}, \{2\}\}$, is formed, then an individual payoff of a player depends on a strategy of another player in *the different* coalition (presence of inter-coalition or inter-group externality).

Multiplicity of equilibria makes both of these externalities co-exist in equilibria, but in different final coalition structures. Thus the game is able to present both intra and inter-coalition externalities, what is impossible in cooperative game theory.

Table 3: Payoff for the family of games with unanimous formation rules. Different partitions have payoff-equal efficient outcomes.

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$
$H_{1,P_{separ}}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{*,**}$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{**}$ $\{\{1\},\{2\}\}$
$L_{1,P_{joint}}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{1,2\}$	$(-5;3)$ $\{1,2\}$
$H_{1,P_{joint}}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{**}$ $\{\{1\},\{2\}\}$	$(3;-5)$ $\{1,2\}$	$(-2;-2)^{**}$ $\{1,2\}$

We can reinstall uniqueness of an equilibrium, what is done in Table 4. If both players are extroverts and prefer be together,[14] then every individual payoff increases by $\epsilon > 0$, if the grand coalition is realized. This means that a change of the game from $\Gamma(1)$ to $\Gamma(2)$ changes the equilibrium in terms of both strategies and the partitions, but not in terms of payoff profiles.

If both players are introverts, and $\delta > 0$ is a markup for *ex post* being alone, then the expansion of the game will not change initial equilibrium in terms of both strategies and the partition. A corresponding payoff matrix is presented in Table 5. A change in a maximum coalition size from $K = 1$ to $K = 2$ does not change the equilibrium strategy profile. cure

[14]what is equivalent to preferences over coalition structures

Table 4: Payoff for two extrovert players who, obtain additional payoffs ϵ being in one coalition P_{joint}, when it is realized. Uniqueness of an equilibrium is reinstalled.

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$
$H_{1,P_{separ}}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)^{*}$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)$ $\{\{1\},\{2\}\}$
$L_{1,P_{joint}}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0+\epsilon;0+\epsilon)}$ $\{1,2\}$	$(-5+\epsilon;3+\epsilon)$ $\{1,2\}$
$H_{1,P_{joint}}$	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$	$(3+\epsilon;-5+\epsilon)$ $\{1,2\}$	$(-2+\epsilon;-2+\epsilon)^{**}$ $\{1,2\}$

Table 5: Payoff for two extrovert players who, obtain additional payoffs ϵ being in one coalition P_{joint}, when it is realized. Uniqueness of an equilibrium is reinstalled.

	$L_{2,P_{separ}}$	$H_{2,P_{separ}}$	$L_{2,P_{joint}}$	$H_{2,P_{joint}}$
$L_{1,P_{separ}}$	$\underline{(0+\delta;0+\delta)}$ $\{\{1\},\{2\}\}$	$(-5+\delta;3+\delta)$ $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	$(-5;3)$ $\{\{1\},\{2\}\}$
$H_{1,P_{separ}}$	$(3+\delta;-5+\delta)$ $\{\{1\},\{2\}\}$	$(-2+\delta;-2+\delta)^{*,**}$ $\{\{1\},\{2\}\}$	(3;-5) $\{\{1\},\{2\}\}$	$(-2;-2)$ $\{\{1\},\{2\}\}$
$L_{1,P_{joint}}$	$\underline{(0;0)}$ $\{\{1\},\{2\}\}$	(-5;3) $\{\{1\},\{2\}\}$	$\underline{(0;0)}$ $\{1,2\}$	$(-5;3)$ $\{1,2\}$
$H_{1,P_{joint}}$	(3;-5) $\{\{1\},\{2\}\}$	(-2;-2) $\{\{1\},\{2\}\}$	$(3;-5)$ $\{1,2\}$	$(-2;-2)$ $\{1,2\}$

In both cases for extroverts and introverts there are no changes in equilibrium payoff profiles. These issues are discussed further.

