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ROUND-ROBIN TOURNAMENTS WITH LIMITED RESOURCES

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Round-Robin Tournaments with Limited Resources

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Abstract

We propose a theoretical model of a round-robin tournament with limited resources motivated by the fact that in a real-world round-robin sport tournament participating teams are sometimes forced to distribute their effort over an extended period. We assume that the participating teams have a limited amount of effort that must be distributed between all matches. We model the outcome of each match as a first-price sealed-bid auction. Results are aggregated after all matches are played with respect to the number of wins. The teams distribute their effort striving to maximize the expected payoff at tournament completion. For a three team tournament, we describe the set of all subgame perfect Nash equilibria in pure strategies. For tournaments with a relatively low first prize, we found two types of equilibria: 'effort-saving' and 'burning out', both leading to unequal payoffs. In contrast, for tournaments with a large first prize a limited budget of effort, in general, does not allow for the first or the last move advantage to be exploited.

Keywords: contest, round-robin tournament, limited resources, first-price auction, first-mover advantage.

JEL codes: C73, D44, Z20.

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1 Introduction

A round-robin tournament is a contest in which participants (teams or individual players) sequentially play against each other, and prizes are distributed according to an overall ranking at the end. Round-robin tournaments are quite widespread in sports today and are included in both their pure form and as a part of more complex formats in soccer competitions, like the FIFA World Cup and the UEFA Champions League, ice hockey, basketball, and volleyball world championships, chess tournaments, and in many other sports. During a long tournament, coaches may sometimes leave their leading players on the bench to give them some rest and let them be better prepared for more important matches. Such decisions can influence the chances of winning this particular game with a less skilled line-up, as well as the chances of winning subsequent games with better prepared and more rested leading players. Thus, a team's choice of effort level for each game is a strategic decision that is influenced by the gains from winning not only the current, but also subsequent games and by decisions about the efforts of other teams. In this paper, we study the economic foundations of teams' strategic behavior in a round-robin contest with limited resources (in this case, effort).

We consider a round-robin tournament with 3 teams in which wins and losses are the only possible results. Each team has a fixed budget of effort normalized to 1. Each team's objective is to maximize the expected payoff at the completion of the tournament given the tournament's prize structure. The payoffs are determined at the end of the tournament based on rankings by the number of wins. We model each match as a *quasi* first-price auction. The word '*quasi*' reflects the fact that the strategic decision of a team is how to split the effort across all matches rather than which level of effort to choose for a single match as happens in a standard contest.

The question we focus on is how the limit of resources and tournament's prize structure influence the so-called first-mover advantage and burning out effects which were found in the previous research. If the first prize is large compared to the second, it is better to split first to third places than to finish second. Vice versa, if the first prize is low, it is better to ensure the second prize than to split all prizes. It appears that in the case of the large first prize, typically all teams have exactly one win which means that the teams split the prizes and that there is no first-mover advantage. The team which does not play in the first match, observes the result and chooses to defeat the winner of the first match and to lose to the loser. This strategy allows to split the prize instead of finishing second as it would otherwise happen. For a relatively low first prize, both teams playing in the first match either spend all or save all their resources resulting in 'effort-saving' and 'burning out' equilibria. The team which does not play in the first match defeats the loser of the first match getting the second place. Thus, in the first match of the tournament the teams either compete as much as they can or they save all the effort to compete with the remaining team.

Although theoretical models clearly predict various strategic patterns in multi-stage tournament participants' behavior, there is ambiguous evidence from empirical literature. Ferrall and Smith (1999) show that sports teams do not act strategically and simply do their best in each game in the MLB, NHL, and NBA. In contrast, various laboratory experiments (for example, Amegashie, Cadsby, and Song (2007)) reveal that individual players who have a fixed budget save a part of the budget for later stages of a competition. Returning to sports teams, Harbaugh and Klumpp (2005) demonstrate that in elimination tournaments underdogs prefer to expend more effort in the first round whereas favorites save their efforts for the final. Using data from the NCAA men's basketball tournament, the authors illustrate that introducing a one-day rest period between a regional semi-final and final game increases the favorite's victory margin in the semi-final. In our paper, we consider teams strategically distributing a fixed amount of effort between matches.

An assumption that prizes are distributed according to overall rankings at the end of the tournament distinguishes a tournament from simply a sequence of independent matches and brings out specific participants' behavior patterns. To the best of our knowledge, most of the existing literature on agents' behavior in these contests (except for (Krumer, Megdish, and Sela 2016; 2017) and Sahm (2017)) ignores this. Making the assumption that each participant plays each other is not enough: it is critical that in a real round-robin, participants maximize the expected payoff after tournament completion rather than the sum of payoffs from separate games.

We consider a model of a round-robin tournament with three participants and perfect information. Following the literature on contest theory (for example, Tullock (1980) or Konrad (2009)), we take participants' effort to be their resources. They decide on how to distribute their effort across all tournament matches. To make this decision, the participants must take into account the type of competition — there are many different ways to define how success in a single game depends on the chosen level of effort. We will also illustrate that the payoff structure is important.

A first-price sealed-bid auction is a common way of modeling auctions; the side with the highest stake wins the prize. According to Muehlheusser (2006), in sports, a first-price auction model is suitable for situations in which there is some objective standard and in which no strategic choice is made to prevent an opponent from winning, for example, in sprint running. We think that it can also be applicable to some team sports tournaments in which coaches select the squad. Coaches could use the squad rotation if they feel some players need rest, thus distributing players' effort over tournament matches.

