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Anna Bogomolnaia, Herve Moulin,
Fedor Sandomirskiy, Elena Yanovskaya

COMPETITIVE DIVISION OF A MIXED MANNA

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Anna Bogomolnaia★, *Hervé Moulin*♠,
Fedor Sandomirskiy♣, *and Elena Yanovskaya*♦

Competitive division of a mixed manna[*†]

A mixed manna contains *goods* (that everyone likes), *bads* (that everyone dislikes), as well as items that are *goods* to some agents, but *bads* or satiated to others.

If all items are goods and utility functions are *homothetic*, concave (and monotone), the *Competitive Equilibrium with Equal Incomes* maximizes the Nash product of utilities: hence it is *welfarist* (determined utility-wise by the feasible set of profiles), single-valued and easy to compute.

We generalize the Gale-Eisenberg Theorem to a mixed manna. The Competitive division is still welfarist and related to the product of utilities or disutilities. If the zero utility profile (before any manna) is Pareto dominated, the competitive profile is unique and still maximizes the product of utilities. If the zero profile is unfeasible, the competitive profiles are the critical points of the product of *dis*utilities on the efficiency frontier, and multiplicity is pervasive. In particular the task of dividing a mixed manna is either good news for everyone, or bad news for everyone.

We refine our results in the practically important case of linear preferences, where the axiomatic comparison between the division of goods and that of bads is especially sharp. When we divide goods and the manna improves, everyone weakly benefits under the competitive rule; but no reasonable rule to divide bads can be similarly *Resource Monotonic*. Also, the much larger set of Non Envious and Efficient divisions of bads can be disconnected so that it will admit no continuous selection.

JEL classification: D61, D63, D82.
Keywords: fair division, mixed manna, goods, bads, competitive equilibrium with equal incomes, Nash product, envy-freeness, resource monotonicity, independence of lost bids.

★ University of Glasgow, UK, and National Research University Higher School of Economics, St.Petersburg, Russia; e-mail: anna.bogomolnaia@glasgow.ac.uk

♠ University of Glasgow, UK, and National Research University Higher School of Economics, St.Petersburg, Russia; e-mail: Herve.Moulin@glasgow.ac.uk

♣ National Research University Higher School of Economics, St.Petersburg, Russia, and St.Petersburg Institute for Economics and Mathematics; e-mail: fsandomirskiy@hse.ru

♦ National Research University Higher School of Economics, St.Petersburg, Russia; e-mail: eyanovskaya@hse.ru

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[†] This paper subsumes [4] and [5].

1 Introduction and main result

The literature on fair division of private commodities, with few exceptions discussed in Section 3, focuses almost exclusively on the distribution of disposable commodities, i. e., desirable *goods* like a cake ([43]), family heirlooms ([36]), the assets of divorcing partners ([7]), office space between co-workers, seats in overdemanded business school courses ([42], [10]), computing resources in peer-to-peer platforms ([19]), and so on. Obviously many important fair division problems involve *bads* (non disposable items generating disutility): family members distribute house chores, workers divide job shifts ([9]) like teaching loads, cities divide noxious facilities, managers allocate cuts within the firm, and so on. Moreover the bundle we must divide (the *manna*) often contains the two types of items: dissolving a partnership involves distributing its assets as well as its liabilities, some teachers relish certain classes that others loathe, the land to be divided may include polluted as well as desirable areas, and so on. And the manna may contain items, such as shares in risky assets, or hours of baby-sitting, over which preferences are single-peaked without being monotone, so they will not qualify as either "good" or "bad", they are "satiable" items. Of course each item may be a good to some agents, a bad to others, and satiable to yet other agents. We speak in this case of dividing a *mixed manna*.

Although the fair division literature pays some attention to the case of a "bad" manna, our paper is, to the best of our knowledge, the first to address the case of a mixed manna.

To see why it is genuinely more complicated to divide a mixed rather than a good or a bad manna, consider the popular fairness test of *Egalitarian Equivalence* (EE) due to Pazner and Schmeidler ([35]). A division of the manna is EE if everyone is indifferent between her share and some common reference share: with mixed items this property may well be incompatible with Efficiency.[1] The news is much better for the division proposed by microeconomists four decades ago ([49]), the *Competitive Equilibrium with Equal Incomes* (here competitive division, for short). Existence is guaranteed when preferences are convex, continuous, but not necessarily monotonic and possibly satiated: see e. g., [40], [28]. And this division retains the key normative properties of Efficiency, No Envy, and Core stability from equal initial endowments (see Lemma 1 Section 4).

A striking result by Gale, Eisenberg, and others ([17], [16], [12], [41]) shows that in the subdomain of *homothetic* (as well as concave and continuous) utilities the competitive division of *goods* obtains by simply maximizing the product of individual utilities. This is remarkable for three reasons. First the "resourcist" concept of competitive division guided by a price balancing Walrasian demands, has an equivalent "welfarist" interpretation as the Nash bargaining solution of the feasible utility set. Second, the competitive utility profile is unique because by the latter definition it solves a strictly convex optimization program; it is also computationally easy to find and continuous with respect

[1] Two agents $1,2$ share (one unit of) two items a,b, and their utilities are linear: $u_1(z_1) = z_{1a} - 2z_{1b}$; $u_2(z_2) = -2z_{2a} + z_{2b}$. The only efficent allocation gives a to 1 and b to 2. In an EE allocation (z_1, z_2) there is some $y \geq 0$ such that $u_i(z_i) = u_i(y)$ for $i = 1,2$. This implies $u_1(z_1) + u_2(z_2) = -(y_a + y_b)$ so that z is not efficient.

to the parameters of individual utilities ([50], [25]); all these properties fail under general Arrow-Debreu preferences. Finally the result is broadly applicable because empirical work relies mostly on homothetic utilities, that include additive, Cobb Douglas, CES, Leontief, and their linear combinations. So the Gale Eisenberg theorem is arguably the most compelling practical vindication of the competitive approach to the fair division of goods.

We generalize this result to the division of a mixed manna under concave, continuous and homothetic preferences. We show that the welfarist interpretation of the competitive division is preserved: the set of feasible utility profiles is still all we need to know to identify the competitive utility profiles (those associated with a competitive division of the items). On the other hand there may be many different such profiles, and in that case they no longer solve a convex program: computational simplicity and continuity as above are lost.

We also show that division problems are of three types, and that a very simple welfarist property determines their type. Keep in mind that, by homotheticity, the zero of utilities corresponds to the ex ante state of the world without any manna to divide. Call an agent "attracted" if there is a share of the manna giving her strictly positive utility, and "repulsed" if there is none, that is to say zero is her preferred share.

If it is feasible to give a positive utility to all attracted agents, and zero to all repulsed ones, we call this utility profile "positive" and speak of a "positive" problem. Then the competitive utility profile is positive for attracted agents and maximizes the product of the attracted agents' utilities over positive profiles; just like in Gale Eisenberg this utility profile is unique and easy to compute. Also, the arrival of the manna is (weakly) good news for everyone.

If on the other hand the efficiency frontier contains allocations where *everyone* gets a strictly negative utility, we call the problem "negative". Then the competitive utility profiles are the *critical points* (for instance local maxima or minima) of the product of all **dis**utilities on the intersection of the efficiency frontier with the (strictly) negative orthant:[2] we may have multiple such profiles, and we expect computational difficulties.[3] Moreover, the arrival of the manna is strictly bad news for everyone.

Finally the "null" problems are those knife-edge cases where the zero utility profile is efficient: it is then the unique competitive utility profile, and the arrival of the manna is no news.

[2]See the precise definition in Section 5.

[3]Selecting the competitive utility profiles *maximizing* the product of *dis*utilities on this part of the efficiency frontier almost surely gives a unique utility profile (Lemmas 3 and 4), but does not eliminate the computational and continuity issues, as explained by Proposition 3.

2 The case of linear preferences

The simplest subdomain of the homothetic domain just discussed is that of linear preferences, represented by additive utilities. Its practical relevance is vindicated by user-friendly platforms like SPLIDDIT or ADJUSTED WINNER[4], computing fair outcomes in a variety of problems including the division of manna. Visitors of these sites must distribute 100 points over the different items, and these "bids" are interpreted as fixed marginal utilities, positive for goods, negative for bads, and zero for a satiated item. At the cost of ignoring complementarities between items, this makes the report of preferences fairly easy, eschewing the complex task of reporting full fledged preferences when we have more than a handful of items.[5] The proof of the pudding is in the eating: tens of thousands of visitors have used these sites since 2014, fully aware of the interpretation of their bids ([20]).

If N is the set of agents and A that of items, a profile of additive utilities is described by a $N \times A$ matrix $u = [u_{ia}]$ with $i \in N, a \in A$; agent i's utility for allocation $z_i \in \mathbb{R}_+^A$ is $u_i(z_i) = \sum_A u_{ia} z_{ia}$. If all items are *goods*, the marginal utilities u_{ia} are all non negative and in the terminology just introduced the problem is *positive*: the classic Gale Eisenberg result applies and the competitive utility profile is the unique Nash bargaining solution. If all items are *bads*, the marginal utilities u_{ia} are all non positive and the problem is *negative*.

In the additive domain we evaluate first the potentially severe multiplicity of competitive divisions in negative problems, illustrated in the numerical example below. Next we propose an invariance property in the spirit of Maskin Monotonicity characterizing the competitive division rule for any problem, positive, negative or null.

On the other hand we prove some strong impossibility results for all-bads problems (hence for negative ones as well): they limit the appeal of any division rule guaranteeing No Envy, or simply a fair share of the manna to every participants. Therefore the contrast between positive and negative problems goes beyond the competitive approach, which is somewhat counter-intuitive: just like labor is time not spent on leisure, allocating z_{ia} units of bad a to i is the same as exempting her from eating $\omega_a - z_{ia}$ units of a (where ω_a is the amount of bad a in the manna). But note that we must distribute $(|N| - 1)\omega_a$ units of the a-exemption, while each agent can eat at most ω_a units of it: these additional capacity constraints create the normative differences that we identify.

A numerical example We start with a two agent, three items sequence of examples illustrating the complicated pattern of competitive allocations in negative problems. We have two agents $N = \{1, 2\}$, three items $A = \{a, b, c\}$, one unit of each item, and marginal

[4] `www.spliddit.org/`; `www.nyu.edu/projects/adjustedwinner/`

[5] Similarly practical combinatorial auctions never ask buyers to report a ranking of all subsets of objects, ([6], [51], [15]).

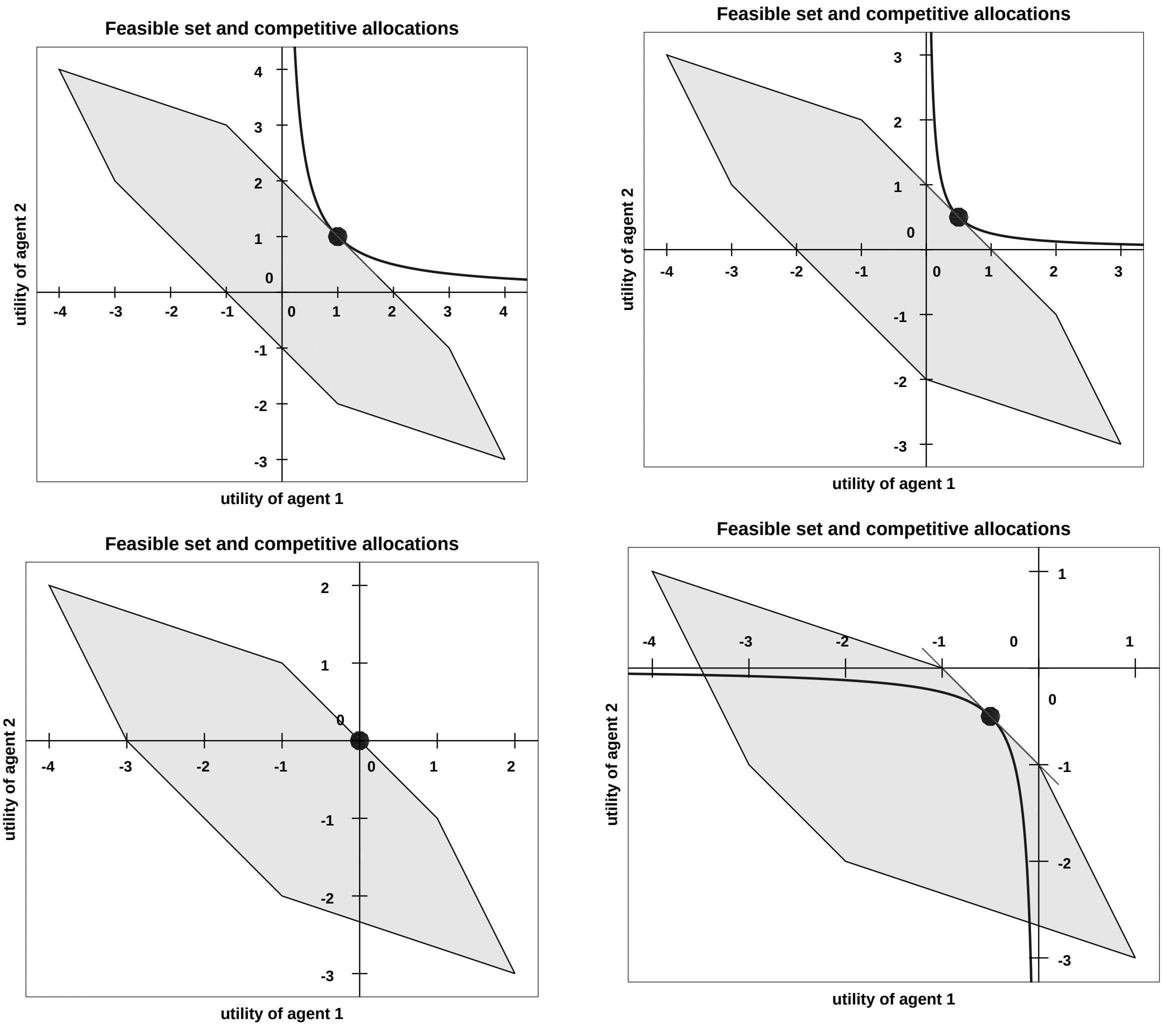


Figure 1 ($\lambda = 4, 3, 2, 1$)

utilities are

	a	b	c
u_1	-1	-3	λ
u_2	-2	-1	λ

Items a, b are bads; as λ takes all integer values from 4 to -3, item c goes from good to satiated ($\lambda = 0$) to bad. For $\lambda = 4, 3$ the problem is positive; it is null for $\lambda = 2$, then negative from $\lambda = 1$ to -3. Figure 1 shows in each case the set of feasible utility profiles, and the competitive utility profiles. Their number varies from 1 to 4. For instance if $\lambda = -1$ all items are bads and the four competitive utility profiles are $(-1, -2)$, $(-1.5, -1.5)$, $(-2, -1)$, $(-2.5, -0.83)$. In Section 5 (Lemmas 3 and 4) we propose to select the profile $(-1.5, -1.5)$ maximizing the product of *dis*utilities; the corresponding allocation is $z_1 = (1, 0, \frac{1}{2})$, $z_1 = (0, 1, \frac{1}{2})$. Note that for $\lambda = 1$ (Figure 1.e) this maximum is achieved by the competitive allocation most favourable to agent 1.