5 Discussion

At the moment there are two game theories: a non-cooperative and a cooperative. A non-cooperative is based on a mapping of a strategy profile of all players (a subset from $\mathbb{R}^N$) into a payoff profile for all players (a bounded subset from $\mathbb{R}^N$). A cooperative game theory is based on a mapping of a subset of integer numbers (a subset from $\mathbb{N}$) into a subset of real numbers (a subset from $\mathbb{R}$).[15]

Insufficiency of cooperative game theory to study coalitions and coalition structures was earlier reported by many authors. Maskin (2011) wrote that "features of cooperative theory are problematic because most applications of game theory to economics involve settings in which externalities are important, Pareto inefficiency arises, and the grand coalition does not form". Myerson (p.370, 1991) noted that "we need some model of cooperative behavior that does not abandon the individual decision-theoretic foundations of game theory". Thus there is a demand for a specially designed non-cooperative game to study coalition structures formation along with an adequate equilibrium concept for this game.

5.1 Referring to existing literature

There is a voluminous literature on the coalition formation, a list of authors is far from complete: Aumann, Hart, Holt, Maschler, Maskin, Moulen, Myerson, Peleg, Roth, Serrano, Shapley, Schmeidler, Weber, Winter, Wooders

[15]The basic assumption is that an individual action of every self-interest agent, located at one coalition can have a non-negligible echo in enjoyment for every other, located at some different coalitions. Construction of a non-cooperative games in this paper disregards possible negligibility of these effects. The presence of negligible echoes leaves a space for a co-existence of non-cooperative and cooperative game theories.

and many others.

However the problem is not in a volume of literature. Before searching for a solution, one need to have a definite answer for the questions: does existing literature well-identify the problem? Thus we need to ask three navigating questions:

Identification What is a problem?

Solution existence Does the problem so far has a solution?

Tool Shall we borrow some existing tool or to construct a new one to solve the problem after it is well-defined?

Let me answer the questions one by one.

5.1.1 Identification

There are different views on problems with non-cooperative formation of coalition structures, two were presented above. There are at least two more recent.

Serrano (2014) wrote on usage non-cooperative, but not cooperative game theory: "the axiomatic route find difficulties identifying solutions", and that for studying coalition formation *"it may be worth to use strategic-form games, as proposed in the Nash program"*. The statement is done almost 60 years after Nash (1953).

Ray and Vohra (2015) wrote on complexity and contradictions in approaches, offer a systematic view on the area, based on "collection of coalitions", or a modified cooperative game theory. "*Yet as one surveys the landscape of this area of research, the first feature that attracts attention is* ***the fragmented nature of the literature***[16]*... The literature on coalition formation embodies two classical approaches that essentially form two parts of this chapter: (i) The blocking approach, in which we require the immunity*

[16] stressed by DL

of a coalitional arrangement to blocking, perhaps subcoalitions or by other groups which intersect the coalition in question... (ii) Noncooperative bargaining, in which individuals make proposals to form a coalition, which can be accepted or rejected...

After all, the basic methodologies differ apparently at ***an irreconcilable level***[17] *over cooperative and noncooperative game approaches...* "

Every view describes the problem partially differently and suggests different ways of solution. Existing variability of views does not let to conclude that the problem is still well-identified and not fragmented. Not well-identified problem can not be well-solved.

The suggested paper has encompassed all the mentioned details into one, and the problem can be pronounced as: how to construct coalition structures from actions of self-interest agents, when co-exist intra and inter coalition externalities.

5.1.2 Existence of a solution

The volume of the literature means the problem is not well-identified so far. Not well-identified problem can not be well-solved. This is the reason for a diversity of contradicting views on coalition (structure) formation from individual behavior. Thus the "problem can not be solved if it's bound are unknown" (Tchekmarev[18])

The current paper suggests another identification of the problem and a consistent, natural solution for it. The proposed game generalizes the noncooperative game of Nash for coalition structure formation, informally the game is a mapping of real tensor into a bounded real tensor.

[17] stressed by DL

[18] Private communication on engineering design.

5.1.3 Tool

How to solve the problem, which was not solved for so long? The answer comes from Albert Einstein: “The significant problems we have cannot be solved at the same level of thinking with which we created them”. This means that to solve the problem one needs to reconsider a basic tool of non-cooperative game theory analysis. The current paper dares to suggest such a tool.