Some papers investigate repeated interactions between two teams. Konrad and Kovenock (2009) propose a (n, m)-contest consisting of repeated battles between two players in which the first player needs to win n battles to claim overall victory, and the second player – m battles. There are both prizes both for winning the whole contest and for winning a single battle. The winner of the battle is determined in an all-pay first-price auction. The authors describe a subgame-perfect equilibrium in mixed strategies. Krumer (2015) considers a (2,2)-contest connected with the 'best-of-three' series which are used in various sports tournaments. He separates home and away matches — each team places a larger value on winning a match at home and pays additional fixed costs for playing an away match. A stronger team plays two matches at home; a weaker team hosts only one home match.[3] For a case in which the teams can choose, the author finds an equilibrium order of home and away games that neither team would like to change.

More complex competitions include a larger number of competitors. Often, each stage of the competition is a contest between two participants (for example, many sports competitions consist of separate matches between two teams). An interesting problem in these tournaments is fairness: whether the expected outcome depends on the order of matches. Fairness gets to the heart of whether all teams have equal chance, irrespective of the tournament calendar. Krumer, Megdish, and Sela (2017) study round-robin tournaments with 3 and 4 participants, in which players play each other consecutively[4], each game is an all-pay first-price auction. Each player maximizes the probability of winning the tournament. The authors show that in the case of 3 players a player who plays in the first and third round gets the maximal payoff, as well as having the highest probability of winning the tournament. In a tournament with four participants with sequential games in each of three rounds, a player who plays in the first game in both the first and second rounds has an advantage. This demonstrates the so-called first-mover advantage phenomenon. After proving the first-mover advantage, the authors question the fairness of these types of tournaments. However, matters are completely different if instead of a first-price auction, the winning probabities are set up as a Tullock contest. Sahm (2017) demonstrates that in a tournament with 3 players and a Tullock contest success function, a player who plays in the first and third round gets the minimal payoff. In our paper, we will show that the limit on resources (effort) and the structure of tournament prizes are the other important factors that determine which player has an advantage. Table 1 summarizes the findings of several 3-players round-robin tournament models.

Though there are many ways to rank a round-robin tournament, usually, in practice, teams are ranked by the number of their wins or by the number of points gained in all matches. As proved in (Rubinstein, 1980), ranking by the number of wins is the only good approach in some sense.[7] In this paper, we follow

[3] A common practice in sport tournaments is that a team that finishes in a higher place in the previous stage is rewarded with the opportunity of playing more matches at home in the next stage.

[4] Note that in some round-robin competitions, like the group stage of the FIFA World Cup, some games are held simultaneously. The model presented in our paper can be generalized to account for this situation.

[5] Contest success function.

[6] The prize structure is a bundle of prizes for the first, the second and the third places respectively.

[7] Namely, ranking by the number of wins is the only aggregation method that satisfies anonymity, positive responsiveness,

	This paper	Krumer et al (2017)	Sahm (2017)
Effort	Limited	Unlimited	Unlimited
Cost function	Opportunity costs only	Linear	Linear
CSF[5]	First-price auction	First-price auction	Tullock contest
Prize structure[6]	$(v, 1-v, 0)$, $v \geq \frac{1}{2}$	(1,0,0)	(1,0,0)
Solution concept	Subgame perfect NE	Subgame perfect NE	Subgame perfect NE
Fairness	**Large first prize — almost fair** **Low first prize — unfair**	**Unfair**	**Unfair**

Table 1: Summary of 3-players round-robin tournament model

a ranking method based on the number of wins.

In a standard contest game formulation, efforts are costly, but, theoretically, not limited. In models of tournaments with more than one match (contest), another setting is possible: participants have a limited effort for all tournament matches. In this case, the participants face the challenge of distributing a fixed amount of effort between the matches they play in. This question is closely connected with the classic Colonel Blotto game (Roberson, 2006), in which two opposing colonels must distribute their soldiers across several battlefields. Each battlefield is won by the side that has more soldiers on this particular battlefield. The player who wins more battlefields wins the game. Kvasov (2007) suggests an all-pay first-price sealed-bid auction for several objects with limited resources. This contest is similar to a Colonel Blotto game due to presence of a budget constraint. In our model of a round-robin tournament, a single match resembles a battlefield. However, the round-robin structure and sequential order of matches makes a round-robin tournament being a more complex contest.

Erez and Sela (2010) investigate a competition with limited and costless efforts. The players play each other; each game is a Tullock contest with an exogenously fixed prize. The authors find equilibrium in pure strategies, as well as an effort-maximizing distribution of prizes between rounds. They show that each of n teams will spend their full budget of effort equally in each of the first $n-2$ matches leaving 0 effort for the last match.[8] Our paper differs from the work of Erez and Sela in two key assumptions. First, in our paper, a team maximizes the expected utility in the round-robin tournament, not the sum of expected values in separate games. As discussed earlier, this assumption reflects the structure of the round-robin tournament. Second, in our model, the winner of an individual match is the team that chose the larger level of effort for this match. Hence, we obtain equilibria which are different from the previous findings (Erez and Sela, 2010).

and the independence of irrelevant alternatives axioms.

[8]This corresponds to the case of $k = n-1$, $\alpha = 1$ in Proposition 1 of the mentioned paper.

In addition to the first-mover advantage, another interesting effect that is found in some types of multiple-stage contests is the burning out effect. It appears that sometimes it is profitable for participants to expend all their efforts in the first stage Amegashie, Cadsby, and Song (2007); Harbaugh and Klumpp (2005). Burning out may potentially lead to additonal Pareto inefficiency in different modifications of all-pay auctions. We show that in a round-robin tournament with 3 players burning out equilibrium appears in the case of low first prize.

Surveys of theoretical and experimental research on contests are provided in (Konrad, 2009) and (Dechenaux, Kovenock, and Sheremeta, 2012), respectively.