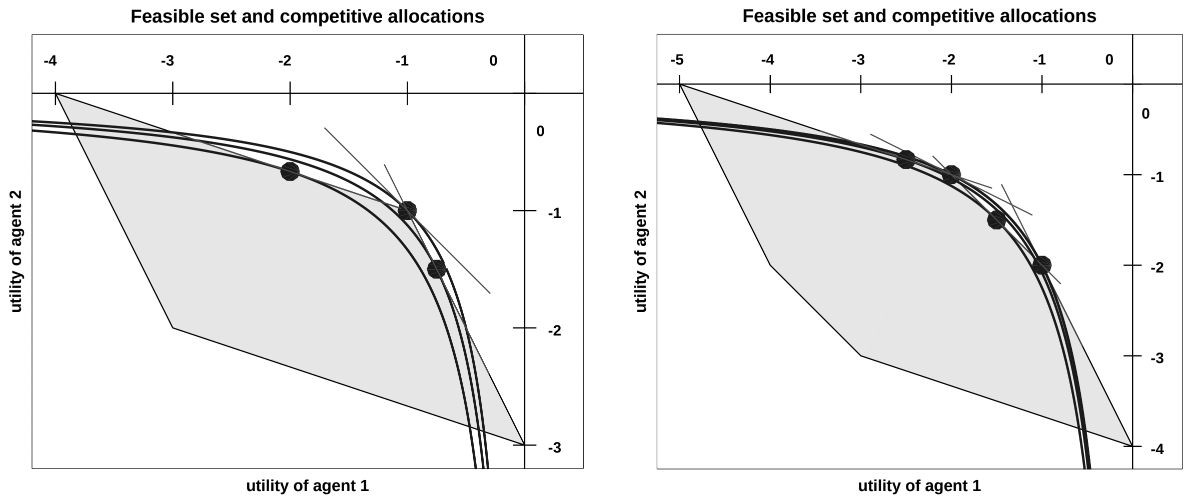


Figure 1 ($\lambda = 0, -1$)

In Subsection 6.1 we estimate the maximal number of welfare-wise different competitive allocations in an *all bads* (or negative) problem. This number grows at least exponentially in the smallest of the number of agents or bads (Proposition 1).

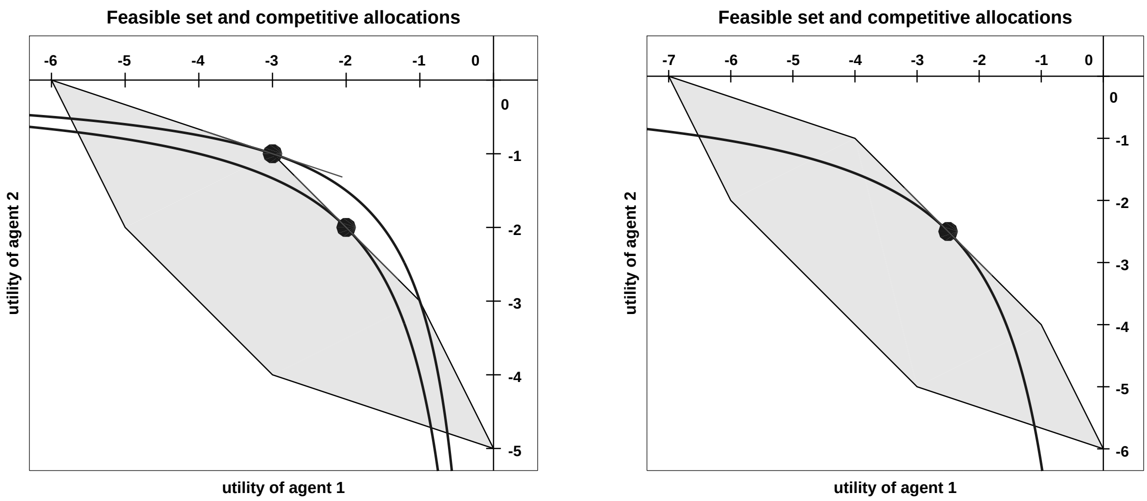


Figure 1 ($\lambda = -2, -3$)

An axiomatic characterization of the competitive rule In the linear domain agents report marginal utilities, which we can interpret as "bids" for the different items (as in [42]). Thus agent i's bid u_{ia} for item a is "losing" if she ends up not consuming any a. *Independence of Lost Bids* (ILB) means that nothing changes when we lower a losing bid: it remains losing and the allocation selected by the rule does not change. The ILB axiom implies a weak incentive property: misreporting on an item which I do not consume anyway (whether I misreport or not) does not pay, and does not affect anyone else either. Here is another consequence of ILB. Suppose item a is a good for agent 1, $u_{ia} > 0$, but a bad for agent 2, $u_{2a} < 0$; by Efficiency agent 2 consumes no a; then ILB says that the selected allocations do not change if agent 2's bid for a was zero instead of u_{2a}. In turn this means that if an item is strictly good for someone, we can assume that it is either good or satiated for everyone else.

The ILB axiom is a weak form of Maskin Monotonicity as explained in Subsection 7.7. In combination with the requirement that all agents end up on the same side of their zero utility, it promptly characterizes the competitive rule for all mixed manna problems (Proposition 2).

Continuity and Monotonicity properties For a general problem with goods and bads, the set of competitive utility profiles is an upper-hemi-continuous correspondence in the matrix of marginal utilities. For positive problems it is single-valued, hence continuous, but for negative problems it does not admit a continuous single-valued selection. Proposition 3 strengthens this statement by weakening the competitiveness requirement to the much less demanding test of No Envy. In an *all bads* problem with three or more agents, there is no continuous single valued selection of the set of efficient and Non Envious allocations; in particular with n agents and two bads the corresponding set of utility profiles can have up to roughly $\frac{2}{3}n$ connected components.

Our last result is also a (simple) impossibility statement. We use the familiar axiom *Resource Monotonicity* (RM) to draw another wedge between positive and negative problems. RM is a solidarity requirement when the manna improves: if we increase the amount of a unanimous good (an item everyone likes), or decrease that of a unanimous bad, everyone should benefit at least weakly.[6] In a positive problem the (single-valued) competitive rule is Resource Monotonic, but in an all bads problems (hence in negative problems as well), no single-valued rule guaranteeing his *Fair Share* to every agent[7] is Resource Monotonic (Proposition 4).

Contents After reviewing the literature (Section 3) and defining the model (Section 4), Section 5 states our generalization of Gale Eisenberg to mixed manna. We focus in Section 6 on the subdomain of linear preferences and the four propositions just described. All substantial proofs are in Section 7.

[6] RM has been applied to many other resource allocation problems with production and/or indivisibilities. See the recent survey [45].

[7] That is, no one is worse off than by consuming a $\frac{1}{n}$-th share of every item. It is an uncontroversial fairness requirement, much weaker than No Envy in the linear domain.

3 Related literature

1. Steinhaus' 1948 "cake-division" model ([43]), assumes linear preferences represented by atomless measures over, typically, a compact euclidean set. It contains our model for goods as the special case where the measures have piecewise constant densities. Sziklai and Segal-Halevi ([38]) show that it preserves the equivalence of the competitive rule and the Nash product maximizer, and that this rule is Resource Monotonic. The cake division literature pays some attention to the division of a *bad* cake, to prove the existence of envy-free divisions of the cake ([44], [3]), or to examine how the classic algorithms by cuts and queries can or cannot be adapted to this case ([7], [37]). It does not discuss the competitive rule for a bad cake.

2. The recent work in computational social choice discusses extensively the fair division of goods (see the survey [8]), recognizing the practical convenience of additive utilities and the conceptual advantages of the competitive solution in that domain (see [31], [50]). For instance Megiddo and Vazirani ([30]) show that the competitive utility profile depends continuously upon the rates of substitution and the total endowment; Jain and Vazirani ([25]) that it can be computed in time polynomial in the dimension $n+m$ of the problem (number of agents and of goods).

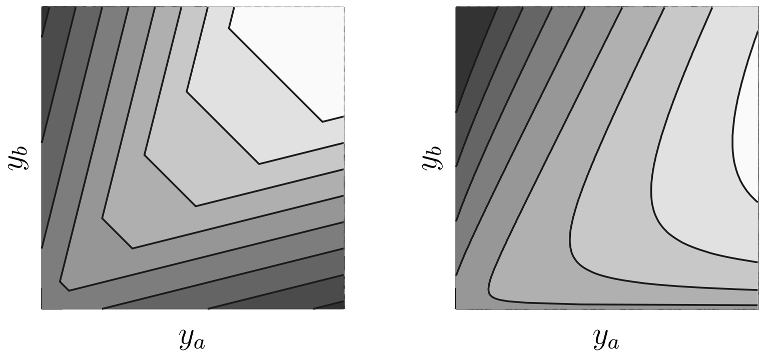
3. The fair division of *indivisible goods* with additive utilities is a much studied variant of the standard model. The maximization of the Nash product loses its competitive interpretation and becomes hard to compute ([26]), however it is envy-free "up to at most one object" ([11]) and can be efficiently approximated for many utility domains ([14], [1], [2], [13]). Also Budish ([9]) approximates the competitive allocation in problems with a large number of copies of several good-types by allowing some flexibility in the number of available copies.

4. Our Proposition 2 is closely related to several axiomatic characterizations of the competitive rule for the fair division of private *goods*, in the much larger domain of Arrow-Debreu preferences. The earliest results by Hurwicz ([23]) and Gevers ([18]) are refined by Thomson ([46]) and Nagahisa ([33]): any efficient and Pareto indifferent rule meeting (some variants of) Maskin Monotonicity (MM) must contain the competitive rule.[8] Our Independence of Lost Bids is weaker than MM in the linear domain, so our Proposition 2 is a variant of these results in the case of mixed items (and homothetic preferences).

5. The probabilistic assignment of goods with von Neuman Morgenstern utilities is another fair division problem with linear and possibly satiated preferences where Hylland and Zeckhauser ([24]) and the subsequent literature recommend (a version of) the competitive rule: e. g., [21]. That rule is no longer related to the maximization of the product of utilities.

6. The purely welfarist axiomatic discussion of non convex bargaining problems identifies the set of critical points of the Nash product among efficient utility profiles as a

[8]Another, logically unrelated characterization combines Consistency and Replication Invariance ([47]) or Consistency and Converse Consistency ([34]).



Figures 2 and 3

natural generalisation of the Nash solution: [22], [39]. This solution stands out also in the rationing model of [27] where we divide utility losses instead of gains. The latter is closer in spirit to our results for the division of bads.

4 The model

The set of agents is N, that of items is A; both are finite. The domain $\mathcal{H}(A)$ consists of all preferences on $\mathbb{R}_+^A$ represented by a real-valued utility function v on $\mathbb{R}_+^A$ that is concave, continuous, and 1-homothetic: $v(\lambda y) = \lambda v(y)$ for all $\lambda \geq 0, y \in \mathbb{R}_+^A$. It is easily checked that if two such utility functions represent the same preference, they differ by a positive multiplicative constant. All our definitions and results are purely ordinal, i. e., independent of the choice of the utility representations; we abuse language by speaking of "the utility function v in $\mathcal{H}(A)$".

The graph of a concave and continuous function v on $\mathbb{R}_+^A$ is the envelope of its supporting hyperplanes, therefore it takes the form $v(y) = \min_{k\in K}\{\alpha_k \cdot y + \beta_k\}$ for some $\alpha_k \in \mathbb{R}^A, \beta_k \in \mathbb{R}$ and a possibly infinite set K. It is easy to see that v is also homothetic if and only if we can choose $\beta_k = 0$ for all k. So the simplest examples are the additive utilities $v(y) = \alpha \cdot y$ and the piecewise linear utilities like $v(y) = \min\{y_a + y_b, 4y_a - y_b, 4y_b - y_a\}$ for $A = \{a, b\}$, of which the indifference contours are represented on Figure 2. Note that this utility is not globally satiated, but for fixed y_b it is satiated at $y_a = y_b$. For a smooth example of a non monotonic function in $\mathcal{H}(A)$ consider for example $v(y) = y_b \ln\{\frac{y_a}{y_b} + \frac{1}{2}\}$, represented in Figure 3.

A *fair division problem* is $\mathcal{P} = (N, A, u, \omega)$ where $u \in \mathcal{H}(A)^N$ is the profile of utility functions, and $\omega \in \mathbb{R}_+^A$ is the manna; we assume $\omega_a > 0$ for all a.

A *feasible allocation* (or simply an *allocation*) is $z \in \mathbb{R}_+^{N\times A}$ such that $\sum_N z_{ia} = \omega_a$ for all a, or in a more compact notation $z_N = \omega$. The corresponding utility profile is $U \in \mathbb{R}^N$ where $U_i = u_i(z_i)$. Let $\mathcal{F}(N, A, \omega)$ be the set of feasible allocations, and $\mathcal{U}(\mathcal{P})$ the corresponding set of utility profiles. We always omit $\mathcal{P}$ or N, A if it creates no confusion.

We call a feasible utility profile U *efficient* if it is not Pareto dominated[9]; a feasible allocation is efficient if it implements an efficient utility profile.

Definition 1: *Given problem $\mathcal{P}$ a competitive division is a triple $(z \in \mathcal{F}, p \in \mathbb{R}^A, \beta \in \{-1, 0, +1\})$ where z is the competitive allocation, p is the competitive price and β the individual budget. The allocation z is feasible and each z_i maximizes i's utility in the budget set $B(p, \beta) = \{y_i \in \mathbb{R}^A_+ | p \cdot y_i \leq \beta\}$:*

$$z_i \in d_i(p, \beta) = \arg \max_{y_i \in B(p,\beta)} \{u_i(y_i)\} \tag{1}$$

Moreover z_i minimizes i's wealth in her demand set

$$z_i \in \arg \min_{y_i \in d_i(p,\beta)} \{p \cdot y_i\} \tag{2}$$

We write $CE(\mathcal{P})$ for the set of competitive allocations, and $CU(\mathcal{P})$ for the corresponding set of utility profiles.

Existence of a competitive allocation can be derived from (much) earlier results that do not require monotonic preferences (e.g., Theorem 1 in [28]; see also [40]), but our main result in the next section gives instead a constructive proof.

In addition to utility maximization (1), property (2) requires demands to be *parsimonious*: each agent spends as little as possible for her competitive allocation. This requirement appears already in [28]: in its absence some satiated agents in N_- may inefficiently eat some items useless to themselves but useful to others.[10]

Recall three standard normative properties of an allocation $z \in \mathcal{F}(N, A, \omega)$. It is *Non Envious* iff $u_i(z_i) \geq u_i(z_j)$ for all i, j. It *Guarantees Fair Share* utility *iff* $u_i(z_i) \geq u_i(\frac{1}{n}\omega)$ for all i. It is in the *Weak Core from Equal Split* iff for all $S \subseteq N$ and all $y \in \mathbb{R}^{S \times A}_+$ such that $y_S = \frac{|S|}{n}\omega$, there is at least one $i \in S$ such that $u_i(z_i) \geq u_i(y_i)$. When we divide goods competitive allocations meet these three properties, even in the much larger Arrow Debreu preference domain. This is still true with mixed items.