5.2 Comparison with existing approaches

5.2.1 A threat

A popular approach to use a “threat ”as a basic concept for coalition formation analysis was suggested by Nash (1953). Consider a strategy profile from a subset of players. Let this profile be a threat to someone, beyond this subset. The threatening players may produce externalities for each other (and negative externalities not excluding!). How credible could be such threat and how to describe it? At the same time there may be some other player beyond the subset of players, who may obtain a bonanza from this threat. But this beneficiant may not join the group due to expected intra-group negative externalities for members or from members of this group. Thus a concept of a threat can not serve as an elementary concept.

There is a parallel argument against using the concept of a ”threat”. Assume there are several agents, who individually can not make harm to some others, possibly allocated in different coalitions. And the threat targets not whole coalitions, but only to some members. But if these small agents make a threat together, it can be credible. Does this mean that the small agents unite in one coalition? The example above demonstrates that it is not necessarily a case. They may have there own contradictions to join. How to describe affect for those, who are not a target for the threat? May be a formal example will be more illustrative here, but the volume of the paper does not let us present it here.

5.2.2 Usage of the non-cooperative approach

The justification of a chosen tool, a non-cooperative game, comes from Maskin (2011) and a remark of Serrano (2014), that for studying coalition formation "it may be worth to use strategic-form games, as proposed in the Nash program". This paper explicitly works with a non-cooperative approach.

However, there is the difference of the research agenda in this paper from the Nash program (Serrano 2004). Nash programs aims to study a non-cooperative formation of *one* coalition, this paper aims to study non-cooperative formation of coalition structures, which may include more than one coalition. The best analogy for the difference is the difference between partial and general *strategic* equilibrium analysis (in terms of strategic market games) in economics. The former isolates the market ignoring cross-market interactions, the latter explicitly studies cross market interactions.

The constructed finite non-cooperative game allows to study what can be a cooperative behavior, when the individuals "rationally further their individual interests" (Olson, 1971).

5.2.3 Novelties of the paper

Nash (1950, 1951) suggested to construct a non-cooperative game as a mapping of a set of strategies into a profile of payoffs, $\times_{i \in N} S_i \to (U_i)_{i \in N}$.

This paper has two contributions in comparison to his paper: construction of a non-cooperative game with an embedding coalition structure formation mechanism, and parametrization of all constructed games by a number of deviators K: $\times_{i \in N} S_i(K) \overset{\mathcal{R}(K)}{\to} \{S(P) \colon P \in \mathcal{P}(K)\} \to \{(\mathcal{U}_i(K))_{i \in N}\}$, where $K \in \{1, \ldots, \#N\}$. The game suggested by Nash becomes a partial case for these games, when coalition structures do not matter. Every constructed game is a mapping of a real tensor into a tensor. ex

The important novelties of the presented model are: an outcome of a game consists of two items 1/ a payoff profile; 2/ an allocation of players over coalitions. Equilibrium in mixed strategies always exists and may be not efficient like in traditional non-cooperative games.

One may ask questions about social desirability of coalition structures, which coalition structure formation mechanisms can form them. This is the new direction to enrich non-cooperative game theory and provide non-cooperative fundamentals for social and mechanism design. This is the challenge for the future research.

The paper suggests the new concept, a family of non-cooperative games for coalition structure formation. This family is parametrized by a maximum coalition size in a set of available coalition structures of every game. Every game in a family has an equilibrium, may be in mixed strategies. This differs from results of cooperative game theory, where games may have no equilibrium, like games with empty cores, etc.

5.2.4 Differences from cooperative game theory concepts

The introduced equilibrium concept differs from the strong Nash, the coalition-proof, the nucleolus, the kernel, the bargaining set.

The common difference is: the presented model let every player make an individual choice, and obtain an individual payoff for every player. A consequence of the self-interest behavior of players is a combination of intra- and inter- coalition externalities. Thus a coalition may have different values (as a sum of individual payoffs of a member of a coalition) in different coalition structures.

There are other more specific differences with every existing cooperative game theory equilibrium concept. Differences from the core approach of Aumann (1960) are clear: a presence of externalities, no restrictions that only one group deviates, no restrictions on the direction of a deviation (inside or outside), and a construction of individual payoffs from a strategy profile of all players. There is no need to assume transferable/non-transferable utilities for players. The suggested approach allows to study coalition structures, which differ from the grand coalition as in Shapley value. Finally the introduced concepts enables to offer a non-cooperative necessary stability criterion based only on an equilibrium of a game, presented in further in this paper. This is impossible to establish with any other equilibrium concept.