The rest of the paper is organized as follows. In Section 2, a general model of a round-robin tournament with limited effort budgets is defined. Thereafter, Section 3 contains a first-price sealed-bid auction specification with 3 participants. For the corresponding game, a pure strategy subgame perfect Nash equilibria are described. The appendix contains all formal proofs of statements in the paper.

2 The Model

Within the contest theory framework, single sports matches are considered to be contests. A contest is a strategic interaction between $N \geq 2$ risk-neutral players (teams, employees etc.) wherein each player i chooses a non-negative effort level e_i, and player i's probability of finishing in k-th place $p_k^i = p_k^i(e_1, \ldots, e_N)$ depends on the effort exerted by all competitors, $i, k = 1, \ldots, N$. The efforts are costly; and the cost functions $c_i(e_i), i = 1, \ldots, N$, are given. Let v_k be the prize for finishing in k-th place, $k = 1, \ldots, N$. Then, each player i maximizes their expected utility $u_i(e_1, \ldots, e_N) = \sum_{k=1}^{N} p_k^i(e_1, \ldots, e_N)v_k - c_i(e_i)$ with respect to e_i. Basic contest models are single shot games which differ in the functional form of winning probabilities p_k^i and the cost functions $c(e_i)$, as well as in the existence of random shocks which affect a player's actual performance by adding to the chosen effort level. The prize structure is often a matter of policy – contest organizers often seek to maximize the overall effort level by choosing an optimal prize distribution.

We model a round-robin tournament as a sequence of matches, in which two teams play a *quasi* contest. In each match, teams exert effort, and a winner is determined; no ties are possible. However, at the moment of stategic choice, the teams take into account not only a particular single match, but also all subsequent matches they must play (and this distinguishes a *quasi* contest from a true contest). The ultimate rankings and respective prizes are determined at the end of the tournament with respect to the number of wins for each team. We present the definition of a round-robin tournament with limited resource below.

Suppose that

- $N = \{1, \dots, n\}$ is a set of players (teams), $n \geq 2$.
- $\{g_t\}$, $t = 1, \dots, \frac{n(n-1)}{2}$, is a sequence of all possible matches — pairs of different elements from the set N. In a round-robin tournament, each team plays each other once according to a predetermined schedule. Sometimes we will refer to the match g_t as match number t.
- Each team has a budget of effort normalized to 1. Efforts are costless (however, there is an alternative to use the efforts later) and perfectly divisible.
- Before each match, teams simultaneously choose their effort level $\{e_i\}, i \in N$ for this particular match subject to their budget constraints. In the end, all teams use their entire budget of effort. Thus, in their last game, a team exerts any remaining effort it has.
- The result of the match between teams i and j, $i \neq j$, is represented by the realization of a random variable R_{ij}, taking the values 1 or 0 with the probabilities p and $1-p$, respectively. If the realization r_{ij} of the random variable R_{ij} is equal to 1, it means that team i won over j, whereas $r_{ij} = 0$ means that j defeated i.
- The probability mass function of R_{ij} is defined by the contest success function $p = p(e_i, e_j)$ given that player i played e_i and player j played e_j. We assume that $p(e_i, e_j) : [0,1]^2 \to [0,1]$, is non-decreasing in e_i and non-increasing in e_j. We also assume that the function p is anonymous in the sense that it does not depend on the names of the teams or the number of the match.
- E_i^t is the set of all possible actions (effort levels) e_i^t of player i, $i = i(t)$, in match t – from 0 to what is left over from 1 after previous matches. Note that the optimal choice of e_i^t is subject to information from matches $1, \dots, t-1$, as observed by all teams. Thus, e_i^t may depend on teams' actions and the realization of the results of matches $1, \dots, t-1$.
- $v = (v_1, \dots, v_n)$ are the prizes that are fixed before the tournament; v_k is the prize for the team which finished in the k-th place. The prizes are allocated at the end of the tournament with respect to the number of wins. In the case of an equal number of wins, the prizes of the teams that are tied are split equally.
- $u_i : \bigcup\limits_{t,i} E_i^t \to \mathbb{R}$, $i = 1, ..., n$, are the expected utilities given the teams' strategies. We consider the teams to be risk-neutral. Note that for a risk-neutral team, the expected utilities are completely defined by the order of games $\{g_t\}$, the prizes v, and the contest success function p.

Thus, we reach the following definition.

A **round-robin tournament with n players** is a game $< N, \{g_t\}, p(\cdot, \cdot), v >$.

Further, we are going to focus on the simple model of a round-robin with just 3 players — teams A, B, and C. The order of matches is as follows: the first match is played between A and B, the second match — between A and C, and the third match between B and C. Before each match starts, participating teams independently decide their level of effort for this match. All information is common knowledge: after each match ends, the result and chosen effort levels are observed by all three teams. Table 2 summarizes the order of games and the distribution of effort.

Round\Team	A	B	C
Round 1 (A vs B)	e_a	e_b	-
Round 2 (A vs C)	$1-e_a$	-	e_c
Round 3 (B vs C)	-	$1-e_b$	$1-e_c$

Table 2: Distribution of effort

The outcome of a single match depends on the level of effort exerted by competing teams. Wins and losses are the only possible outcomes. Denote as $p(e_x, e_y)$ the probability of winning for a team which exerts e_x units of effort against an opponent that exerts e_y units of effort. In this paper, we consider the first-price auction specification of a contest success function:

$$p_1^1(e_1, e_2) = \begin{cases} 1 & \text{if } e_1 > e_2 \\ \frac{1}{2} & \text{if } e_1 = e_2 \\ 0 & \text{if } e_1 < e_2 \end{cases}.$$

The timing of the tournament is as follows:

1. Teams A and B choose their effort level for the match A vs B, e_a and e_b respectively, $e_a, e_b \in [0, 1]$. All three teams learn the result and the level of effort chosen by all the teams.