Lemma 1 *A competitive allocation is efficient; it is No Envious, Guarantees Fair Share, and is in the Weak Core from Equal Split.*

Proof. No Envy is clear. Fair Share Guaranteed holds because $B(p, \beta)$ contains $\frac{1}{n}\omega$. We check Efficiency. If (z, p, β) is a competitive division and z is Pareto-dominated by some $z' \in \mathcal{F}$, then for all $i \in N$ we must have $(p, z'_i) \geq (p, z_i)$ because otherwise i can either benefit or save money by switching to z'_i (property (2)). Since z' dominates z, some agent j strictly prefers z'_j to z_j, and therefore z'_j is outside his budget set, i.e., $(p, z'_j) > (p, z_j)$. Summing up these inequalities over all agents we get the contradiction $(p, \omega) > (p, \omega)$. The argument for the Weak Core property is similar. ■

[9] That is $U \leq U'$ and $U' \in \mathcal{U}(\mathcal{P}) \Longrightarrow U' = U$.

[10] For instance $N = \{1, 2\}, A = \{a, b\}, \omega = (1, 1)$ and $u_1(z_1) = 6z_{1a} + 2z_{1b}$, $u_2(z_2) = -z_{2b}$. The inefficient allocation $z_1 = (\frac{1}{3}, 1), z_2 = (\frac{2}{3}, 0)$ meets (1) for $p = (\frac{3}{2}, \frac{1}{2})$ and $\beta = 1$. But $z'_2 = (0, 0)$ also gives zero utility to agent 2 and costs zero, so z_2 fails (2). The unique competitive division according Definition 1 is efficient: $z_1 = (1, 1), z_2 = 0$, and $p = (\frac{1}{2}, \frac{1}{2})$.

Remark 1: A competitive allocation may fail the standard Core from Equal Split property, where coalition S blocks allocation z if it can use its endowment $\frac{|S|}{n}e^A$ to make everyone in S weakly better off and at least one agent strictly more. This is because "equal split" may give resources to agents who have no use for them. Say three agents share one unit of item a with $u_i(z_i) = z_i$ for $i = 1, 2$ and $u_3(z_3) = -z_3$. The competitive allocation splits a equally between agents 1 and 2, which coalition $\{1, 3\}$ blocks by giving $\frac{2}{3}$ of a to agent 1.

5 Main result

We define formally the partition of division problems alluded to in the Introduction. Given a problem $\mathcal{P}$ we partition N as follows:

$$N_+ = \{i \in N | \exists z \in \mathcal{F} : u_i(z_i) > 0\} \ ; \ N_- = \{i \in N | \forall z \in \mathcal{F} : u_i(z_i) \leq 0\}$$

We call agents in N_+ *attracted to* the manna, and those in N_- *repulsed by* it. All agents in N_-, and only those, are globally satiated, and for them $z_i = 0$ is a global maximum, not necessarily unique.

The partition is determined by the relative position of the set $\mathcal{U}$ of feasible utility profiles and the cone $\Gamma = \mathbb{R}_+^{N_+} \times \{0\}^{N_-}$, where attracted agents benefit while repulsed agents do not suffer. Let $\Gamma^* = \mathbb{R}_{++}^{N_+} \times \{0\}^{N_-}$be the relative interior of Γ.

Lemma 2 *Each problem $\mathcal{P}$ is of (exactly) one of three types:*
positive if $\mathcal{U} \cap \Gamma^* \neq \varnothing$; negative if $\mathcal{U} \cap \Gamma = \varnothing$; null if $\mathcal{U} \cap \Gamma = \{0\}$.

Given a smooth function f and a closed convex C we say that *$x \in C$ is a critical point of f in C* if the upper contour of f at x has a supporting hyperplane that supports C as well:

$$\forall y \in C : \partial f(x) \cdot y \leq \partial f(x) \cdot x \text{ and/or } \forall y \in C : \partial f(x) \cdot y \geq \partial f(x) \cdot x \quad (3)$$

This holds in particular if x is a local maximum or local minimum of f in C.

In the next statement we write $\mathcal{U}^{eff}$ for the set of efficient utility profiles, and $\mathbb{R}^N_=$ for the interior of $\mathbb{R}^N_-$.

Theorem *Competitive divisions exist in all problems $\mathcal{P}$. Moreover*
i) If $\mathcal{P}$ is positive their budget is $+1$; an allocation is competitive iff its utility profile maximizes the product $\Pi_{N_+}U_i$ over $\mathcal{U} \cap \Gamma^*$; so $CU(\mathcal{P})$ contains a single utility profile, positive in N_+ and null in N_-.
ii) If $\mathcal{P}$ is negative their budget is -1; an allocation is competitive iff its utility profile is in $\mathcal{U}^{eff} \cap R^N_=$ and is a critical point of the product $\Pi_N|U_i|$ in $\mathcal{U}$; so all utility profiles in $CU(\mathcal{P})$ are negative.
iii) If $\mathcal{P}$ is null their budget is 0; an allocation is competitive iff its utility profile is 0.

We see that the competitive utility profiles are entirely determined by the set of feasible utility profiles: the competitive approach still has a welfarist interpretation when we divide a mixed manna.

Moreover the Theorem implies that the task of dividing the manna is either good news (at least weakly) for everyone, or strictly bad news for everyone.

The possible multiplicity of $CU(\mathcal{P})$ for negative problems with linear preferences is the subject of Subsection 6.1. Without backing up this proposal by specific normative arguments, we submit that a natural selection of $CU(\mathcal{P})$ obtains by maximizing the Nash product of individual *dis*utilities on the negative efficiency frontier.[11]

Lemma 3 *If $\mathcal{P}$ is a negative problem, the profile U^* maximizing the Nash product $\prod_{i\in N}|U_i|$ over $\mathcal{U}^{eff}\cap\mathbb{R}^N_-$ is a critical point of the product on $\mathcal{U}$ and $U^*_i<0$ for all $i\in N$; hence $U^*\in CU(\mathcal{P})$.*

This selection is almost always unique: we prove this in the linear domain.

Lemma 4 *Fix N, A and ω. For almost all negative problems $\mathcal{P}=(N,A,u,\omega)$ with additive utilities (w.r.t. the Lebesgue measure on the space $R^{N\times A}$ of utility matrices) the utility profile U^* defined in Lemma 3 is unique.*

Remark 2 The Competitive Equilibrium with Fixed Income Shares (CEFI for short) replaces in Definition 1 the common budget β by individual budgets $\theta_i\beta$, where the positive weights θ_i are independent of preferences. It is well known that in an all goods problem, this asymmetric generalization of the competitive solution obtains by maximizing the weighted product $\Pi_N U_i^{\theta_i}$ of utilities, so that it preserves the uniqueness, computational and continuity properties of the symmetric solution. The same is true of our Theorem that remains valid word for word for the CEFI divisions upon raising U_i to the power θ_i. In particular the partition of problems in positive, negative or null is unchanged.

6 Additive utilities

A utility function is now a vector $u_i\in\mathbb{R}^A$ and corresponding utilities are $U_i=u_i\cdot z_i=\sum_A u_{ia}z_{ia}$. For agent i item a is a good (resp. a bad) if $u_{ia}>0$ (resp. $u_{ia}<0$); if $u_{ia}=0$ she is satiated with any amount of a. Given a problem $\mathcal{P}$ the following partition of items is key to understanding the competitive divisions.

$$A_+=\{a|\exists i:u_{ia}>0\}\ ;\ A_-=\{a|\forall i:u_{ia}<0\}\ ;\ A_0=\{a|\max_i u_{ia}=0\} \tag{4}$$

We call an item in A_+ a *collective good*, one in A_- a *collective bad*, and one in A_0 a *neutral* item. In an efficient allocation an item in A_+ is consumed only by agents for whom it is a good, and a neutral item in A_0 is consumed only by agents who are indifferent to it. We note that the above partition determines the sign of competitive prices.

Fact: if (z,p,β) is a competitive division, we have

$$p_a>0 \text{ if } a\in A_+\ ;\ p_a<0 \text{ if } a\in A_-\ ;\ p_a=0 \text{ if } a\in A_0 \tag{5}$$

[11] Note that minimizing the $\prod_{i\in N}|U_i|$ on $\mathcal{U}\cap\mathbb{R}^N_-$ picks a boundary point where this product is null, not a competitive allocation.

The proof is simple. If the first statement fails an agent who likes a would demand an infinite amount of it; if the second fails no one would demand b. If the third fails with $p_a > 0$ the only agents who demand a have $u_{ia} = 0$, so that eating some a violates (2); if it fails with $p_a < 0$ an agent such that $u_{ia} = 0$ gets an arbitrarily cheap demand by asking large amounts of a, so (2) fails again.

6.1 The multiplicity issue

Proposition 1 *If utilities are additive in problem $\mathcal{P}$, the number $|CU(\mathcal{P})|$ of distinct competitive utility profiles is finite. Set $n = |N|$ and $m = |A|$, then*
i) If $n = 2$ the upper bound of $|CU(\mathcal{P})|$ is $2m - 1$.
ii) If $m = 2$ the upper bound of $|CU(\mathcal{P})|$ is $2n - 1$.
iii) For general n, m, $|CU(\mathcal{P})|$ can be as high as $2^{\min\{n,m\}} - 1$ if $n \neq m$, and $2^{n-1} - 1$ if $n = m$.
We offer no guess about the upper bound of $|CU(\mathcal{P})|$ for general n, m.

Three examples follow to illustrate the Proposition. For statement $i)$ the agents in $N = \{1, 2\}$ share five bads $A = \{a, b, c, d, e, f\}$, one unit of each; utilities are

	a	b	c	d	e	f
u_1	-1	-1	-2	-4	-8	-17
u_2	-17	-8	-4	-2	-1	-1

Here $|CE(\mathcal{P})| = |CU(\mathcal{P})| = 11$. In five competitive allocations no bad is split between the agents; agent 1 eats all the bads in a left interval of A, and agent 2 all those in the complement right interval of A. For instance $\{a, b\}$ for 1 and $\{c, d, e, f\}$ for 2 is sustained by the price $p = -(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ and $\beta = -1$. In addition we have six competitive allocations where exactly one bad is shared between 1 and 2, while 1 gets the bads to its left, if any, and 2 those to its right, if any. For instance if we split f agent 1 gets the five other bads and $\frac{1}{34}$ of f, while 2 eats $\frac{33}{34}$ of f; the price is $p = -\frac{1}{33}(2, 2, 4, 8, 16, 34)$. Notice that agent 1 gets exactly his Fair Share utility (from eating $\frac{1}{2}$ of every item).

For statement $ii)$ we take $N = \{1, 2, 3, 4, 5, 6\}$, two bads $A = \{a, b\}$, one unit of each, and the utilities

	u_1	u_2	u_3	u_4	u_5	u_6
a	-1	-1	-2	-3	-3	-6
b	-6	-3	-3	-2	-1	-1

Again $|CE(\mathcal{P})| = |CU(\mathcal{P})| = 11$. The five allocations where the left-most agents divide a equally and eat no b, while the right-most ones divide b equally and eat no a, are competitive. For instance 1 and 2 share a while $3, 4, 5, 6$ share b corresponds to $p = -(2, 4)$ and $\beta = -1$. In the other six competitive divisions one agent eats some of both bads, agents to his left eat only a and agents to his right only b. For instance the allocation

	z_1	z_2	z_3	z_4	z_5	z_6
a	5/12	5/12	1/6	0	0	0
b	0	0	1/6	5/18	5/18	5/18

is sustained by the price $p = -(\frac{12}{5}, \frac{18}{5})$.

Finally for statement *iii)* we set $N = \{1,2,3,4,5,6\}$, $A = \{a,b,c,d,e\}$, one unit of each bad, and the utilities

	a	b	c	d	e
u_1	-1	-3	-3	-3	-3
u_2	-3	-1	-3	-3	-3
u_3	-3	-3	-1	-3	-3
u_4	-3	-3	-3	-1	-3
u_5	-3	-3	-3	-3	-1
u_6	-1	-1	-1	-1	-1

We check that $|CE(\mathcal{P})| = |CU(\mathcal{P})| = 31$. The symmetric competitive division with uniform price $\frac{6}{5}$ for each bad gives to each of the first five agents $\frac{5}{6}$ units of her preferred bad, while agent 6 eats $\frac{1}{6}$ of every bad, precisely his Fair Share. Now for each strict subset of the first five agents, for instance $\{3,4,5\}$, there is a competitive allocation where each such agent eats "his" bad in full, while agent 1 shares the rest with the other agents:

	a	b	c	d	e
z_1	2/3	0	0	0	0
z_2	0	2/3	0	0	0
z_3	0	0	1	0	0
z_4	0	0	0	1	0
z_5	0	0	0	0	1
z_6	1/3	1/3	0	0	0

Here prices are $p = -(\frac{3}{2}, \frac{3}{2}, 1, 1, 1)$. This construction can be adjusted for each non trivial partition of the first five agents. Note that agent 6's utility goes from -1 (his Fair Share) to $-\frac{1}{2}$, when he shares a single bad with a single other agent; utilities of other agents vary also between -1 and $-\frac{1}{2}$.

Remark 3. It is easy to show that for $n = 2$ and/or $m = 2$, $|CU(\mathcal{P})|$ is odd in almost all problems (excluding only those where the coefficients of u satisfy certain simple equations). We conjecture that a similar statement holds for any n, m.

6.2 Independence of Lost Bids

We offer a compact axiomatic characterization of competitive fair division. Because our axioms compare the selected allocations across different problems, we define first division rules. Notation: when we rescale each utility u_i as $\lambda_i u_i$, the new utility matrix is written $\lambda * u$.

Definition 2 *A division rule f associates to every problem $\mathcal{P} = (N, A, u, \omega)$ a set of feasible allocations $f(\mathcal{P}) \subset \mathcal{F}(N, A, \omega)$ such that for any rescaling λ, $\lambda_i > 0$ for all i, we* *have:* $f(N, A, \lambda * u, \omega) = f(N, A, u, \omega)$. *Moreover* f *meets Pareto-Indifference (PI). For every* $\mathcal{P}$ *and* $z, z' \in \mathcal{F}(N, A, \omega)$

$$\{z \in f(\mathcal{P}) \text{ and } u_i \cdot z_i = u_i \cdot z'_i \text{ for all } i\} \Longrightarrow z' \in f(\mathcal{P})$$

Note that PI implies that f is entirely determined by its utility correspondence $F(\mathcal{P}) = \{u \cdot z | z \in f(\mathcal{P})\}$. The invariance to rescaling property makes sure that division rules are ordinal constructs, they only depend upon the underlying linear preferences.

The competitive division rule $\mathcal{P} \rightarrow CE(\mathcal{P})$ meets Definition 2. We give other examples after Proposition 2. Definition 2 is not restricted to linear preferences, but our next axiom is.