The approach does not need a concept of a blocking coalition. The reason is: if there is one coalition, which can not block, but there are two, separate coalition, which do can block. How to describe actions of these coalitions from individual actions of their members? This is the same argument, as why not to use a threat as an elementary concept. This situation can be studied by the suggested model.

The suggested approach is different in a role for a central planner offered by Nash, who "argued that cooperative actions are the result of some process of bargaining" Myerson (p.370, 1991). The central planner offers a predefined family of coalition structure formation mechanisms, what is indexed by a maximum number of deviators, family of eligible partitions and a family of rules to construct these partitions from individual strategies of players.

6 Formal definition of cooperation

This section formalizes an idea of cooperation presented in the example above. We demonstrate only one way for defining cooperation, when it is intentional and complete.

Definition 6 (complete cooperation in a coalition)**.** *In a game $\Gamma(K)$ with an equilibrium $\sigma^*(K) = (\sigma_i^*)_{i\in N}$ a set of players g, ex ante* ***completely cooperate in the coalition*** *g if for every player $i \in g$ there is*

ex ante *: for every i in g, a desirable coalition g always belongs to a chosen coalition structure, i.e such if s_i is chosen by i, then $s_i \in S_i(P_i)$, $g \in P_i$,* *where P_i is a coalition structure chosen by i.* [19]

ex post 1 *: every realized coalition structure contains g, i.e. $g \in \forall P^*$, where P^* is a formed equilibrium partition of $\Gamma(K)$,*

First of all, cooperation is defined for a game $\Gamma(K)$. If a game changes due to a change in the number of deviators K, the cooperation may evaporate.

Every player chooses partitions, every chosen partition contains the desirable coalition g. Individually chosen coalition structures by all players may be different.

After the game is over the coalition g always belongs to every final equilibrium coalition structure, disregard allocation of players in other coalitions. A final partition may differ from a chosen one, but in any case it will contain the desirable coalition.

The defined cooperation assumes agents are acting from self-interest motivations. The lunch game example further expands the case, where there is imperfect cooperation.

The dinner game example above has the complete cooperation for players C_1 and C_2. The definition does not exclude inter and intra-coalition interaction. If we relax some conditions of the definition we will obtain weaker conditions for cooperation. Cooperation in repeated games is of special interest and will be addressed in one of the next papers.

7 Application: Bayesian games

In this section we demonstrate how intra-coalition externalities of the grand coalition may happen from equilibrium mixed strategies. In order to show that, a standard Battle of Sexes game is modified.

Let there be two players, Ann and Bob. Each has two options: to be together with another or to be alone. In every option each can choose where

[19] However coalition structures chosen by different players may be different.

to go: to Box or to Opera. Hence every player has four strategies. A set of strategies of the game leads to 16 outcomes. Every outcome consists of a payoff profile and a partition (or a coalition structure). Both players have preferences over coalition structures: they prefer to be together, then be separated.

The rules of coalition structure formation mechanism are:

1. If they both choose to be together, i.e. both choose the coalition structure $P_{joint} = \{Ann, Bob\}$ then:
   (a) if both choose the same action for P_{joint} (i.e. both choose Box or both choose Opera), then they go to where they both choose to go,
   (b) otherwise they do not go anywhere, but enjoy just being together;
2. if at least one of them chooses to stay alone, i.e. chooses a partition $P_{separ} = \{\{Ann\}, \{Bob\}\}$, then each goes alone to where she/he chooses, may be to a different Opera or to a different Box performances.