2. For the match A vs C, team C chooses their effort level $e_c \in [0, 1]$, whereas team A expends $1 - e_a$. The outcome of the second match is realized.

3. For the match B vs C, teams B and C exert effort levels of $1 - e_b$ and $1 - e_c$, respectively. Then, the outcome of the tournament is determined.

The teams' payoffs are determined after all the games are played and the tournament ends. The team that collects the maximum number of wins gets the payoff v_1, the second best team gets v_2, and the worst team gets v_3. If several teams collect an equal number of wins, they share the corresponding payoffs equally. Note that teams get nothing for winning a match per se. We assume that teams are risk-neutral and that they maximize their expected payoff.

In the following chapter, we analyze teams' equilibrium strategies and payoffs. We also study the comparative statics of the model with respect to the prize structure.

3 First-price sealed-bid auction with limited effort

3.1 The model

In a first-price auction specification, the team which exerts more effort wins the match with certainty. If the two opponents exert the same amount of effort, each wins with probability equal to $\frac{1}{2}$. The corresponding game described above is an extenisve-form game, and we use the subgame perfect equilibrium solution concept to solve it. We start solving the game by backward induction.

First, we compute the best response of team C to the pair of effort from teams A and B, as well as to the lottery outcome in the case of equal efforts. Let $t(x) : [0, 1] \to \{A, B\}$ be Nature's strategy (the tie-breaking function) that determines the winner of a match between A and B if they exert the same effort x. For any x, both $t(x) = A$ and $t(x) = B$ can happen with an equal probability of $\frac{1}{2}$. When making a decision in Round 1, the teams know only the distribution of winning probabilities; after Round 1 is played, they observe the realization of $t(x)$ if they both chose an effort level of x. So, by the structure of the game, team C knows t when it makes a decision, whereas teams A and B did not. We do not introduce tie-breaking functions for the other rounds because they do not influence the chosen strategies (since the teams have already decided on their strategies when Round 2 begins). In the following text, by ex-ante payoffs, we mean the expected payoffs that teams think of before a tie-break in Round 1 is realized. By interim payoffs, we mean the expected payoffs when the result of a tie-break in Round 1 is known, but the results of the other two games have not yet been determined. Ex-ante and interim payoffs differ only when teams' efforts in Round 1 are equal.

Let the prizes be $(v_1, v_2, v_3) = (v, 1 - v, 0)$, $\frac{1}{2} \leq v \leq 1$. Given the strategies (e_a, e_b) of teams A and B, we calculate the best response $BR_c(e_a, e_b, t(e_a))$ of team C and the payoffs of the teams in the profile $(e_a, e_b, e_c(e_a, e_b, t(e_a)))$, where $e_c(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a))$.

Proposition 1. *For any pair (e_a, e_b), the best response $BR_c(e_a, e_b, t(e_a))$ of team C and the corresponding payoffs of teams A, B, and C are the following (see also Figures 1,2, and 3, and Tables 3,4, and 5):*

- *If $v > \frac{2}{3}$, then*

$$BR_c(e_a, e_b, t(e_a)) = \begin{cases} (1-e_a, e_b), & \text{if } e_a + e_b > 1 \\ [1-e_a, 1], & \text{if } (e_a > e_b = 1 - e_a) \text{ or } (e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = A) \\ [0, e_b], & \text{if } (e_b > e_a = 1 - e_b) \text{ or } (e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = B) \\ (1-e_a, 1], & \text{if } (0 < e_a + e_b < 1 \text{ and } e_a > e_b) \text{ or } (0 < e_a = e_b < \frac{1}{2} \text{ and } t(e_a) = A) \\ [0, e_b), & \text{if } (0 < e_a + e_b < 1 \text{ and } e_b > e_a) \text{ or } (0 < e_a = e_b < \frac{1}{2} \text{ and } t(e_a) = B) \\ 1, & \text{if } e_a = e_b = 0 \text{ and } t(0) = A \\ 0, & \text{if } e_a = e_b = 0 \text{ and } t(0) = B \end{cases} \tag{1}$$

- *If $v = \frac{2}{3}$, then*

$$BR_c(e_a, e_b, t(e_a)) = \begin{cases} (1-e_a, e_b), & \text{if } e_a + e_b > 1 \\ [0, 1], & \text{if } e_a + e_b = 1 \\ [0, e_b) \cup (1 - e_a, 1], & \text{if } 0 < e_a + e_b < 1 \\ \{0, 1\}, & \text{if } e_a = e_b = 0 \end{cases} \tag{2}$$

- *If $\frac{1}{2} < v < \frac{2}{3}$, then*

$$BR_c(e_a, e_b, t(e_a)) = \begin{cases} (1-e_a, e_b), & \text{if } e_a + e_b > 1 \\ [0, e_b), & \text{if } (e_a > e_b > 0 \text{ and } e_a + e_b \leq 1) \text{ or } (1 > e_b > e_a = 0) \\ & \text{or } (e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = A) \\ (1-e_a, 1], & \text{if } (e_b > e_a > 0 \text{ and } e_a + e_b \leq 1) \text{ or } (1 > e_a > e_b = 0) \\ & \text{or } (e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = B) \\ [0, 1] & \text{if } (e_a = 1,\ e_b = 0) \text{ or } (e_a = 0,\ e_b = 1) \\ 0, & \text{if } e_a = e_b = 0 \text{ and } t(0) = A \\ 1, & \text{if } e_a = e_b = 0 \text{ and } t(0) = B \end{cases} \tag{3}$$