Definition 3 *The division rule* f *is Independent of Lost Bids (ILB) if for any two problems* $\mathcal{P}, \mathcal{P}'$ on N, A, ω *where* u, u' *are additive, differ only in the entry* ia*, and* $u'_{ia} < u_{ia}$*, we have*

$$\forall z \in f(\mathcal{P}) : z_{ia} = 0 \Longrightarrow z \in f(\mathcal{P}') \quad (6)$$

Recall from Section 2 our interpretation of u_{ia} as agent i's bid for item a. ILB says that the bid u_{ia} only matters if it is winning, i. e., agent i eats some of item a. It can be shown that for a generic utility matrix u an efficient allocation z has no more than $n + m - 1$ non zero coordinates (see Lemma 1 in [4]): then ILB reduces considerably the number of parameters relevant to describe the outcome selected by the rule.

That the competitive rule $\mathcal{P} \rightarrow CE(\mathcal{P})$ meets ILB is clear by Definition 1: as a becomes less attractive to i in the shift from $\mathcal{P}$ to $\mathcal{P}'$, i's Walrasian demand can only shrink, and it still contains z_i.

The characterization requires the uncontroversial fairness property known as **Equal Treatment of Equals** (ETE): for all $\mathcal{P}$

$$u_i = u_j \Longrightarrow U_i = U_j \text{ for all } U \in F(\mathcal{P}) \text{ and all } i, j \in N$$

We also impose the solidarity property uncovered in our Theorem. **Solidarity** (SOL): for all $\mathcal{P}$

$$U_i \cdot U_j \geq 0 \text{ for all } U \in F(\mathcal{P}) \text{ and all } i, j \in N$$

Finally we call the rule f **Efficient** (EFF) if it selects only efficient allocations in every problem $\mathcal{P}$.

Proposition 2 *If a division rule meets Equal Treatment of Equals, Solidarity, Efficiency and Independence of Lost Bids, it contains the competitive rule.*

If problem $\mathcal{P}$ involves only goods ($u_{ia} \geq 0$ for all i, a) or only bads ($u_{ia} \leq 0$ for all i, a), Solidarity is automatically true, so the characterization boils down to ETE, EFF and ILB.

We show after the proof (Subsection 7.7) that ILB is a strictly weaker requirement than Maskin Monotonicity in the linear domain, thus connecting Proposition 2 to earlier results mentioned in point 4 of Section 3.

We discuss the tightness of our characterization.

Drop ETE. The CEFI division rule (Remark 1 Section 5) fails ETE for general weights. It is straightforward to check that it meets ILB either by suitably adapting Lemma 6 or directly in the general Definition 1. Solidarity follows from our (adapted) Theorem.

Drop ILB. Inspired by the Kalai-Smorodinsky bargaining solution we construct now an efficient welfare rule F meeting SOL and ETE but failing ILB. Observe that if $\mathcal{P}$ is positive we have $U_i^{\max} = \max_{U\in\mathcal{U}} U_i > 0$ for all $i \in N_+$, and if $\mathcal{P}$ is negative $U_i^{\min} = \min_{U\in\mathcal{U}} U_i < 0$. In a positive problem the rule picks the unique efficient utility profile U such that $\frac{U_i}{U_i^{\max}}$ is constant for $i \in N_+$, and $U_i = 0$ in N_-; in a negative problem it picks the efficient profile such that $\frac{U_i}{U_i^{\min}}$ is constant for all i; and the null utility at a null problem.

We do not know if the statement is tight with respect to SOL, but recall that SOL is not needed for all goods or all bads problems. We conjecture that the statement is tight with respect to EFF. We know at least that we cannot drop both EFF and SOL, because a constrained version of the competitive rule, where we impose $\sum_A z_{ia} = \frac{1}{n}\sum_A \omega_{ia}$ as in [24], satisfies ETE and ILB.

6.3 Single-valued Efficient and Envy-Free rules

In this section and the next we uncover some negative features of the competitive division rule in negative problems. It will be enough to state them for "all bads" problems. The first result follows from a careful analysis of the set $\mathcal{A}$ of efficient and envy-free allocations in problems with two bads a, b, and any number of agents.

Lemma 5 *If we divide at least two bads between at least three agents, there are problems $\mathcal{P}$ where the set $\mathcal{A}$ of efficient and envy-free allocations, and the corresponding set of disutility profiles, have $\lfloor \frac{2n+1}{3} \rfloor$ connected components.*

In a two-agent problem (even with mixed manna), No Envy coincides with Fair Share Guarantee, so the set $\mathcal{A}$ is clearly connected.

The proof of Lemma 5 makes clear that in a problem with exactly two bads the maximal number of connected components of $\mathcal{A}$ is indeed $\lfloor \frac{2n+1}{3} \rfloor$. But we have no clue about the maximal number of components in general all-bads problems. Nor do we know the answer for the division of goods: if we divide exactly two goods, one can easily check that $\mathcal{A}$ is connected. But beyond this simple case we do not know if $\mathcal{A}$ remains connected in every "all goods" problem.

We call the division rule f **Continuous** (CONT) if for each choice of N, A, the corresponding welfare rule $(N, A, u, \omega) \to F(N, A, u, \omega)$ is a continuous function of $u \in \mathbb{R}^{N\times A}$. If the division rule does not depend upon the units of items in A,[12] CONT implies that $\mathcal{P} \to F(\mathcal{P})$ is also continuous in $\omega \in \mathbb{R}_+^A$.

We call the rule f **Envy-Free** (EVFR) if $f(\mathcal{P})$ contains at least one envy-free allocation for every problem $\mathcal{P}$.

[12]That is, for each $\lambda > 0$ the set $F(\mathcal{P})$ is unchanged if we replace ω_a by $\lambda\omega_a$ and u_{ia} by $\frac{1}{\lambda}u_{ia}$. Clearly *CU* meets this property.

Proposition 3 *If we divide at least two bads between at least four agents, no single-valued rule can be Efficient, Envy-Free and Continuous.*

This incompatibility result is tight. The equal division rule, $F_i(\mathcal{P}) = \{\frac{1}{n}u_i \cdot \omega\}$ for all $\mathcal{P}$, is EVFR and CONT. A single-valued selection of the competitive rule CU meets EFF and EVFR. The Egalitarian rule defined at the end of the previous subsection meets EFF and CONT.

6.4 Resource Monotonicity

Adding more of an item that everyone likes to the manna, or removing some of one that everyone dislikes, should not be bad news to anyone: the agents own the items in common and welfare should be comonotonic to ownership. When this property fails someone has an incentive to sabotage the discovery of new goods, or add new bads to the manna.

We say that problem $\mathcal{P}'$ improves problem $\mathcal{P}$ on item $a \in A$ if they only differ in the amount of item a and either $\{\omega_a \leq \omega'_a$ and $u_{ia} \geq 0$ for all $i\}$ or $\{\omega_a \geq \omega'_a$ and $u_{ia} \leq 0$ for all $i\}$.

Resource Monotonicity (RM): if $\mathcal{P}'$ improves upon $\mathcal{P}$ on item $a \in A$, then $F(\mathcal{P}) \leq F(\mathcal{P}')$

Proposition 4
i) With two or more agents and two or more bads, no efficient single-valued rule can be Resource Monotonic and Guarantee Fair Share ($u_i \cdot z_i \geq \frac{1}{n}u_i \cdot \omega$).
ii) The competitive rule to divide goods is Resource Monotonic (as well as single-valued, efficient and GFS).

The proof of statement *i*) is by means of a simple two-person, two-bad example. Fix a rule F meeting EFF, RM and GFS. Consider the problem $\mathcal{P}$ with $\begin{array}{ccc} & a & b \\ u_1 & -1 & -4 \\ u_2 & -4 & -1 \end{array}$ and $\omega = (1,1)$, and set $U = F(\mathcal{P})$. As $-(1,1)$ is an efficient utility profile, one of U_1, U_2 is at least -1, say $U_1 \geq -1$. Now let $\omega' = (\frac{1}{9}, 1)$ and pick $z' \in f(\mathcal{P}')$. By GFS and feasibility:

$$-z'_{2b} \geq u_2 \cdot z'_2 \geq \frac{1}{2}u_2 \cdot \omega' = -\frac{13}{18}$$

$$\Longrightarrow z'_{1b} \geq \frac{5}{18} \Longrightarrow u_1 \cdot z'_1 = U'_1 \leq -\frac{10}{9} < U_1$$

contradicting RM. Extending this argument to the general case $n \geq 3, m \geq 2$ is straightforward.

We omit for brevity the proof of statement *ii*), available in [4] as well as in [38] for the more general cake-division model. It generalizes easily to positive problems, when we add a unanimous good to an already positive problem.

We stress that this positive result applies only to the linear domain, it does not extend to general homothetic, convex and monotonic preferences. On the latter domain, precisely the same combination of axioms as in Proposition 4 cannot be together satisfied: see [32] and [48] . This makes the goods versus bads contrast in the case of linear preferences all the more intriguing.

7 Appendix: Proofs

7.1 Lemma 2

The three cases are clearly mutually exclusive; we check they are exhaustive. It is enough to show that if $\mathcal{U}$ intersects $\Gamma_{\neq 0} = \Gamma \diagdown \{0\}$ then it intersects Γ^* as well. Let $z \in \mathcal{F}$ be an allocation with $u(z) \in \Gamma_{\neq 0}$ and i_+ be an agent with $u_{i_+}(z_{i_+}) > 0$. Define a new allocation z' with $z'_{i_+} = z_{i_+} + \varepsilon \sum_{j \neq i_+} z_j$ and $z'_j = (1-\varepsilon)z_j$ for $j \neq i_+$. By continuity we can select a small $\varepsilon > 0$ such that $u(z') \in \Gamma_{\neq 0}$. By construction $z'_{i_+a} > 0$ for all $a \in A$.

For any $j \in N_+ \setminus \{i_+\}$ we can find $y_j \in \mathbb{R}^A$ such that $u_j(z'_j + \delta y_j) > 0$ for small $\delta > 0$. Indeed if $u_j(z'_j)$ is positive we can take $y_j = 0$. And if $u_j(z'_j) = 0$, assuming that y_j does not exist implies that z'_j is a local maximum of u_j. By concavity of u_j it is then a global maximum as well, which contradicts the definition of N_+.

Consider an allocation z'': $z''_{i_+} = z'_{i_+} - \delta \sum_{j \in N_+ \setminus \{i_+\}} y_j$, $z''_j = z'_j + \delta y_j$ for $j \in N_+ \setminus \{i_+\}$ and $z''_k = z'_k$ for $k \in N_-$. For small $\delta > 0$ this allocation is feasible and yields utilities in Γ^*.

7.2 Main Theorem

Throughout the proof it is convenient to consider competitive divisions (z, p, β) with arbitrary budgets $\beta \in \mathbb{R}$ (not only $\beta \in \{-1, 0, 1\}$); this clearly yields exactly the same set of competitive allocations $CE(\mathcal{P})$ and utility profiles $CU(\mathcal{P})$.

7.2.1 Positive problems: statement i)

Let $\mathcal{N}(V) = \prod_{i \in N_+} V_i$ be the Nash product of utilities of the attracted agents. We fix a positive problem $\mathcal{P}$ and proceed in two steps.

Step 1. *If U maximizes $\mathcal{N}(V)$ over $V \in \mathcal{U} \cap \Gamma^*$ and $z \in \mathcal{F}$ is such that $U = u(z)$, then z is a competitive allocation with budget $\beta > 0$.*

Let $\mathcal{C}_+$ be the convex cone of all $y \in \mathbb{R}_+^{N \times A}$ with $u(y) \in \Gamma$. For any $\lambda > 0$ put

$$\mathcal{C}_\lambda = \left\{ y \in \mathcal{C}_+ \mid \mathcal{N}(u(y)) \geq \lambda^{|N_+|} \right\}.$$

Since $\mathcal{P}$ is positive the set $\mathcal{C}_\lambda$ is non-empty for any $\lambda > 0$. Continuity and concavity of utilities imply that $\mathcal{C}_\lambda$ is closed and convex. Homogeneity of utilities give $\mathcal{C}_\lambda = \lambda \mathcal{C}_1$.

Set $\lambda^* = (\mathcal{N}(U))^{\frac{1}{|N_+|}}$. The set $\mathcal{C}_\lambda$ does not intersect $\mathcal{F}$ for $\lambda > \lambda^*$, and $\mathcal{C}_{\lambda^*}$ touches $\mathcal{F}$ at z.

Step 1.1 There exists a hyperplane H separating $\mathcal{F}$ from $\mathcal{C}_{\lambda^*}$.

Consider a sequence λ_n converging to λ^* from above. Since $\mathcal{C}_{\lambda_n}$ and $\mathcal{F}$ are convex sets that do not intersect, they can be separated by a hyperplane H_n. The family $\{H_n\}_{n\in\mathbb{N}}$ has a limit point H. The hyperplane H separates $\mathcal{F}$ from $\mathcal{C}_{\lambda^*}$ by continuity of u. Thus there exist $q \in \mathbb{R}^{N\times A}$ and $Q \in \mathbb{R}$ such that $\sum_{i,a} q_{ia}y_{ia} \le Q$ for $y \in \mathcal{F}$ and $\sum_{i,a} q_{ia}y_{ia} \ge Q$ on $\mathcal{C}_{\lambda^*}$. The coefficients q_{ia} will be used to define the vector of prices p.

By the construction z maximizes $\mathcal{N}(u(y))$ over $\mathcal{B}^N(q,Q) = \{y \in \mathcal{C}_+ \mid \sum_{i,a} q_{ia}y_{ia} \le Q\}$. Think of the latter as a "budget set with agent-specific prices".

Define the vector of prices p by $p_a = \max_{i\in N} q_{ia}$ and $\mathcal{B}^*(p,Q) = \{y \in \mathcal{C}_+ \mid \sum_i p\cdot y_i \le Q\}$. We show now that we do not need agent-specific pricing.

Step 1.2 The allocation z maximizes $\mathcal{N}(u(y))$ over $y \in \mathcal{B}^*(p,Q)$.

It is enough to show the double inclusion $z \in \mathcal{B}^*(p,Q) \subset \mathcal{B}^N(q,Q)$. The second one is obvious since $\sum_{i,a} y_{ia}p_a \le Q$ implies $\sum_{i,a} y_{ia}q_{ia} \le Q$. Let us check the first inclusion. Taking into account that $z \in \mathcal{F}$ and $\sum_{i,a} q_{ia}y_{ia} \le Q$ for $y \in \mathcal{F}$, we get

$$\sum_i p\cdot z_i = \sum_a p_a \sum_i z_{ia} = \sum_a p_a = \sum_a \max_i q_{ia} = \max_{y\in\mathcal{F}} \sum_{i,a} q_{ia}y_{ia} \le Q.$$

Step 1.3 (z,p,β) is a competitive division for some $\beta > 0$.