Formally the rules are:

$$\begin{aligned}&\mathcal{R}(K=1)\colon S(K=1) \mapsto S(\{\{1\},\{2\}\}),\\ &\quad \forall s \in S_i(K=1) = \{O_{Ann,P_{separ}}, B_{Ann,P_{separ}}\} \times \{O_{Bob,P_{separ}}, B_{Bob,P_{separ}}\}\end{aligned}$$

and

$$\mathcal{R}(K=2)\colon S(K=2) \mapsto \begin{cases} S(\{1,2\}), \\ \text{ if } s \in \{O_{Ann,P_{separ}}, B_{Ann,P_{separ}}\} \times \{O_{Bob,P_{separ}}, B_{Bob,P_{separ}}\} \\ S(K=2) \setminus S(\{1,2\}\}), \quad \text{otherwise} \end{cases}$$

The whole Table 6 corresponds to the game with $K = 2$, where the game for $K = 1$ is a nested component. If Ann and Bob play the game with $K = 1$

Table 6: Payoff for the Bach-or-Stravinsky game. B is for Box, O is for Opera. If the players choose to be together, and it is realized due to the rule of coalition structure formation, then each obtains an additional fixed payoff $\epsilon > 0$.

	$B_{Bob,P_{separ}}$	$O_{Bob,P_{separ}}$	$B_{Bob,P_{joint}}$	$O_{Bob,P_{joint}}$
$B_{Ann,P_{separ}}$	$(2;1)^*$ $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	(2;1) $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$
$O_{Ann,P_{separ}}$	(0;0) $\{\{1\},\{2\}\}$	$(1;2)^{**}$ $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	(1;2) $\{\{1\},\{2\}\}$
$B_{Ann,P_{joint}}$	(2;1) $\{\{1\},\{2\}\}$	(0;0) $\{\{1\},\{2\}\}$	$(2+\epsilon;1+\epsilon)^{**}$ $\{1,2\}$	$(\epsilon;\epsilon)$ $\{1,2\}$
$O_{Ann,P_{joint}}$	(0;0) $\{\{1\},\{2\}\}$	(1;2) $\{\{1\},\{2\}\}$	$(\epsilon;\epsilon)$ $\{1,2\}$	$(1+\epsilon;2+\epsilon)^{**}$ $\{1,2\}$

with a single coalition structure $\{\{Ann\},\{Bob\}\}$, then the payoffs for this game are in the two-by-two top-left corner of Table 6. If Ann and Bob are together, then each obtains an additional payoff ϵ, and the corresponding cells make a two-by-two bottom-right corner.

Every game with $K = 1$ and $K = 2$ has only one mixed strategies equilibrium and only one equilibrium partition. Mixed strategies equilibrium for $K = 1$ is described in every textbook. A change in K from $K = 1$ to $K = 2$ results in a changes of an equilibrium strategy profile and an equilibrium partition.

For $K = 2$ the game still has a mixed strategies equilibrium like for $K = 1$. The differences are: in another domain of pure strategies, the different coalition structure and the different payoff profile. Mixed strategies equilibrium for Ann is: $\sigma^*(B_{Ann,P_{joint}}) = (1+\epsilon)/(3+2\epsilon)$, $\sigma^*(O_{Ann,P_{joint}}) = (2+\epsilon)/(3+2\epsilon)$, $i \in \{\text{Ann}, \text{Bob}\}$.

Equilibrium mixed strategies may appear in any coalition structure, for every $K = 1, 2$. When they appear within a coalition, then there is equilibrium additional intra-coalition activity. Such games are not described in literature.

The presented game allows to construct intra-coalition externalities from mixed strategies within one partition. Mixed intra-coalition externality means that players are exposed to equilibrium fluctuations from strategic actions of another player.

A game as above can not be constructed within any cooperative game equilibrium concept. It is impossible to construct Shapley value (1953) here even if the grand coalition is in the equilibrium: players have equilibrium mixed strategies inside it.

8 Application: Stochastic games

Shapley (1953) defined stochastic games as " the play proceeds by steps from position to position, according to transition probabilities controlled jointly by the two player". This section demonstrates how this type of games with coalition structures as states of a game may appear. The example differs from example above as a set of the equilibrium mixed strategies induces more than one equilibrium coalition structure. We use a game, similar to corporate dinner game, but with identical players.

8.1 Corporate lunch game

There is a set of four identical players $N = \{A, B, C, D\}$. An individual strategy is a coalition structure, or an allocation of all players across tables during lunch. A coalition structure is an allocation of players over no more than four tables, where possibly empty tables do not matter. A rule of coalition formation is a unanimous agreement to form a coalition. If player a chooses a coalition, but other members of the coalition did not choose him, the player eats alone.