- *If $v = \frac{1}{2}$, then*

$$
BR_c(e_a,e_b,t(e_a)) = \begin{cases} [0,e_b), & \text{if } (e_a > e_b > 0 \text{ and } e_a+e_b \leq 1) \text{ or } (1 > e_b > e_a = 0) \text{ or} \\ & \text{or } (e_a > e_b \text{ and } e_a+e_b > 1) \text{ or } (e_a+e_b \geq 1 \text{ and } e_a = e_b \text{ and } t(\frac{1}{2}) = A) \\ (1-e_a,1], & \text{if } (e_b > e_a > 0 \text{ and } e_a+e_b \leq 1) \text{ or } (1 > e_a > e_b = 0) \text{ or} \\ & \text{or } (e_b > e_a \text{ and } e_a+e_b > 1) \text{ or } (e_a+e_b \geq 1 \text{ and } e_a = e_b \text{ and } t(\frac{1}{2}) = B) \\ [0,1] & \text{if } (e_a = 1,\ e_b = 0) \text{ or } (e_a = 0,\ e_b = 1) \\ 0, & \text{if } e_a = e_b = 0 \text{ and } t(0) = A \\ 1, & \text{if } e_a = e_b = 0 \text{ and } t(0) = B \end{cases} \tag{4}
$$

Figures 1,2, and 3 depict the payoffs after team C plays their best response to teams A and B's strategies (e_a and e_b). Point R has the coordinates of $e_a = \frac{1}{2}$, $e_b = \frac{1}{2}$. Tables 3,4, and 5 consist of payoffs at the boundaries from figures 1,2, and 3. Payoffs on the boundaries depicted by the thin lines are the same as in the corresponding interior region.

All proofs are provided in the Appendix.

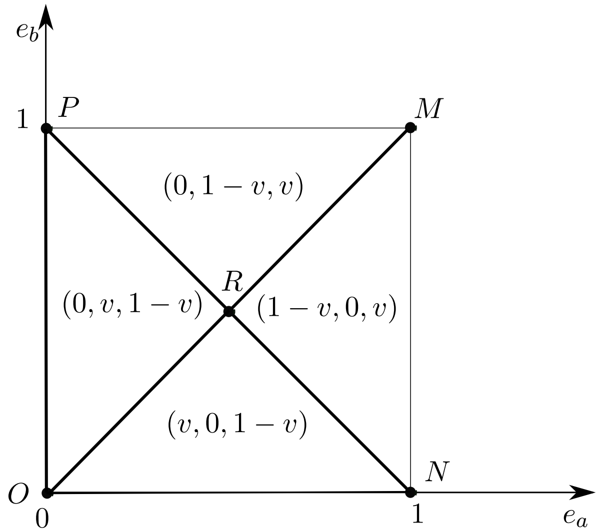

Figure 1: Case $v \in [\frac{1}{2}, \frac{2}{3})$.

P	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3}\right)$
PR	$(0, v, 1-v)$
N	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{3} + \frac{1}{4}v, \frac{1}{3} - \frac{1}{4}v, \frac{1}{3}\right)$
RN	$(v, 0, 1-v)$
RM and M	$\left(\frac{1-v}{2}, \frac{1-v}{2}, v\right)$
O	$\left(\frac{1+v}{4}, \frac{1+v}{4}, \frac{1-v}{2}\right)$
OP	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
OR and R	$(\frac{v}{2}, \frac{v}{2}, 1-v)$
ON	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Table 3: Case $v \in [\frac{1}{2}, \frac{2}{3})$.

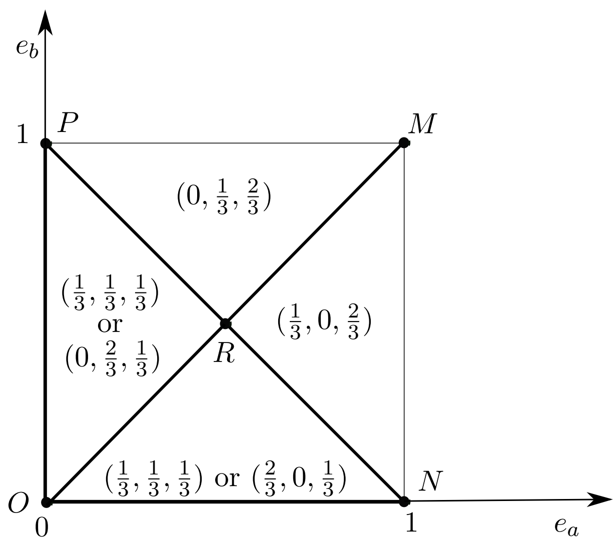

Figure 2: Case $v = \frac{2}{3}$.

PR	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$ or $\left(0, \frac{2}{3}, \frac{1}{3}\right)$
P	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$
RN	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$or $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$
N	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$
RM and M	$\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$
OR	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$ or $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$
R	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$ or $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$ or $\left(\frac{5}{12}, \frac{1}{4}, \frac{1}{3}\right)$ or $\left(\frac{1}{4}, \frac{5}{12}, \frac{1}{3}\right)$
O	$\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{6}\right)$ or $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$ or $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$
PO	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$
ON	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Table 4: Case $v = \frac{2}{3}$.