Consider an agent i from N_+. Check that the bundle z_i belongs to his competitive demand $d_i(p,\beta_i)$, where $\beta_i = p\cdot z_i$. Indeed if there exists $z_i' \in \mathbb{R}^A_+$ such that $p\cdot z_i' \le \beta_i$ and $u_i(z_i') > u_i(z_i)$, then switching the consumption of agent i from z_i to z_i' gives an allocation in $\mathcal{B}^*(p,Q)$ and increases the Nash product, contradicting Step 1.2. Note that $\beta_i > 0$ for $i \in N_+$ because otherwise we can take $z_i' = 2z_i$. Check now that z_i is parsimonious: it minimizes $p\cdot y_i$ over $d_i(p,\beta_i)$. If not, pick $y_i \in d_i(p,\beta_i)$ with $p\cdot y_i < p\cdot z_i$, then for δ small enough and positive, the bundle $z_i' = (1+\delta)y_i$ meets $p\cdot z_i' \le \beta_i$ and $u_i(z_i') > u_i(z_i)$.

We use now the classic equalization argument ([16]) to check that β_i does not depend on $i \in N_+$. We refer to the fact that the geometric mean is below the arithmetic one as "the inequality of means".

Assume $\beta_i \ne \beta_j$ and consider a new allocation z', where the budgets of i and j are equalized: $z_i' = \frac{\beta_i+\beta_j}{2\beta_i}z_i$ and $z_j' = \frac{\beta_i+\beta_j}{2\beta_j}z_j$. This allocation belongs to $\mathcal{B}^*(p,Q)$ and homogeneity of utilities implies

$$\mathcal{N}(u(z')) = \mathcal{N}(U)\left(\frac{\beta_i+\beta_j}{2\beta_i}\right)\left(\frac{\beta_i+\beta_j}{2\beta_j}\right).$$

Now the (strict) inequality of means gives $\frac{\beta_i+\beta_j}{2} > \sqrt{\beta_i\beta_j}$, therefore $\mathcal{N}(u(z')) > \mathcal{N}(U)$ contradicting the optimality of z. Denote the common value of β_i by β.

Turning finally to the repulsed agents we check that for any $i \in N_-$ there is no z_i' such that $u_i(z_i') = 0$ and $\beta_i' = p\cdot z_i' < p\cdot z_i = \beta_i$, i.e., i can not decrease his spending. Assuming that z_i' exists we can construct an allocation $z' \in \mathcal{B}^*(p,Q)$, where agent i switches to z_i', consumption of other agents from N_- remains the same, and $z_j' = z_j\frac{N_+\beta+\beta_i-\beta_i'}{N_+\beta}$ for $j \in N_+$. In other words, money saved by i are redistributed among positive agents. By homogeneity $\mathcal{N}(u(z')) > \mathcal{N}(U)$, contradiction. A corollary is that β_i must be zero: take $z'_i = 0$ if $\beta_i > 0$, and $z'_i = 2z_i$ if β_i is negative. At z_i agent i reaches his maximal welfare of zero. Therefore, if i can afford z_i, then z_i is in the demand set. Since the price β_i of z_i is zero, we conclude $z_i \in d_i(p, \beta)$. The proof of Step 1.3 and of Step 1 is complete.

Step 2 *If (z, p, β) is a competitive division, then $\beta > 0$, and $U = u(z)$ belongs to $\mathcal{U} \cap \Gamma^*$ and maximizes $\mathcal{N}$ over this set.*
Check first $\beta > 0$. If $\beta \leq 0$ the budget set $B(p, \beta)$ contains z_i and $2z_i$ for all i, therefore $u_i(2z_i) \leq u_i(z_i)$ implies $U_i \leq 0$. Then U is Pareto-dominated by any $U' \in \mathcal{U} \cap \Gamma^*$, contradicting the efficiency of z (Lemma 1).

Now $\beta > 0$ implies U belongs to Γ^*: every $i \in N_+$ has a y_i with $u_i(y_i) > 0$ and can afford δy_i for small enough $\delta > 0$; every $i \in N_-$ can afford $y_i = 0$, hence $u_i(z_i) = 0$ and $p \cdot z_i \leq 0$ (by (2)).

Consider $U' = u(z')$ that maximizes $\mathcal{N}$ over $\mathcal{U} \cap \Gamma^*$. For any $i \in N_+$ his spending $\beta'_i = p \cdot z'_i$ must be positive. Otherwise $\delta z'_i \in B(p, \beta)$ for any $\delta > 0$ and agent i can reach unlimited welfare. Similarly $\beta'_i < 0$ for $i \in N_-$ implies $\delta z'_i \in d_i(p, \beta)$ for any $\delta > 0$, so the spending in $d_i(p, \beta)$ is arbitrarily low, in contradiction of parsimony (2).

For attracted agents $\frac{\beta}{\beta'_i} z'_i \in B(p, \beta)$ gives $\frac{\beta}{\beta'_i} U'_i = u_i\left(\frac{\beta}{\beta'_i} z'_i\right) \leq U_i$. Therefore if U is not a maximizer of $\mathcal{N}$, we have

$$\mathcal{N}(U) < \mathcal{N}(U') \leq \mathcal{N}(U) \prod_{i \in N_+} \frac{\beta'_i}{\beta} \Longrightarrow 1 < \left(\prod_{i \in N_+} \frac{\beta'_i}{\beta}\right)^{\frac{1}{|N_+|}} \leq \frac{\sum_{i \in N_+} \beta'_i}{|N_+|\beta}$$

where we use again the inequality of means. Now we get a contradiction from

$$\sum_{i \in N_+} \beta'_i \leq \sum_{i \in N} \beta'_i = p \cdot \omega = \sum_{i \in N} p \cdot z_i \leq \sum_{i \in N_+} \beta + \sum_{i \in N_-} 0 = |N_+|\beta$$

7.2.2 Negative problems: statement *ii*)

The proof is simpler because we do not need to distinguish agents from N_+ and N_-. We define the Nash product for negative problems by $\mathcal{N}(V) = \prod_{i \in N} |V_i|$ and focus now on its critical points in $\mathcal{U}^{eff}$. We start by the variational characterization of such points. If $V \in \mathbb{R}^N_=$ we have $\frac{\partial}{\partial V_i} \mathcal{N}(V) = \frac{1}{V_i} \mathcal{N}(V)$. Therefore $U \in \mathcal{U} \cap \mathbb{R}^N_=$ is a critical point of $\mathcal{N}$ on $\mathcal{U}$ that lay on $\mathcal{U}^{eff}$ iff

$$\sum_{i \in N} \frac{U'_i}{|U_i|} \leq -|N| \text{ for all } U' \in \mathcal{U} \tag{7}$$

The choice of the sign in this inequality is determined by Efficiency. Set $\varphi_U(U') = \sum_{i \in N} \frac{U'_i}{|U_i|}$: inequality (7) says that $U' = U$ maximizes $\varphi_U(U')$ on $\mathcal{U}$.

We fix a negative problem $\mathcal{P}$ and proceed in two steps.

Step 1. *If a utility profile $U \in \mathcal{U}^{eff} \cap \mathbb{R}^N_=$ is a critical point of $\mathcal{N}$ on $\mathcal{U}$, then any $z \in \mathcal{F}$ implementing U is a competitive allocation with budget $\beta < 0$.*
By (7) for any $y \in \mathcal{F}$ we have $\varphi_U(u(y)) \leq -|N|$. Define

$$\mathcal{C}_\lambda = \left\{y \in \mathbb{R}^{N\times A}_+ \mid \varphi_U(u(y)) \geq \lambda\right\}$$

For $\lambda \leq 0$ it is non-empty (it contains 0), closed and convex. For $\lambda > -|N|$ the set $\mathcal{C}_\lambda$ does not intersect $\mathcal{F}$ and for $\lambda = -|N|$ it touches $\mathcal{F}$ at z. Consider a hyperplane $\sum_{i,a} q_{ia}y_{ia} = Q$ separating $\mathcal{F}$ from $\mathcal{C}_{-|N|}$ and fix the sign by assuming $\sum_{i,a} q_{ia}y_{ia} \leq Q$ on $\mathcal{F}$ (existence follows as in Step 1.1 for positive problems). By the construction z maximizes $\varphi_U(u(y))$ on $\mathcal{B}^N(q,Q) = \{y \in \mathbb{R}^{N\times A}_+ \mid \sum_{i,a} q_{ia}y_{ia} \leq Q\}$. Defining prices by $p_a = \max_{i\in N} q_{ia}$ and mimicking the proof of Step 1.2 for positive problems we obtain that z belongs to $\mathcal{B}^*(p,Q) = \{y \in \mathbb{R}^{N\times A}_+ \mid \sum_{i,a} p_a y_{ia} \leq Q\}$ and maximizes $\varphi_U(u(y))$ there.

We check now that z is a competitive allocation with negative budget. For any agent $i \in N$ the bundle z_i belongs to his demand $d_i(p,\beta_i)$ (as before $\beta_i = p \cdot z_i$). If not, i can switch to any $z'_i \in B(p,\beta_i)$ with $u_i(z'_i) > U_i$, thus improving the value of φ_U and contradicting the optimality of z. The maximal spending β_i must be negative, otherwise i can afford $y_i = 0$ and $u_i(z_i) < u_i(y_i)$. If there is some $z'_i \in d_i(p,\beta_i)$ such that $p \cdot z'_i < \beta_i$, the bundle $z''_i = \frac{\beta_i}{p\cdot z'_i} z'_i$ is still in $B(p,\beta_i)$ and $u_i(z''_i) > U_i$: therefore $p \cdot z_i = \beta_i$ and z_i is parsimonious ((2)).

Finally, $\beta_i = \beta_j$ for all $i,j \in N$. If $\beta_i \neq \beta_j$, we use an *un*equalization argument dual to the one in Step 1.3 for positive problems. Assume for instance $\beta_i > \beta_j \Leftrightarrow |\beta_i| < |\beta_j|$ and define z' from z by changing only z'_i to $\frac{1}{2}z_i$ and z'_j to $\frac{2\beta_j+\beta_i}{2\beta_j} z_j$. Clearly $z' \in \mathcal{B}^*(p,Q)$ and we compute

$$\varphi_U(u(z')) - \varphi_U(U) = -\frac{1}{2} - \frac{2\beta_j + \beta_i}{2\beta_j} + 2 = \frac{1}{2} - \frac{\beta_i}{2\beta_j} > 0$$

But we showed that z maximizes $\varphi_U(u(y))$ in $\mathcal{B}^*(p,Q)$: contradiction.

Step 2. *If (z,p,β) is a competitive division, then $\beta < 0$ and the utility profile $U = u(z)$ is a critical point of the Nash product on U that belongs to $\mathcal{U}^{eff} \cap \mathbb{R}^N_=$.*
Check first that $\beta < 0$. If not each agent can afford $y_i = 0$ so $U_i \geq 0$ for all i, which is impossible in a negative problem. Assume next $U_i \geq 0$ for some i: we have $2z_i \in B(p,\beta)$, $u_i(2z_i) \geq u_i(z_i)$, and $p \cdot (2z_i) < p \cdot z_i$, which contradicts (1) and/or (2) in Definition 1. Therefore U belongs to $\mathbb{R}^N_=$. Finally $p \cdot z_i < \beta$ would imply $u_i(z_i) < u_i(\lambda z_i)$ for $\lambda \in [0,1[$, and $\lambda z_i \in B(p,\beta)$ for λ close enough to 1, a contradiction. Summarizing we have shown $U \in \mathcal{U}^{eff} \cap \mathbb{R}^N_=$ and $p \cdot z_i = \beta < 0$ for all i.

To prove that U is a critical point it is enough to check that it maximizes $\varphi_U(u(y))$ on $\mathcal{F}$. Fix $z' \in \mathcal{F}$, set $U' = u(z')$ and $p \cdot z'_i = \beta'_i$. To show $\varphi_U(U') \leq \varphi_U(U)$ we will prove

$$U'_i \leq \frac{\beta'_i}{\beta} U_i \text{ for all } i \tag{8}$$

This holds if $\beta_i' < 0$ because $\frac{\beta}{\beta_i'} z_i' \in B(p, \beta)$ so $\frac{\beta}{\beta_i'} U_i' = u_i \left(\frac{\beta}{\beta_i'} z_i' \right) \leq U_i$. If $\beta_i' \geq 0$ we set $z_i'' = \alpha z_i' + (1 - \alpha) z_i$, where $\alpha > 0$ is small enough that $p \cdot z_i'' < 0$. We just showed (8) holds for $u_i(z_i'')$, therefore

$$u_i(z_i'') \leq \frac{p \cdot z_i''}{\beta} U_i = \alpha \frac{\beta_i'}{\beta} U_i + (1 - \alpha) U_i.$$

Concavity of u_i gives $\alpha U_i' + (1 - \alpha) U_i \leq u_i(z_i'')$ and the proof of (8) is complete. Now we sum up these inequalities and reach the desired conclusion

$$\varphi_U(U') = \sum_{i \in N} \frac{U_i'}{|U_i|} \leq -\frac{\sum_{i \in N} \beta_i'}{\beta} = -\frac{\sum_{i \in N} p \cdot z_i'}{\beta} = -\frac{p \cdot \omega}{\beta} = -|N| = \varphi_U(U)$$

7.2.3 Null problems: statement *iii*)

The proof resembles that for positive problems, as we must distinguish N_+ from N_-, but the Nash product no longer plays a role. Fix a null problem $\mathcal{P}$.

Step 1. *Any $z \in \mathcal{F}$ such that $u(z) = 0$ is competitive with $\beta = 0$.*

Suppose first all agents are repulsed, $N = N_-$. Then $u_i(y_i) \leq 0$ for all $i \in N$ and $y_i \in \mathbb{R}_+^A$ and $(z, 0, 0)$ is a competitive division: everybody has zero money, all bundles are free and all agents achieve the best possible welfare with the smallest possible spending. We assume from now on $N_+ \neq \varnothing$.

Define $\psi(y) = \min_{i \in N_+} u_i(y_i)$ for $y \in \mathbb{R}_+^{N \times A}$ and the sets $\mathcal{C}_\lambda = \{y \in \mathcal{C}_+ \mid \psi(y) \geq \lambda\}$, where $\mathcal{C}_+ = \{y \in \mathbb{R}_+^{N \times A} | u(y) \in \Gamma\}$ (as in the positive proof). For $\lambda \geq 0$ the set $\mathcal{C}_\lambda$ is non-empty, closed and convex. If $\lambda > 0$, the sets $\mathcal{C}_\lambda$ and $\mathcal{F}$ do not intersect. As in Step 1.1 of the positive proof we construct a hyperplane separating $\mathcal{F}$ and $\mathcal{C}_0$, define the set $\mathcal{B}^N(q, Q)$, the vector of prices p, and the set $\mathcal{B}^*(p, Q)$. Similarly we check that the allocation z maximizes $\psi(y)$ over $y \in \mathcal{B}^*(p, Q)$, and $\psi(z) = 0$.

We set $\beta_i = p \cdot z_i$ and show that $(z, p, 0)$ is a competitive division in three substeps.

Step 1.1 for all $i \in N$ and $x_i \in \mathbb{R}_+^A$: $p \cdot x_i < \beta_i \Longrightarrow u_i(x_i) < 0$.