A player has preferences over coalition structures: she/he prefers to eat with someone and other players eat individually. If one eats alone he is hurt by a possible formed coalition of others. Coalition structures, and payoff profiles for the highest cases payoffs are presented in Table 7:

Table 7: Office lunch game: strategies and payoff profiles. Full set of equilibrium mixed strategies are indicated only for player A.

num	Coalition structure	Payoff profile U_A, U_B, U_C, U_D	Coalition values as in cooperative game the
1*	$\{A, B\}, \{C\}, \{D\}$: $\sigma^*_A = \sigma^*_B = 1/3$	(10,10,3,3)	$20_{A,B}$, 3_C, 3_D
2*	$\{A, C\}, \{B\}, \{D\}$: $\sigma^*_A = \sigma^*_C = 1/3$	(10,3,10,3)	$20_{A,C}$, 3_B, 3_D
3*	$\{A, D\}, \{C\}, \{B\}$: $\sigma^*_A = \sigma^*_D = 1/3$	(10,3,3,10)	$20_{A,C}$, 3_C, 3_B
4	$\{A\}, \{B\}, \{C, D\}$	(3,3,10,10)	3_A, 3_B, $20_{C,D}$
5	$\{A\}, \{D\}, \{B, C\}$	(3,10,10, 3)	3_A, 3_D, $20_{C,B}$
6	$\{A\}, \{C\}, \{B, D\}$	(3,10,3,10)	3_A, 3_C, $20_{B,D}$
7	$\{A\}, \{B\}, \{C\}, \{D\}$	(3,3,3,3)	3_A, 3_B, 3_C, 3_D
8	$\{A, B\}, \{C, D\}$	(3,3,3,3)	$6_{A,B}$, $6_{C,D}$
9	$\{A, C\}, \{B, D\}$	(3,3,3,3)	$6_{A,B}$, $6_{C,D}$
10	$\{A, D\}, \{B, C\}$	(3,3,3,3)	$6_{A,D}$, $6_{B,C}$
11	all other with $K = 3, 4$	(0,0,0,0)	$= 0$

Payoffs in Table 7 are organized in the way that formation of a coalition by other players deteriorates payoffs for the rest. Thus the same coalition may have different values in different coalition structures, compare payoff for the coalition $\{A, B\}$ in different coalition structures.

If there are two 2-player coalitions every player receives only three units of payoffs, while if a player is in a coalition of two, while others are separate the player obtains a ten unit payoff. An increase in a size of a maximum coalition only decreases payoffs for all players.

Outcomes of the game are coalition structures or states of a stochastic game. An increase of $K = 2$ to $K = 3, 4$ does not change an equilibrium in mixed strategies, hence we can speak about robustness of an equilibrium for $K = 2$ to an increase in K. This issue is addressed later in this paper.

It is clear that the game does not have an equilibrium in pure strategies. This is a Bayesian game, with a probability distribution of equilibrium mixed strategies. Equilibrium mixed strategies are indicated only for player A in the first three lines of the Table 7.

8.2 A formal definition of a stochastic game of coalition structure formation

Let $\Gamma(K)$ be a non-cooperative game as defined above.

Definition 7. *A game $\Gamma(K)$ is a stochastic game if a set of equilibrium partitions is bigger than two, $\#(\{P^*\}(K)) \geq 2$, where a state is an equilibrium partition $P^* \in \{P^*\}(K)$.*

Studying properties of stochastic games with non-cooperative coalition structure formation and families of such games are left for future.

9 Application: non-cooperative criterion for stability

Aumann (1990) used "stag and hare" game to demonstrate absence of a self-enforcement property of Nash equilibrium. We use an extended version of the same game to demonstrate how by modifying the game we can reach a focal point of the game, unavailable within standard Nash equilibrium approach. Introduction of a family of games allows to modify strategy sets and to demonstrate what means self-enforcement in terms of individual behavior. Then we construct a non-cooperative coalition structure stability criteria.