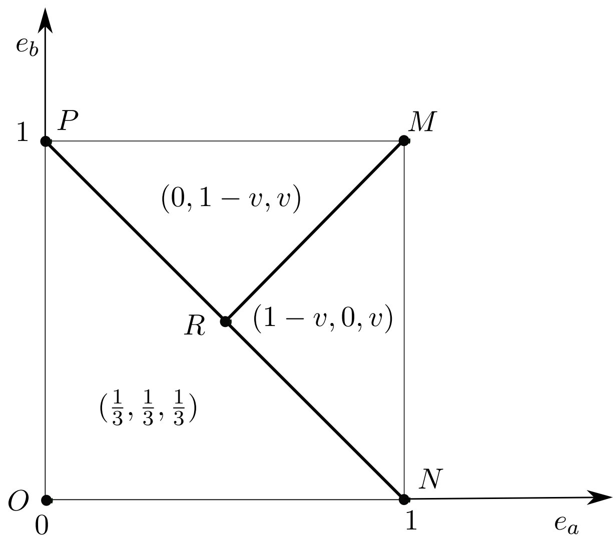

Figure 3: Case $v \in (\frac{2}{3}, 1]$.

PR and P	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3}\right)$
RN and N	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{3} + \frac{1}{4}v, \frac{1}{3} - \frac{1}{4}v, \frac{1}{3}\right)$
R	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{3} - \frac{1}{8}v, \frac{1}{3} + \frac{1}{8}v, \frac{1}{3}\right)$ or or $\left(\frac{1}{3} + \frac{1}{8}v, \frac{1}{3} - \frac{1}{8}v, \frac{1}{3}\right)$
RM and M	$\left(\frac{1-v}{2}, \frac{1-v}{2}, v\right)$
O	$\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{6}\right)$

Table 5: Case $v \in (\frac{2}{3}, 1]$.

We can now characterize the set of all equilibrium profiles $(e_a^*, e_b^*, e_c^*(e_a, e_b, t(e_a)))$, where $e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a))$, and the corresponding equilibrium payoffs. Let M_1 be the set of the best responses of team C, such that:

$$e_c^*(e_a, e_b, t(e_a)) = \begin{cases} \frac{1}{2}, & \text{when } e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = A \\ z_1, & \text{when } e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = B \end{cases},$$

where $z_1 \in [0, \frac{1}{2})$. Let M_2 be the set of the best responses of team C, such that:

$$e_c^*(e_a, e_b, t(e_a)) = \begin{cases} \frac{1}{2}, & \text{when } e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = B \\ z_2, & \text{when } e_a = e_b = \frac{1}{2} \text{ and } t(\frac{1}{2}) = A \end{cases},$$

where $z_2 \in (\frac{1}{2}, 1]$. Define the following sets of strategy profiles:

$$\mathcal{S}_1 = \{(e_a^*, e_b^*, e_c^*(e_a, e_b, t(e_a)))|0 < e_a^*, e_b^* < \tfrac{1}{2}, e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(e_a^*, 1 - e_a^*, t(e_a^*)) < 1 - e_a^*, e_c^*(1 - e_b^*, e_b^*, t(e_a^*)) > e_b^*\},$$

$$\mathcal{S}_2 = \{(e_a^*, e_b^*, e_c^*(e_a, e_b, t(e_a)))|e_a^* + e_b^* < 1, 0 < e_a^* < \tfrac{1}{2}, e_b^* > \tfrac{1}{2}, e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(e_a^*, 1 - e_a^*) < 1 - e_a^*\},$$

$$\mathcal{S}_3 = \{(e_a^*, e_b^*, e_c^*(e_a, e_b, t(e_a)))| e_a^* + e_b^* < 1, 0 < e_b^* < \tfrac{1}{2}, e_a^* > \tfrac{1}{2}, e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(1 - e_b^*, e_b^*) > e_b^*\},$$

$$\mathcal{S}_4 = \{(e_a^*, \tfrac{1}{2}, e_c^*(e_a, e_b, t(e_a)))| 0 < e_a^* < \tfrac{1}{2}, e_c^*(e_a^*, 1 - e_a^*) < 1 - e_a^*, e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)) \backslash M_1\},$$

$$\mathcal{S}_5 = \{(\tfrac{1}{2}, e_b^*, e_c^*(e_a, e_b, t(e_a)))| 0 < e_b^* < \tfrac{1}{2}, e_c^*(1 - e_b^*, e_b^*) > e_b^*, e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)) \backslash M_2\},$$

$$\mathcal{S}_6 = \{(e_a^*, e_b^*, e_c^*(e_a, e_b, t(e_a)))| e_a^* + e_b^* = 1, e_a^* > e_b^* > 0, e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(e_a^*, e_b^*) > 1 - e_a^*\},$$

$$\mathcal{S}_7 = \{(e_a^*, e_b^*, e_c^*(e_a, e_b, t(e_a))| e_a^* + e_b^* = 1, 0 < e_a^* < e_b^*, e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(e_a^*, e_b^*) < 1 - e_a^*\},$$

$$\mathcal{S}_8 = \{(\tfrac{1}{2}, \tfrac{1}{2}, e_c^*(e_a, e_b, t(e_a))| e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(\tfrac{1}{2}, \tfrac{1}{2}, A) = 1 - e_c^*(\tfrac{1}{2}, \tfrac{1}{2}, B)\},$$

$$\mathcal{S}_9 = \{(0, 0, e_c^*(e_a, e_b, t(e_a)))| e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(0, 1) \neq 1, e_c^*(1, 0) \neq 0\},$$

$$\mathcal{S}_{10} = \{(1, 0, e_c^*(e_a, e_b, t(e_a)))| e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(1, 0) = 0\},$$

$$\mathcal{S}_{11} = \{(0, 1, e_c^*(e_a, e_b, t(e_a)))| e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(0, 1) = 1\},$$

$$\mathcal{S}_{12} = \{(1, 1, e_c^*(e_a, e_b, t(e_a)))| e_c^*(e_a, e_b, t(e_a)) \in BR_c(e_a, e_b, t(e_a)), e_c^*(0, 1) = 1, e_c^*(1, 0) = 0\}.$$

Theorems 1 and 2 describe the set of all equilibrium profiles if the first prize is large and low, respectively. To avoid excessive formalization and save journal space, for the knife-edge case $v = \frac{2}{3}$, we describe only equilibrium payoffs. One can find the set of all equilibria in that case with the help of Proposition 1 and Figure 2. Figure 4 and Table 6 summarize the findings.