Suppose $p \cdot x_i < \beta_i$ and $u_i(x_i) \geq 0$ for some $i \in N_+$. For each j in N_+ pick a bundle y_j^+ such that $u_j(y_j^+) > 0$ and construct the allocation z' as follows: $z_i' = x_i + \delta y_i^+$; $z_j' = z_j + \delta y_j^+$ for any other $j \in N_+$; $z_j' = z_j$ for $j \in N_-$. If $\delta > 0$ is small enough $z' \in \mathcal{B}^*(p, Q)$ and for any $j \in N_+$ we have $u_j(z_j') > 0$, by concavity and homogeneity of u_j. For instance

$$\frac{1}{2} u_i(z_i') = u_i \left(\frac{1}{2} x_i + \frac{1}{2} \delta y_i^+ \right) \geq \frac{1}{2} u_i(x_i) + \frac{1}{2} \delta u_i(y_i^+) > 0$$

Therefore $\psi(z') > 0$ contradicting the optimality of z.

The proof when $p \cdot x_i < \beta_i$ and $u_i(x_i) \geq 0$ for some $i \in N_-$ is similar and left to the reader.

Step 1.2 $\beta_i = 0$ for all $i \in N$.

If $\beta_i > 0$ then $x_i = 0$ is such that $p \cdot x_i < \beta_i$ and $u_i(x_i) = 0$, which we just ruled out. If $\beta_i < 0$ then $p \cdot (2z_i) < \beta_i$ yet $u_i(2z_i) = 0$, contradicting Step 1.1.

From Steps 1.1, 1.2 we see that for all i if $y_i \in d_i(p, 0)$ then $p \cdot y_i = 0$: so if we show $z_i \in d_i(p, 0)$ the parsimony property (2) is automatically satisfied. Therefore our next substep completes the proof of Step 1.

Step 1.3 $z_i \in d_i(p, 0)$ *for all* $i \in N$.

For $i \in N_-$ this is obvious since such agent reaches his maximal welfare $u_i = 0$. Pick now $i \in N_+$ and assume $z_i \notin d_i(p, 0)$. Then $d_i(p, 0)$ contains some y_i with $u_i(y_i) > 0$. Let w be a bundle with negative price. Such bundle exists since $p \cdot \omega = \sum_{i \in N} \beta_i = 0$ and $p \neq 0$. Hence the bundle $x_i = y_i + \delta w$ with small enough $\delta > 0$ has negative price $p \cdot x_i < 0$ and $u_i(x_i) > 0$. Contradiction.

Step 2. *If* (z, p, β) *is a competitive division, then* $u(z) = 0$ *and* (z, p, β') *with* $\beta' = 0$ *is also competitive.*

If $\beta < 0$ we have $u_i(z_i) < 0$ for all $i \in N$. Otherwise $u_i(z_i) \geq 0$ and $p \cdot z_i < 0$ implies as before that $z_i' = 2z_i$ improves U_i (at least weakly) while remaining in $B(p, \beta)$ and lowering i's spending. But $U \in \mathbb{R}^N_=$ is not efficient in a null problem.

Thus $\beta \geq 0$, hence $u_i(z_i) \geq 0$ for all $i \in N$ because the bundle 0 is in the budget set. The problem is null therefore $u(z) = 0$, implying $0 \in d_i(p, \beta)$ and by parsimony (2) $p \cdot z_i \leq 0$, for all i. Hence $z_i \in d_i(p, 0)$ therefore $(z, p, 0)$ is clearly a competitive division.

7.3 Lemma 3

We have $u_i(\omega) < 0$ for every $i \in N$, else the allocation z with $z_i = \omega$ and $z_j = 0$ for $j \neq i$ yields utilities in Γ.

Consider the set of utility profiles dominated by $\mathcal{U} \cap \mathbb{R}^N_-$: $\mathcal{U}_\leq = \{U \in \mathbb{R}^N_- | \exists U' \in \mathcal{U} \cap \mathbb{R}^N_- : U \leq U'\}$. This set is closed and convex and contains all points in $\mathbb{R}^N_-$ that are sufficiently far from the origin. Indeed, any $U \in \mathbb{R}^N_-$ such that $U_N < \min_i u_i(\omega)$, where $U_N = \sum_i U_i$, is dominated by the utility profile $z : z_i = \frac{U_i}{U_N}\omega, \ i \in N$.

Fix $\lambda \geq 0$ and consider the upper contour of the Nash product at λ: $C_\lambda = \{U \in \mathbb{R}^N_- \mid \Pi_N |U_i| \geq \lambda\}$. For sufficiently large λ the closed convex set C_λ is contained in $\mathcal{U}_\leq$. Let λ^* be the minimal λ with this property. Negativity of $\mathcal{P}$ implies that $\mathcal{U}_\leq$ is bounded away from 0 so that λ^* is strictly positive. By definition of λ^* the set C_{λ^*} touches the boundary of $\mathcal{U}_\leq$ at some U^* with strictly negative coordinates. Let H be a hyperplane supporting $\mathcal{U}_\leq$ at U^*. By the construction, this hyperplane also supports C_{λ^*}, therefore U^* is a critical point of the Nash product on $\mathcal{U}_\leq$: that is, U^* maximizes $\sum_{i \in N} \frac{U_i}{|U_i^*|}$ over all $U \in \mathcal{U}_\leq$. So U^* belongs to the Pareto frontier of $\mathcal{U}_\leq$, which is contained in the Pareto frontier of $\mathcal{U}$. Thus U^* is a critical point of the Nash product on $\mathcal{U}$ and belongs to $\mathcal{U}^{eff} \cap \mathbb{R}^N_=$. By the construction any U in the interior of C_{λ^*} is dominated by some $U' \in \mathcal{U} \cap \mathbb{R}^N_-$: so U^* maximizes the Nash product on $\mathcal{U}^{eff} \cap \mathbb{R}^N_-$.

7.4 Lemma 4

In the previous proof note that the supporting hyperplane H to $\mathcal{U}$ at U^* is unique because it is also a supporting hyperplane to C_{λ^*} that is unique. Hence, if $\mathcal{U}$ is a polytope (e.g., for additive utilities), U^* belongs to a face of maximal dimension.

When utilities are additive, both sets $\mathcal{U}$ and $\mathcal{U}_{\leq}$ are polytopes. Let $D \subset \mathbb{R}^{N \times A}$ be the set of all u such that the problem (N, A, u) is negative and U^* is not unique. By the above remark if $u \in D$ then for some $\lambda > 0$ the set $\mathcal{U}_{\leq}$ has at least two faces F and F' of maximal dimension that are tangent to the surface S_λ: $\prod_{i \in N} |U_i| = \lambda$, $U \in \mathbb{R}^N_{=}$. The condition that F is tangent to S_λ fixes λ. The set of all hyperplanes tangent to a fixed surface S_λ has dimension $|N| - 1$ (for every point on S there is one tangent hyperplane) though the set of all hyperplanes in $\mathbb{R}^N$ is $|N|$-dimensional. Hence tangency of F' and S_λ cuts one dimension. So D is contained in a finite union of algebraic surfaces and, therefore, has Lebesgue-measure zero.

7.5 KKT conditions for additive utilities

The first order characterization of the competitive allocations is very useful in the proof of Propositions 1,2 and 3. Recall the partition of A (4) and the correponding signs of the competitive prices (5).

Lemma 6

i) If $\mathcal{P}$ is positive $(z, p, +1)$ is a competitive division iff p meets (5) and:
for all $i \in N_-$: $U_i = 0$ and $p \cdot z_i = 0$;
for all $i \in N_+$: $U_i > 0$ and $p \cdot z_i = 1 = \frac{1}{|N_+|} p \cdot \omega$; moreover

$$\text{for all } a \in A_+ \cup A_- \ \{z_{ia} > 0\} \Longrightarrow \frac{u_{ia}}{U_i} = p_a = \max_{j \in N_+} \frac{u_{ja}}{U_j} \tag{9}$$

ii) If $\mathcal{P}$ is negative $(z, p, -1)$ is a competitive division iff for all $i \in N$: $U_i < 0$ and $p \cdot z_i = -1 = \frac{1}{|N|} p \cdot \omega$; moreover

$$\text{for all } a \in A_+ \cup A_- \ \{z_{ia} > 0\} \Longrightarrow \frac{u_{ia}}{|U_i|} = p_a = \max_{j \in N} \frac{u_{ja}}{|U_j|} \tag{10}$$

iii) If $\mathcal{P}$ is null z is a competitive allocation iff $U_i = 0$ for all $i \in N$. Then $(p, 0)$ is a corresponding price and budget iff $p \cdot z_i = 0 = p \cdot \omega$ and there exists $\lambda \in \mathbb{R}^{N_+}_{++}$ such that

$$\text{for all } a \in A_+ \cup A_- \ \{z_{ia} > 0\} \Longrightarrow \lambda_i u_{ia} = p_a = \max_{j \in N_+} \lambda_j u_{ja} \tag{11}$$

A consequence of Lemma 6 is that for all $a \in A_+$ and $b \in A_-$ and all $i \in N$ we have

$$\frac{u_{ia}}{p_a} \leq |U_i| \leq \frac{u_{ib}}{p_b}$$

with equality on the left if $z_{ia} > 0$ and on the right if $z_{ib} > 0$. Also an agent $i \in N_-$ eats only in A_0 if $\mathcal{P}$ is positive or null, and only in $A_- \cup A_0$ if $\mathcal{P}$ is negative.

Proof *Statement i)*. Assume $\mathcal{P}$ is positive. Assume $(z, p, +1)$ is competitive: by our Theorem it maximizes the product of utilities over $\mathcal{U} \cap \Gamma^*$. Therefore $U_j = 0$ in N_- and $p \cdot z_j = 0$ because such agent eats only in A_0 which is free. Agent $i \in N_+$ clearly spends all his budget so $p \cdot z_i = 0$. If $z_{ia} > 0$ and we transfer a vanishingly small amount of item a to agent $j \in N_+$, inequality $\frac{u_{ia}}{U_i} \geq \frac{u_{ja}}{U_j}$ guarantees that the product of utilities does not increase; this implies $\frac{u_{ia}}{U_i} = \max_{j \in N_+} \frac{u_{ja}}{U_j}$. Now for any $a, b \in A_+ \cup A_-$ such that $z_{ia} > 0$ and $z_{ib} > 0$ (1) implies $\frac{u_{ia}}{p_a} = \frac{u_{ib}}{p_b}$. Multiplying numerator and denominator by z_{ia} and summing up over the support of z_i in $A_+ \cup A_-$ we get

$$\frac{u_{ia}}{p_a} = \frac{\sum_{A_+ \cup A_-} u_{ib} z_{ib}}{\sum_{A_+ \cup A_-} p_b z_{ib}} = \frac{U_i}{1} \tag{12}$$

where the second equality holds because extending the sum to A_0 changes nothing because $p_b = 0$ and $z_{ib} > 0$ can only happen if $u_{ib} = 0$. This proves (9).

Conversely pick $(z, p, +1)$ meeting (9) and the two properties just before. Items in $A_+ \cup A_-$ are eaten exclusively by agents in N_+. Check that an item $a \in A_0$ can only be eaten by i if $u_{ia} = 0$: for $i \in N_-$ this follows from $U_i = 0$, and for $i \in N_+$ it follows from writing (9) as $\frac{u_{ia}}{p_a} = U_i$ for $a \in A_+ \cup A_-$ such that $z_{ia} > 0$ and summing up as in (12) to get $U_i = \sum_{A_+ \cup A_-} u_{ib} z_{ib}$.

Therefore the maximization of $\Pi_{N_+} U_i$ in $\mathcal{U} \cap \Gamma^*$ is equivalent to that of $\Pi_{N_+} u_i \cdot z_i$ when those agents share the items in $A_+ \cup A_-$, up to an arbitrary distribution of A_0 to agents who do not mind. The KKT optimality conditions of this latter problem obtain from (9) by ignoring the prices p_a. We conclude by our Theorem that $(z, p, +1)$ is a competitive division.

Statement ii) The proof is similar and simpler, because we do not have to distinguish between agents in N_+ and N_-. It is omitted for brevity.

Statement iii) The first sentence, and the fact that the competitive budget is 0 with $p \cdot z_i = 0$ for all i come from our Theorem. We check now that if $\mathcal{P}$ is null, and $z \in \mathcal{F}$ implements the zero utility profile, there exists λ, p meeting (11). Recall that $\mathcal{U}$ intersects $\Gamma = \mathbb{R}_+^{N_+} \times \{0\}^{N_-}$ only at 0 (Lemma 2), and in fact $\mathcal{U} \cap \mathbb{R}_+^N = \{0\}$ as well because no agent in N_- can get a positive utility. As $\mathcal{U}$ is a polytope, we can separate it from $\mathbb{R}_+^N$ by a strictly positive vector λ. The separation property is $\sum_N \lambda_i (u_i \cdot y_i) \leq 0 = \sum_N \lambda_i (u_i \cdot z_i)$ for any $y \in \mathcal{F}$, which implies $\lambda_i u_{ia} = \max_{j \in N} \lambda_j u_{ja}$ whenever $z_{ia} > 0$. So if we define $p_a = \max_{j \in N} \lambda_j u_{ja}$ for all $a \in A_+ \cup A_-$, property (11) follows at once. Clearly p_a is null if $a \in A_0$ and the price of other items is non zero: then (11) implies that z_i is agent i's competitive demand at p (property (1)).

We omit the simple proof of the converse statement: if $(z, p, 0)$ is a competitive division, there exists λ meeting (11).

7.6 Proposition 1

Step 1. Check that $CU(\mathcal{P})$ is finite. If $\mathcal{P}$ is positive or null, this follows from the Theorem, even without assuming linear preferences. If $\mathcal{P}$ is negative the set $B = \mathcal{U}(\mathcal{P}) \cap \mathbb{R}^N_-$ is a polytope and the Theorem says that at each profile $U \in CU(\mathcal{P})$ the function $\Pi_N |U_i|$ is critical in B. Clearly this function has at most one critical point in the interior of each face of B, and there is a finite number of such interiors.

Step 2. Statement iii). We generalize the numerical example with 6 agents and 5 bads just after the Proposition to n agents and m bads with $n > m$. We set $A = \{a_1, \cdots, a_m\}$ and use the notation e^S for the vector in $\mathbb{R}^A$ with $e^S_i = 1$ if $i \in S$ and zero otherwise. For agent $i, 1 \le i \le m$, set as before $u_{ia_i} = -1, u_{ia_j} = -3$ for $j \neq i$, and for agents $m+1$ to n pick $u_{ia} = -1$ for all a. Then for any $q, 1 \le q \le m$, the allocation

$$z_i = \frac{m}{n} e^{a_i} \text{ for } 1 \le i \le q \ ; \ z_j = e^{a_j} \text{ for } q+1 \le j \le m \ ; \ z_j = \frac{1}{n} e^{\{a_1, \cdots, a_q\}} \text{ for } m+1 \le j \le n$$

is competitive for the prices $p_{a_i} = -\frac{q+1}{q}$ for $1 \le i \le q$ and $p_{a_j} = -1$ for $q+1 \le j \le m$.