There are two hunters $i = 1, 2$. If players can hunt only individually, then $K = 1$, and the only partition is $P_{separ} = \{\{1\}, \{2\}\}$. An individual strategy set of i is $S_i(K=1) = \Big\{(P_{separ}, hare), (P_{separ}, stag)\Big\}$ with a general element $s_i \equiv s_i(K)$. Every s_i consists of two terms: who is a hunting partner and what is an animal to hunt. For example, a strategy $s_i = (P_{separ}, hare)$ is interpreted as player i chooses to hunt alone for a hare.

A set of a corresponding strategies for the game with $K = 1$ is $S(K = 1) = S_1(K = 1) \times S_1(K = 1)$.

For a game with $K = 2$ every hunter can choose either to hunt alone, in a coalition structure $P_{separ} = \{\{1\}, \{2\}\}$, or together, $P_{joint} = \{1, 2\}$. For every hunting partition a player chooses a target for hunting: a hare or a stag, as in the game for $K = 1$. A set of strategies of i is

$$S_i(K=2) = \Big((P_{separ}, hare), (P_{separ}, stag), (P_{joint}, hare), (P_{joint}, stag)\Big),$$

where a strategy consists of two terms. A set of strategies of the game is a direct (Cartesian) product, $S(K = 2) = S_1(K = 2) \times S_2(K = 2)$.

We do not rewrite the rules for coalition structure formation, as they are the same as in the BoS game above. The difference is renaming strategies.

Every player knows, which game is played, either with $K = 1$ or with $K = 2$. A case with uncertainty in parameter K is not addressed here and left for the future.

We assume that there is a unanimous coalition formation rule, i.e. hunters can hunt together only if both choose to be together. Payoffs for the both games $\Gamma(K = 1)$ and $\Gamma(K = 2)$ are presented in Table 8. Some payoffs outcomes have the special interpretation: $(8; 8)$ every hunter obtains a hare, $(4; 4)$ hunters obtain one hare for two, $(100; 100)$ both hunters obtain one hare.

If hunters play a game with $K = 1$, then a maximum achievable payoff is $(8, 8)$, when each hunts individually for a hare. An an equilibrium strategy profile is $\Big((P_{separ}, hare), (P_{separ}, hare)\Big)$ with the payoff $(8; 8)$. In the game $\Gamma(K = 1)$ the players can not reach the efficient outcome $(100, 100)$. It is available only if of both hunters decide to hunt together. This focal point (in terminology of Schelling) can be reached only within the game $\Gamma(K = 2)$. This is the explanation for the problem posed by Aumann - there is a point, which seems to be attractive, but one can not describe it in terms of a Nash equilibrium of a traditional non-cooperative game. The focal point is feasible in the game $\Gamma(K = 2)$, but not in the game $\Gamma(K = 1)$.

Self-enforcing property of the equilibrium is that both players realize individual gain from a change of a game from $K = 1$ to $K = 2$ and neither can deviate. But players can not reach the outcome $(100; 100)$ without a change in a game.

In literature a self-enforcement property of an equilibrium is not well-defined, but intuitively it depends on what players think about willings of many others to deviate from an equilibrium.

Table 8: Expanded stag and hare game

	P_{separ}, hare	P_{separ}, stag	P_{joint},hare	P_{joint}, stag
P_{separ},hare	$(8;8)^*$; $\{\{1\},\{2\}\}$	(8;0); $\{\{1\},\{2\}\}$	(8;8);$\{\{1\},\{2\}\}$	(8;0); $\{\{1\},\{2\}\}$
P_{separ}, stag	(0;8); $\{\{1\},\{2\}\}$	(0;0); $\{\{1\},\{2\}\}$	(0;8); $\{\{1\},\{2\}\}$	(0;0); $\{\{1\},\{2\}\}$
P_{joint}, hare	(8;8); $\{\{1\},\{2\}\}$	(8;0); $\{\{1\},\{2\}\}$	(4;4); $\{1,2\}$	(8;0); $\{1,2\}$
P_{joint}, stag	(0;8); $\{\{1\},\{2\}\}$	(0;0) ;$\{\{1\},\{2\}\}$	(0;8);$\{1,2\}$	$(100;100)^{**}$; $\{1,2\}$

If there is an uncertainty, which game is played, either $\Gamma(K = 1)$ or $\Gamma(K = 2)$, then players will randomize between two strategies: $P_{separ}, hare$ and $P_{joint}, stag$. In this case the game becomes a stochastic game as described earlier.