Theorem 1. *Let $v > \frac{2}{3}$. Then, in the first-price sealed-bid auction specification of the round-robin tournament with a limited effort model with prizes $(v_1, v_2, v_3) = (v, 1 - v, 0)$, the set of all SPE in pure strategies is $\bigcup_{i=1}^{11} \mathcal{S}_i$. At all SPE from the set $\bigcup_{i=1}^{8} \mathcal{S}_i$ all teams get a payoff of $\frac{1}{3}$. At all SPE from the set $\mathcal{S}_9$, teams get $\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{6}\right)$. At all SPE from the set $\mathcal{S}_{10}$, teams get $\left(\frac{1}{3} + \frac{1}{4}v, \frac{1}{3} - \frac{1}{4}v, \frac{1}{3}\right)$, and at all SPE from the set $\mathcal{S}_{11}$, teams get $\left(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3}\right)$.*

Theorem 1 shows that in the case of a large first prize, $v > \frac{2}{3}$, there are three types of equilibria. In the first type of equilibria ($\mathcal{S}_1 - \mathcal{S}_8$), neither team is disadvantaged. In the second type of equilibria ($\mathcal{S}_9$), teams that play the first match get larger payoffs at the expense of team C. Teams A and B can try to exploit their first move by exerting 0 effort in Round 1, which we refer to as effort-saving behavior. It means that playing 0 is a focal point, and there is only a sort of cooperative first-mover advantage for teams A and B. In the third type of equilibria ($\mathcal{S}_{10}$ and $\mathcal{S}_{11}$), one of the teams that participates in the first match gets the largest payoff at the expense of its first round opponent. Comparative statics with respect to a first prize v suggests that the payoffs in only the third type of equilibria is sensitive to a change of v in the zone where $v > \frac{2}{3}$. Setting a large first prize produces positive incentives for teams to commit to exert full effort in the first round if they have such an opportunity.

Theorem 2. *Let $\frac{1}{2} \leq v < \frac{2}{3}$. Then, in the first-price sealed-bid auction specification of the round-robin tournament with a limited effort model with prizes $(v_1, v_2, v_3) = (v, 1-v, 0)$, the set of all SPE in pure strategies is $\mathcal{S}_9 \cup \mathcal{S}_{12}$. At SPE from the set $\mathcal{S}_9$, teams get $(\frac{1+v}{4}, \frac{1+v}{4}, \frac{1-v}{2})$, whilst in SPE from the set $\mathcal{S}_{12}$, teams get $(\frac{1-v}{2}, \frac{1-v}{2}, v)$.*

If the first prize v is relatively low ($v \in [\frac{1}{2}, \frac{2}{3})$), then according to Theorem 2 there are only two types of equilibria: $\mathcal{S}_9$ and $\mathcal{S}_{12}$. For teams A and B, equilibria from set $\mathcal{S}_9$ are better for these teams than those from set $\mathcal{S}_{12}$. While the former equilibria allow teams A and B in the case of successful coordination to obtain a payoff larger than team C, the latter equilibria could be considered as a burning out equlibria with team C getting the highest payoff. Comparative statics with respect to parameter v is different for these equilibria: in the effort-saving equilibria $\mathcal{S}_9$, a higher v means higher profits for teams A and B, whereas in the burning out equlibria $\mathcal{S}_{12}$, a higher v means higher payoffs for team C.

Theorem 3. *Let $v = \frac{2}{3}$. Then, in the first-price sealed-bid auction specification of the round-robin tournament with a limited effort model with prizes $(v_1, v_2, v_3) = (v, 1-v, 0)$, the following pairs of strategies for teams A and B enter in at least one SPE:*

1. $(e_a^*, e_b^*) = (1, 1)$. *The payoffs in the corresponding SPE are* $\left(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}\right)$.
2. $(e_a^*, e_b^*), 0 < e_a^* + e_b^* \leq 1, (e_a^*, e_b^*) \notin \{(0,1), (1,0)\}$. *The payoffs in the corresponding SPE are* $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.
3. $(e_a^*, e_b^*) = (0, 0)$. *The payoffs in the corresponding SPE are* $\left(\frac{5}{12}, \frac{5}{12}, \frac{1}{6}\right)$, or $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$, or $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$.
4. $(e_a^*, e_b^*) = (0, 1)$. *The payoffs in the corresponding SPE are* $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$.
5. $(e_a^*, e_b^*) = (1, 0)$. *The payoffs in the corresponding SPE are* $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{3}\right)$.