Similarly in the case $m > n$ we set $u_{ia_k} = -1$ for $k = i$ or $n+1 \le k \le m$, and $u_{ia_k} = -3$ for $k \le n, k \neq i$. Then for any subset of agents $N^* \subseteq N$ the allocation where those agents share equally the bads $a_{n+1}, \cdots, a_m$, while bad $a_i, 1 \le i \le n$ goes to agent i, is competitive with prices $p_{a_{n+1}} = p_{a_i} = -\frac{n^*}{n^*+1}$ for $i \in N^*$, $p_{a_j} = -1$ for $j \in N \diagdown N^*$.

This construction can be repeated for any subset of m bads, thus generating $2^m - 1$ different competitive divisions. We omit for brevity the similar argument for the case $m > n$.

For the longer proof of the statements $i)$ and $ii)$ Lemma 6 is critical.

Step 3. Statement i)

We fix a negative problem $\mathcal{P} = (N = \{1, 2\}, A, u, \omega)$. If A_0 is non empty, it is easy to check that $CU(\mathcal{P})$ does not change if we simply drop those items; so we assume from now on that A_+ and A_- partition A, and A_- is non empty. If u_{1a} and u_{2a} are of strictly opposite signs for some item a, for instance $u_{1a} > 0 > u_{2a}$, then by Efficiency item a goes entirely to agent 1 and if we replace u_{2a} by $u'_{2a} = 0$ then again $CU(\mathcal{P})$ is unchanged, so we can assume that all items in A_+ have $u_{ia} \ge 0$ for $i = 1, 2$ with at least one strictly (for an item with $u_{1a} = u_{2a} = 0$ can be discarded as well). For all items in A_- we have as usual $u_{ia} < 0$ for $i = 1, 2$. If $a \in A_+$ (resp. A_-) we say for clarity that a is a good (resp. a bad). We keep in mind that prices are positive for goods and negative for bads.

We label the items $k \in \{1, \cdots, m\}$ so that the ratios $\frac{u_{1k}}{u_{2k}}$ increase weakly in k. with the convention $\frac{1}{0} = \infty$. We will prove the statement first when the sequence $\frac{u_{1k}}{u_{2k}}$ increases strictly in k.

Step 3.1 We fix a competitive division $(z, p, -1)$ and show three properties of z:
$a)$ if k, k' are bads and $z_{1k}, z_{2k'} > 0$ then $k < k'$
$b)$ if ℓ, ℓ' are goods and $z_{1\ell}, z_{2\ell'} > 0$ then $\ell' < \ell$
$c)$ if k is a bad, ℓ is a good, and $z_{1k}, z_{1\ell} > 0$ then $k < \ell$; if $z_{2k}, z_{2\ell} > 0$ then $\ell < k$

Condition a), b) follow directly from Efficiency. Pick k, k' as in the premises of a) but such that $k' < k$: then transferring ε units of k from 1 to 2, against δ units of k' from 2 to 1 (which is feasible for ε, δ small enough) is beneficial to both if $\frac{u_{1k'}}{u_{1k}} < \frac{\varepsilon}{\delta} < \frac{u_{2k'}}{u_{2k}}$ which is feasible as $k' < k$. The argument for b) is similar.

For condition c) we must use the competitiveness assumption in particular property (10).[13] Fix a bad k and a good ℓ s. t. $z_{1k}, z_{1\ell} > 0$. Buying $\frac{1}{|p_k|}$ unit of k and $\frac{1}{p_\ell}$ unit of ℓ is budget neutral so by competitiveness it is not profitable to agent 2: $\frac{u_{2k}}{|p_k|} + \frac{u_{2\ell}}{p_\ell} \leq 0 \iff \frac{u_{2\ell}}{u_{2k}} \geq \frac{p_\ell}{p_k}$. But z_1 is 1's competitive demand so $\frac{u_{1\ell}}{p_\ell} = \frac{u_{1k}}{p_k}$: combining the last two inequalities gives $\frac{u_{1\ell}}{u_{1k}} \leq \frac{u_{2\ell}}{u_{2k}}$ implying $k < \ell$ as claimed.

Together these three properties imply that at most one item a can be shared by both agents in the sense $z_{1a}, z_{2a} > 0$. Moreover if G_i (resp. B_i) is the set of goods (resp. bads) consumed by agent i, then all items in B_1 and G_2 are ranked below all items in B_2 and G_1, with at most one common item to both B_i-s or to both G_i-s.

Step 3.2 We show $|CU(\mathcal{P})| \leq 2m - 1$

Consider a competitive allocation where no item is shared. By Step 1 there is an index k such that agent 2 eats all goods in $\{1, \cdots, k\}$ and all bads in $\{k+1, \cdots, m\}$, while agent 1 eats the bads of $\{1, \cdots, k\}$ and the goods of $\{k+1, \cdots, m\}$. There are at most $m - 1$ such allocations.

Now consider a competitive allocation where our agents split (only) item k. The assignment of all other items to one agent or the other is determined as in the previous paragraph. Thus the relative prices of all items eaten by agent i are determined by her marginal utilities, and the equality of both budgets clinches the price and a single division of item k. Hence there are at most m competitive divisions splitting an item.

Step 3.3 An example where $|CU(\mathcal{P})| = 2m - 1$

There are only bads and the utilities generalize the example given just after Proposition 1 in Subsection 6.1. We use the notation $(x)_+ = \max\{x, 0\}$:

$$u_{1k} = -2^{(k-2)_+} \text{ for } 1 \leq k \leq m-1 \text{ ; } u_{1m} = -2^{m-2} + 1$$

$$u_{21} = -(2^{m-2} + 1) \text{ ; } u_{2k} = -2^{(m-1-k)_+} \text{ for } 2 \leq k \leq m$$

We let the reader check that giving bads 1 to k to agent 1 and the rest to agent 2 is a competitive allocation for budget -1 and the price:

$$p = -(\frac{u_{11}}{2^k}, \cdots, \frac{u_{1k}}{2^k}, \frac{u_{2(k+1)}}{2^{m-k}}, \cdots, \frac{u_{2m}}{2^{m-k}})$$

whereas splitting equally bad k, giving bads 1 to $k-1$ to agent 1, and bads $k+1$ to m to agent 2, is a competitive allocation for budget -1 and the price:

$$p = -\frac{1}{3}(\frac{u_{11}}{2^{k-3}}, \cdots, \frac{u_{1(k-1)}}{2^{k-3}}, \frac{u_{1k}}{2^{k-3}} = \frac{u_{2k}}{2^{m-2-k}}, \frac{u_{2(k+1)}}{2^{m-2-k}}, \cdots, \frac{u_{2m}}{2^{m-2-k}})$$

[13] Efficiency would only imply that a bad k and a good ℓ cannot be both consumed by both agents.

This example is clearly robust: small perturbations of the disutility matrix preserve the number of competitive allocations.

Step 3.4 It remains to consider the case where the sequence $\frac{u_{1k}}{u_{2k}}$ increase weakly but not strictly. Suppose $\frac{u_{1k}}{u_{2k}} = \frac{u_{1(k+1)}}{u_{2(k+1)}}$. Then in an competitive allocation with price p we have

$$\frac{u_{ik}}{p_k} = \frac{u_{i(k+1)}}{p_{k+1}} \text{ for } i = 1, 2$$

Indeed if one of $i = 1, 2$ eats both k and $k+1$, this follows by (1). If on the contrary i eats item k and j eats item $k+1$, then (1) again implies $\frac{u_{ik}}{p_k} \leq \frac{u_{i(k+1)}}{p_{k+1}}$ and $\frac{u_{j(k+1)}}{p_{k+1}} \leq \frac{u_{jk}}{p_k}$.

So for a given amount of money spent by i on items k and $k+1$, she gets the same utility no matter how she splits this expense between the two items.[14] Therefore there is an interval of competitive allocations obtained by shifting the consumption of k and $k+1$ while keeping the total expense on these two items fixed for each agent. They all give the same utility profile and use the same price. If we merge k and $k+1$ with endowments ω_k, ω_{k+1} into an item k^* with one unit of endowment, $\omega_{k^*} = 1$, and utilities $u_{ik^*} = u_{ik}\omega_k + u_{i(k+1)}\omega_{k+1}$, the above interval of competitive allocations becomes a single competitive allocation for the new price $p_{k^*} = p_k\omega_k + p_{k+1}\omega_{k+1}$, with p unchanged elsewhere. By successively merging all the items sharing the same ratio $\frac{u_{1k}}{u_{2k}}$, we do not change the number of competitive allocations distinct utility-wise, and reach a problem with fewer items where the ratios $\frac{u_{1k}}{u_{2k}}$ increase strictly in k. So we only need to prove the statement in this case.

Step 4. Statement ii)

We fix a negative problem $\mathcal{P} = (N, A = \{a, b\}, u)$. By our Theorem there is at least one bad, i. e., A_- is non empty. Suppose first that b is a bad, while a is a good: $a \in A_+$. As in the previous proof we can assume $u_{ia} \geq 0$ for all i, with at least one strict inequality. In a competitive allocation z everyone consumes b because all utilities are negative. Some agent i consumes a as well, and by genericity and Efficiency no other agent does. Moreover i must have the highest ratio $\frac{u_{ia}}{u_{ib}}$: this determines the competitive price and it is then easy to check that $CU(\mathcal{P})$ is unique, no matter how many agents have the highest ratio $\frac{u_{ja}}{u_{jb}}$. We omit the details.

We turn the case where both items are bads. We label the agents $i \in \{1, \cdots, n\}$ in such a way that the ratios $\frac{u_{ia}}{u_{ib}}$ increase weakly in i. We describe first the efficient and non envious allocations (which will be useful in the proof of Proposition 3), then the competitive allocations in step 4.3.

Step 4.1. Assume $\frac{u_{ia}}{u_{ib}}$ increases strictly in i. If z is an efficient allocation, then for all i, j, $\{z_{ia} > 0$ and $z_{jb} > 0\}$ implies $i \leq j$. In particular at most one agent is eating both bads, and we have two types of efficient and envy-free allocations. For $1 \leq i \leq n-1$ the *$i/i+1$-cut* $z^{i/i+1}$ is the allocation $z_j^{i/i+1} = (\frac{1}{i}, 0)$ for $j \leq i$, and $z_j^{i/i+1} = (0, \frac{1}{n-i})$ for

[14] If one item is a good and the other a bad, shifting money between them either increase both consumptions or decrease both.

$j \geq i+1$. For $2 \leq i \leq n-1$ the allocation z is an *i-split* if there are numbers x, y such that

$$z_j = (\frac{1-x}{i-1}, 0) \text{ for } j \leq i-1 \; ; \; z_j = (0, \frac{1-y}{n-i}) \text{ for } j \geq i+1 \quad (13)$$

$$z_i = (x, y) \text{ with } 0 \leq x \leq \frac{1}{i}, \; 0 \leq y \leq \frac{1}{n-i+1} \quad (14)$$

Also, z is a 1-split if $z_1 = (1, y)$ and $z_j = (0, \frac{1-y}{n-1})$ for $j \geq 2$; and z is a n-split if $z_n = (x, 1)$ and $z_j = (\frac{1-x}{n-1}, 0)$ for $j \leq n-1$. Note that the cut $z^{i/i+1}$ is both an i-split and an $i+1$-split.

We have shown that, if $\frac{u_{ia}}{u_{ib}}$ increases strictly, an efficient and envy-free allocation must be an i-split. We turn to the case where the increase is not strict.

Step 4.2. Assume the sequence $\frac{u_{ia}}{u_{ib}}$ increases only weakly, for instance $\frac{u_{ia}}{u_{ib}} = \frac{u_{(i+1)a}}{u_{(i+1)b}}$. Then if z is efficient and envy-free we may have $z_{(i+1)a} > 0$ and $z_{ib} > 0$, however we can find z' delivering the same utility profile and such that one of $z'_{(i+1)a}$ and z'_{ib} is zero. Indeed No Envy and the fact that u_i and u_{i+1} are parallel gives $u_i \cdot z_i = u_i \cdot z_{i+1}$ and $u_{i+1} \cdot z_{i+1} = u_{i+1} \cdot z_i$, from which the claim follows easily. We conclude that the i-split allocations contain, utility-wise, all efficient and envy-free allocations.

Step 4.3. If the cut $z^{i/i+1}$ is a competitive allocation, the corresponding price is $p = -(i, n-i)$, and property (10) reads $\frac{u_{ja}}{i} \geq \frac{u_{jb}}{n-i}$ for $j \leq i$, $\frac{u_{jb}}{n-i} \geq \frac{u_{ja}}{i}$ for $j \geq i+1$, which boils down to

$$\frac{u_{ia}}{u_{ib}} \leq \frac{i}{n-i} \leq \frac{u_{(i+1)a}}{u_{(i+1)b}} \text{ for } 1 \leq i \leq n-1 \quad (15)$$

Next for $2 \leq i \leq n-1$ if the i-split allocation z (13) is competitive, the (normalized) price must be $p = -n(\frac{u_{ia}}{u_{ia}+u_{ib}}, \frac{u_{ib}}{u_{ia}+u_{ib}})$ and each agent must be spending exactly -1:

$$p_a \frac{1-x}{i-1} = p_b \frac{1-y}{n-i} = p_a x + p_b y = -1$$

which gives

$$x = \frac{1}{nu_{ia}}((n-i+1)u_{ia} - (i-1)u_{ib}) \; ; \; y = \frac{1}{nu_{ib}}(iu_{ib} - (n-i)u_{ia}) \quad (16)$$

We let the reader check that these formulas are still valid when $i = 1$ or $i = n-1$.

An i-split allocation z is *strict* if it is not a cut, i. e., both x, y in (13) are strictly positive. By (16), for any $i \in \{1, \cdots, n\}$ there is a strict i-split allocation that is competitive if and only if

$$\frac{i-1}{n-i+1} < \frac{u_{ia}}{u_{ib}} < \frac{i}{n-i} \quad (17)$$

(with the convention $\frac{1}{0} = \infty$).

Step 4.4. Counting competitive allocations. There are at most n competitive (strict) i-split allocations, and $n-1$ cuts $z^{i/i+1}$, hence the upper bound $2n-1$. An example where the bound is achieved uses any sequence $\frac{u_{ia}}{u_{ib}}$ meeting (17) for all $i \in \{1, \cdots, n\}$, as these inequalities imply (15) for all $i \in \{1, \cdots, n-1\}$.

7.7 Proposition 2

We pick a division rule f with associated welfare rule F meeting ETE, SOL and ILB, and we fix an arbitrary problem $\mathcal{P} = (u, \omega)$ (omitting N, A that stay constant throughout the proof). We choose a competitive division (z, p, β) at $\mathcal{P}$ with associated utility profile $\overline{U}$, and must show that $z \in f(\mathcal{P})$.

Case 1: $\mathcal{P}$ is null. Then there is no feasible profile U' in $\mathbb{R}_+^N \diagdown \{0\}$ so SOL implies $F(\mathcal{P}) \in \mathbb{R}_-^N$. There the null utility if Pareto dominant, so $F(\mathcal{P}) = \{0\}$ by Efficiency.