9.1 Criterion of coalition structure (a partition) stability

There is a nested family of games

$$\Gamma = \{\Gamma(K=1), \ldots, \Gamma(K), \ldots \Gamma(K=N)\} \colon \Gamma(K=1) \subset \Gamma(K) \subset \Gamma(K=N).$$

It is characterized by a list of equilibrium strategy profiles,

$$\Big(\sigma^*(1), \ldots, \sigma^*(K), \ldots, \sigma^*(K=N)\Big),$$

where $\sigma^*(K) = (\sigma_i^*(K))_{i \in N}$ and by a list of equilibrium partitions

$$\Big(\{P^*\}(1), \ldots, \{P^*\}(K), \ldots, \{P^*\}(K=N)\Big),$$

$\{P^*\}(K) \subset \mathcal{P}(K)$. Every game $\Gamma(K)$ from a family Γ has an equilibrium may be in mixed strategies.

The family of games has an equilibrium expected payoff profiles:

$$\Big((EU_i^{\Gamma(1)})^*_{i \in N}, \ldots, (EU_i^{\Gamma(K)})^*_{i \in N}, \ldots, (EU_i^{\Gamma(K=N)})^*_{i \in N}\Big),$$

where $(EU_i^{\Gamma(K)})^*_{i \in N} \equiv (EU_i^{\Gamma(K)}(\sigma^*))_{i \in N}$.

Let us take a game $\Gamma(K_0) \in \Gamma$ with $\sigma^*(K_0)$ as an equilibrium mixed strategy set. The question is: what is a condition when an equilibrium strategy profile does not change with an increase in a maximum coalition size K_0? The criterion is based on the idea that a set of mixed strategies should not change with an increase in K, i.g. when a game $\Gamma(K)$ is sequentially changed for $\Gamma(K+1), \ldots, \Gamma(N)$. The criterion is a sufficient criterion and defines a maximum coalition size, when an equilibrium strategy profile for a less K_0 still supports a an equilibrium for a bigger K.

Definition 8. ***Partition stability criterion*** *for a game* $\Gamma(K_0)$ *is a maximum coalition size* K^* *when an equilibrium still holds true, i.e. for all* $i \in N$

there is a maximum number K^* such that

1.

$$K^* = \max_{\substack{K=K_0,\dots,N \\ \Gamma(K_0)\dots,\Gamma(K=N)}} \left\{ EU_i^{\Gamma(K_0)}\left(\sigma_i^*(K_0), \sigma_{-i}^*(K_0)\right) \geq EU_i^{\Gamma(K)}\left(\sigma_i^*(K), \sigma_{-i}^*(K)\right)\right\},$$

2. *Dom* $\sigma^*(K^*) =$ *Dom* $\sigma^*(K_0)$

where $\sigma^*(K_0)$ is an equilibrium in the game $\Gamma(K_0)$, $\sigma^*(K)$ is an equilibrium in a game $\Gamma(K)$, $K = K_0, \dots, N$, and Dom is a domain of equilibrium mixed strategies set.

The definition is operational, it can be constructed directly from a definition of a family of games. This definition guarantees stability of both payoffs and partitions, and is a sufficient criterion of stability. Some applications may require weaker forms of the criterion.

Now it is clear that the statement of Aumann (1990) that Nash equilibrium is generally not self-enforcing is correct. In the extended version of stag and hare game we have seen that an increase in K changed an equilibrium. The same took place in a variation of Battle of Sexes game. However this did not happen in the Corporate Dinner or the Corporate Lunch game.

The proposed criterion may serve as a measure of trust to an equilibrium or as a test for self-enforcing of an equilibrium. This criterion can be applied to study opportunistic behavior in coalition partitions. If players in a coalition g of a game $\Gamma(K_1)$ have perfect cooperation, this does not mean that in a wider game $\Gamma(K_2)$, $K_1 < K_2$, they will still cooperate.

10 Conclusion

The current paper presents a non-cooperative game for coalition structure formation. Using the example, the paper offers a way to construct cooperation in coalition formation on self-interest fundamentals. The paper offers also a non-cooperative criterion to measure stability of coalition structures families of nested games and demonstrates some applications.

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