Case 2: $\mathcal{P}$ is positive. Then $\beta = 1$ and $p \cdot z_i = 1$ or 0, respectively when i is in N_+ or N_-. Consider the positive problem $\mathcal{Q} = (w, \omega)$ where $w_i = p$ for $i \in N_+$, and $w_i = 0$ for $i \in N_-$. Efficiency implies that at least one coordinate of $F(\mathcal{Q})$ is strictly positive, so by SOL they are all non negative. Thus $F_i(\mathcal{Q}) = 0$ in N_-. By ETE $F_i(\mathcal{Q})$ does not depend on $i \in N_+$, moreover W equal to 1 in N_+ and 0 in N_- is Pareto optimal at $\mathcal{Q}$: we conclude that $F(\mathcal{Q}) = W$.

Now we set $\overline{w}_i = \overline{U}_i p$ for all $i \in N$ (so $\overline{w}_i = w_i = 0$ in N_-), and $\overline{\mathcal{P}} = (\overline{w}, \omega)$. By the scale invariance property in Definition 2 we have $F(\overline{\mathcal{P}}) = \overline{U}$, moreover $\overline{w}_i \cdot z_i = \overline{U}_i(p \cdot z_i) = \overline{U}_i$ in N. By Pareto-Indifference (Definition 2) we conclude $z \in F(\overline{\mathcal{P}})$.

We compare now u and $\overline{w}$. Fix $a \in A_+ \cup A_-$; for all $i \in N$ we claim

$$z_{ia} > 0 \Longrightarrow u_{ia} = \overline{U}_i p_a = \overline{w}_i \; ; \; z_{ia} = 0 \Longrightarrow u_{ia} \leq \overline{U}_i p_a = \overline{w}_{ia}$$

Both claims are from (10) in Lemma 6 for $i \in N_+$; for $i \in N_-$ we must have $z_{ia} = 0$ and we know $u_{ia} \leq 0$. The two statements remain true for $a \in A_0$ because if i eats some a then $u_{ia} = 0$, and $p_a = 0$ implies $\overline{w}_{ia} = 0$ for all i.

Finally we apply ILB by lowering each $\overline{w}_{ia}$ to u_{ia} whenever possible and $z \in f(\mathcal{P})$ follows.

Case 3: $\mathcal{P}$ is negative. The omitted proof is similar, only simpler because we do not need to distinguish between N_+ and N_-.

We check finally that ILB is a consequence of Maskin Monotonicity (MM; see [29]) in the additive domain. We do this in the case of bads only, as both cases are similar. Individual allocations z_i vary in the rectangle $[[0, \omega]]$ $(0 \leq z \leq \omega)$ and utilities in $\mathbb{R}_-^A$, so MM for the division rule f means that for any two problems $\mathcal{P}, \mathcal{P}'$ on N, A and $z \in f(\mathcal{P})$ we have

$$\forall i \in N \{\forall w \in [[0, \omega]]\colon u_i \cdot z_i > u_i \cdot w \Longrightarrow u_i' \cdot z_i > u_i' \cdot w\} \Longrightarrow z \in f(\mathcal{P}') \qquad (18)$$

We fix $\mathcal{P}, i \in N$ and $z \in f(\mathcal{P})$. We write $A^0 = \{a | z_{ia} = 0\}, A^1 = \{a | z_{ia} = \omega_a\}$ and $A^2 = A \diagdown (A^0 \cup A^1)$. The implication in the premises of (18) reads

$$\forall w \in [[0, \omega]] \; u_i \cdot (w - z_i) < 0 \Longrightarrow u_i' \cdot (w - z_i) < 0$$

The cone generated by the vectors $w - z_i$ when w covers $[[0, \omega]]$ is $C = \{\delta \in \mathbb{R}^A | \delta_a \geq 0$ for $a \in A^0$, $\delta_a \leq 0$ for $a \in A^1\}$. By Farkas Lemma the implication $\{\forall \delta \in C : u_i \cdot \delta < 0 \Longrightarrow u_i' \cdot \delta < 0\}$ means that, up to a positive scaling factor,

$$u_{ia}' = u_{ia} \text{ on } A^2 \; ; \; u_{ia}' \leq u_{ia} \text{ on } A^0 \; ; \; u_{ia}' \geq u_{ia} \text{ on } A^1$$

Thus MM says that after lowering a lost bid on item a, or increasing one that gets the all of a, the initial allocation will remain in the selected set. Now ILB only considers lowering a lost bid, so it is only "half" of MM. The competitive rule fails the other half.

7.8 Lemma 5

Step 1 Only two bads
We use the notation and results in Step 4 of the proof of Proposition 1. Fix a problem $(N, \{a, b\}, u)$ where the ratios $r_i = \frac{u_{ia}}{u_{ib}}$ increase strictly in $i \in \{1, \cdots, n\}$ and write S^i for the closed rectangle of i-split allocations (13), (14): we have $S^i \cap S^{i+1} = \{z^{i/i+1}\}$ for $i = 1, \cdots, n-1$, and $S^i \cap S^j = \varnothing$ if i and j are not adjacent. We saw that envy-free and efficient allocations must be in the connected union $\mathcal{B} = \cup_{i=1}^n S^i$ of these rectangles. Writing $\mathcal{EF}$ for the set of envy-free allocations, we describe now the connected components of $\mathcal{A} = \mathcal{B} \cap \mathcal{EF}$. Clearly the set of corresponding utility profiles has the same number of connected components.

We let the reader check that the cut $z^{i/i+1}$ is EF (envy-free) if and only if it is competitive, i. e. inequalities (15) hold, that we rewrite as:

$$r_i \leq \frac{i}{n-i} \leq r_{i+1} \tag{19}$$

If $z^{i/i+1}$ is EF then both $S^i \cap \mathcal{EF}$ and $S^{i+1} \cap \mathcal{EF}$ are in the same component of $\mathcal{A}$ as $z^{i/i+1}$, because they are convex sets containing $z^{i/i+1}$. If both $z^{i-1/i}$ and $z^{i/i+1}$ are EF, so is the interval $[z^{i-1/i}, z^{i/i+1}]$; then these two cuts as well as $S^i \cap \mathcal{EF}$ are in the same component of $\mathcal{A}$. And if $z^{i/i+1}$ is EF but $z^{i-1/i}$ is not, then the component of $\mathcal{A}$ containing $z^{i/i+1}$ is disjoint from any component of $\mathcal{A}$ in $\cup_1^{i-1} S^j$ (if any), because $S^i \cap \cup_1^{i-1} S^j = \{z^{i-1/i}\}$; a symmetrical statement holds if $z^{i-1/i}$ is EF but $z^{i/i+1}$ is not.

Finally if $S^i \cap \mathcal{EF} \neq \varnothing$ while neither $z^{i-1/i}$ nor $z^{i/i+1}$ is in $\mathcal{EF}$, the convex set $S^i \cap \mathcal{EF}$ is a connected component of $\mathcal{A}$ because it is disjoint from $S^{i-1} \cap \mathcal{EF}$ and $S^{i+1} \cap \mathcal{EF}$, and all three sets are compact. In this case we speak of an interior component of $\mathcal{A}$. We claim that S^i contains an interior component if and only if

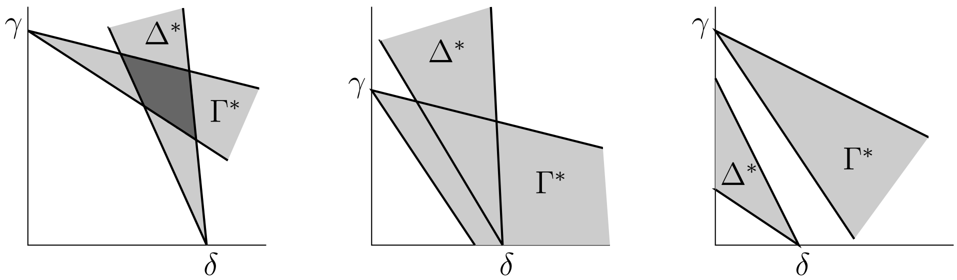
$$\frac{i-1}{n-i+1} < r_{i-1} < r_i < r_{i+1} < \frac{i}{n-i}$$

where for $i = 1$ this reduces to the two right-hand inequalities, and for $i = n$ to the two left-hand ones. The claim is proven in the next Step.

Now consider a problem with the following configuration:

$$r_1 < r_2 < \frac{1}{n-1} < \frac{3}{n-3} < r_3 < r_4 < r_5 < \frac{4}{n-4} <$$

$$< \frac{6}{n-6} < r_6 < r_7 < r_8 < \frac{7}{n-7} < \frac{9}{n-9} \cdots$$



Figures 4A, 4B, 4C

By inequalities (19) we have $z^{i/i+1} \in \mathcal{EF}$ for $i = 3q-1$, and $1 \leq q \leq \lfloor \frac{n}{3} \rfloor$, and no two of those cuts are adjacent so they belong to distinct components. Moreover S^i contains an interior component of $\mathcal{A}$ for $i = 3q-2$, and $1 \leq q \leq \lfloor \frac{n+2}{3} \rfloor$, and only those. So the total number of components of $\mathcal{A}$ is $\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n+2}{3} \rfloor = \lfloor \frac{2n+1}{3} \rfloor$ as desired.

We let the reader check that we cannot reach a larger number of components.

Step 2: $\{S^i$ *contains an interior component*$\} \Longleftrightarrow \{$*inequalities (2) hold*$\}$
Pick $z \in S^i$ as in (13), (14) and note first that for $2 \leq i \leq n-1$, the envy-freeness inequalities reduce to just four inequalities: agents $i-1$ and i do not envy each other, and neither do agents i and $i+1$ (we omit the straightforward argument). Formally

$$\frac{1}{r_{i+1}}(\frac{1}{n-i} - \frac{n-i+1}{n-i}y) \leq x \leq \frac{1}{r_i}(\frac{1}{n-i} - \frac{n-i+1}{n-i}y) \tag{20}$$

$$r_{i-1}(\frac{1}{i-1} - \frac{i}{i-1}x) \leq y \leq r_i(\frac{1}{i-1} - \frac{i}{i-1}x)$$

In the (non negative) space (x,y) define the lines $\Delta(\lambda)$: $y = \lambda(\frac{1}{i-1} - \frac{i}{i-1}x)$ and $\Gamma(\mu)$: $x = \mu(\frac{1}{n-i} - \frac{n-i+1}{n-i}y)$. As shown on Figure 4 when λ varies $\Delta(\lambda)$ pivots around $\delta = (\frac{1}{i}, 0)$, corresponding to $z^{i/i+1}$, and similarly $\Gamma(\mu)$ pivots around $\gamma = (0, \frac{1}{n-i+1})$, corresponding to $z^{i-1/i}$. The above inequalities say that (x,y) is in the cone Δ^* of points below $\Delta(r_i)$ and above $\Delta(r_{i-1})$, and also in the cone Γ^* below $\Gamma(\frac{1}{r_i})$ and above $\Gamma(\frac{1}{r_i+1})$. Thus $\delta \in \Gamma^*$ if and only if $z^{i/i+1}$ is EF, and $\gamma \in \Delta^*$ if and only if $z^{i-1/i}$ is EF. If neither of these is true γ is above or below Δ^* on the vertical axis and δ is to the left or to the right of Γ^* the horizontal axis. But if γ is below Δ^* while δ is right of Γ^*, the two cones do not intersect and $S^i \cap \mathcal{EF} = \varnothing$; ditto if γ is above Δ^* while δ is left of Γ^* (see Figures 4A,4B,4C). Moreover γ above Δ^* and δ right of Γ^* is impossible as it would imply

$$\frac{1}{n-i+1} > \frac{r_i}{i-1} \text{ and } \frac{1}{i} > \frac{1}{r_i(n-i)}$$

a contradiction. We conclude that $\{S^i \cap \mathcal{EF} \neq \varnothing$ and $z^{i-1/i}, z^{i/i+1} \notin \mathcal{EF}\}$ holds if and only if γ is below Δ^* and δ is to the left of Γ^*, which is exactly the system (2).

In the case $i = 1$ the EF property of z reduces to (20) and the i-split allocation has $x = 1$. If $r_1 > \frac{1}{n-1}$ the right-hand inequality in (20) is impossible with $x = 1$, therefore

$r_1 < \frac{1}{n-1}$; but then the fact that $z^{1/2}$ is not EF gives (see (19)) $r_2 < \frac{1}{n-1}$ as desired. A similar argument applies for the case $i = n$.

Step 3: Any number of bads

Fix a problem $(N, \{a,b\}, u)$ with $\lceil \frac{2n+1}{3} \rceil$ connected components as in Step 1. Given any $m \geq 3$, construct a problem $(N, \widetilde{A}, \widetilde{u})$ with $\widetilde{A} = \{a, b_1, \cdots, b_{m-1}\}$ and for all agents i

$$\widetilde{u}_{ia} = u_{ia} \ ; \ \widetilde{u}_{ib_k} = \frac{1}{m-1} u_{ib} \text{ for all } 1 \leq k \leq m-1$$

The bads b_k are smaller size clones of b. If some $\widetilde{z}$ is efficient and EF in the new problem, then the following allocation z is efficient and EF in the initial problem:

$$z_{ib} = \sum_{1}^{m-1} \widetilde{z}_{ib_k} \ ; \ z_{ia} = \widetilde{z}_{ia}$$

and $z, \widetilde{z}$ deliver the same disutility profile. Therefore in the two problems the sets of efficient and EF allocations have the same number of components.

7.9 Proposition 3

Fix a division rule f meeting EFF and EVFR, and such that F is single-valued. In the problems discussed below, no two efficient and envy-free allocations have the same utility profile, so f is single valued as well. Assume first $n = 4$, $m = 2$. Consider $\mathcal{P}^1$ where, with the notation in the previous proof, we have

$$r_1 < r_2 < \frac{1}{3} < 1 < 3 < r_3 < r_4$$

(note that the numerical example at the beginning of Section 5 is of this type)

By (19) and (2) $\mathcal{A}$ has three components: one interior to S^1 (excluding the cut $z^{1/2}$), one around $z^{2/3}$ intersecting S^2 and S^3, and one interior to S^4 excluding $z^{3/4}$. Assume without loss that f selects an allocation in the second or third component just listed, and consider $\mathcal{P}^2$ where r_1, r_2 are unchanged but the new ratios r'_3, r'_4 are

$$r_1 < r_2 < 3 < r'_3 < 1 < r'_4 < \frac{1}{3}$$

Here, again by (19) and (2), $\mathcal{A}$ has a single component interior to S^1, the same as in $\mathcal{P}^1$: none of the cuts $z^{i/i+1}$ is in $\mathcal{A}$ anymore, and there is no component interior to another S^i. When we decrease continuously r_3, r_4 to r'_3, r'_4, the allocation $z^{1/2}$ remains outside $\mathcal{A}$ and the component interior to S^1 does not move. Therefore the allocation selected by f cannot vary continuously in the ratios r_i, or in the underlying utility matrix u.

We can clearly construct a similar pair of problems to prove the statement when $n \geq 5$ and $m = 2$. And for the case $m \geq 3$ we use the cloning technique in Step 3 of the previous proof.

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Fedor Sandomirskiy
National Research University Higher School of Economics, St.Petersburg, Russia, and St.Petersburg
Institute for Economics and Mathematics; e-mail: fsandomirskiy@hse.ru