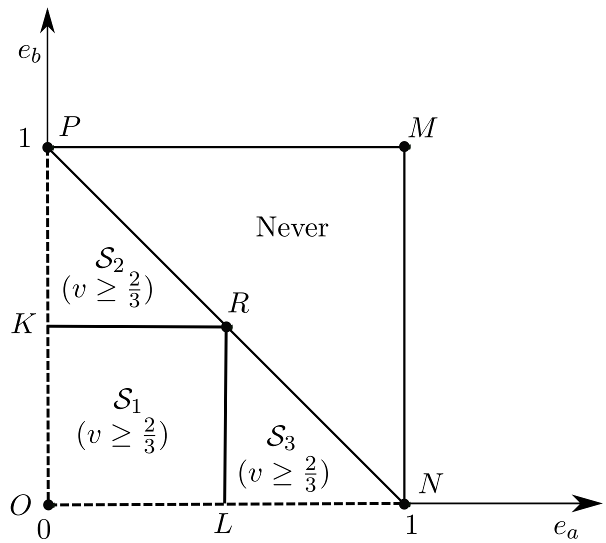

Figure 4: Projections of equilibria on (e_a, e_b), see also Table 6

	$v \in [\frac{1}{2}, \frac{2}{3})$	$v = \frac{2}{3}$	$v \in (\frac{2}{3}, 1]$
KR		$+$	$+$
RL		$+$	$+$
RN		$+$	$+$
PR		$+$	$+$
R		$+$	$+$
O	$+$	$+$	$+$
N		$+$	$+$
P		$+$	$+$
M	$+$	$+$	

Table 6: Projections of equilibria on (e_a, e_b), see also Figure 4

4 Conclusions

In this paper, we consider the model of a round-robin tournament with 3 participants, a limited budget of effort and a contest success function in the form of a first-price auction. In contrast to (Krumer, Megdish, and Sela, 2017), under the winner-takes-all tournament prize structure, we did not find any evidence that team B (which plays the first and third match) has an advantage. Moreover, if the first prize is large enough, then we find three types of pure strategy subgame perfect Nash equilibria. In the first type of equilibria, team C (which plays the second and third match) typically defeats the winner of the first match, which leads to exactly one win for each team, and thus, equal payoffs for all three teams. Second, there exists an 'effort-saving' equilibria in which teams playing in the first match waste zero effort and save all their efforts for the match against team C. In these equilibria, team C is disadvantaged. Third, one of the teams playing in the first match can get the largest payoff if it uses all its effort in the first match, while the opponent saves all its effort for the next match. To make this profile an equilibrium, team C must equalize opponents' efforts in both matches. The principal assumption that distinguishes our model from the one in (Krumer, Megdish, and Sela, 2017) and explains the difference in the outcome is the limited budget of effort.

In addition, for tournaments with a relatively low first prize, we found two types of equilibria: 'effort-saving' and 'burning out', with teams playing in the first match getting more and less than team C, respectively. This illustrates that the burning out effect may arise not only in elimination multi-stage contests as shown in (Amegashie, Cadsby, and Song, 2007) and Harbaugh and Klumpp (2005), but also in a round-robin tournament with limited resources (effort). These results imply that in a real-world, a round-robin subtournament of three teams with limted resources and one prize (ticket to the next tournament phase) is much more fair due to absence of the first or the last mover advantage compared to the same subtournament with two prizes (tickets).

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Appendix

The appendix contains formal proofs of statements made in the paper.

Proof of Proposition 1. When $v > \frac{1}{2}$, the teams are strictly interested in winning a tournament. When $v = \frac{1}{2}$, the teams are instead interested in not being last. Consider the following cases:

1. If $e_a + e_b > 1$, then C can win the tournament with certainty by allocating efforts $e_c > 1 - e_a$ in the second round against A, and $1 - e_c > 1 - e_b$ in the third round against B. Then, for $v > \frac{1}{2}$, all $e_c \in (1 - e_a, e_b)$ are the best responses for team C. Team C receives payoff v, the winner of the first round receives $1 - v$, and the loser of the first round receives 0.

Now, let $v = \frac{1}{2}$. To finish among the top two teams with a probability of 1, it is necessary and sufficient for team C to defeat the loser of the match between teams A and B. Consequently, if A won over B in Round 1, any e_c, such that $1 - e_c > 1 - e_b$, or $e_c < e_b$, guarantees that team C will finish in at least second place; if B won over A in Round 1, any $e_c > 1 - e_a$ works.

2. Let $e_a + e_b = 1$ and $e_a > \frac{1}{2}$. The latter inequality means that A defeats B with certainty. If team C exerts less than $1 - e_a$ of effort in Round 2, then team A wins the tournament, and team C's payoff is $1 - v$. Otherwise, team C can either tie the opponents' efforts in the second and third rounds or win the second round and lose in the third. The latter strategy allows team C to obtain the expected payoff of $\frac{1}{3}$, and the two other teams also get $\frac{1}{3}$. Tying the opponents' efforts in both rounds by playing $e_c = 1 - e_a$ means:

- with a probability of $\frac{1}{4}$, C wins in both rounds and gets a payoff of v; teams A and B get $1 - v$ and 0, respectively;
- with a probability of $\frac{1}{4}$, C wins in Round 2 and loses in Round 3, which means that each team has 1 win and gets a payoff of $\frac{1}{3}$;
- with a probability of $\frac{1}{4}$, C wins in Round 3, but loses in Round 2. Team C finishes second and gets $1 - v$, whereas team A receives v and team B — 0;
- with a probability of $\frac{1}{4}$, team C loses in both rounds and gets a payoff of 0. Teams A and B get v and $1 - v$, respectively.

Thus, the expected payoff of team C is $\frac{1}{4} \cdot v + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot (1 - v) = \frac{1}{3}$; the expected payoffs of teams A and B are $\frac{1}{3} + \frac{1}{4}v$ and $\frac{1}{3} - \frac{1}{4}v$, respectively. Compare the outcomes:

- If $v > \frac{2}{3}$, all strategies of $e_c \geq 1 - e_a$ are the best responses for team C. The corresponding payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ if C chooses $e_c > 1 - e_a$ and $(\frac{1}{3} + \frac{1}{4}v, \frac{1}{3} - \frac{1}{4}v, \frac{1}{3})$ if C chooses $e_c = 1 - e_a$.

- If $\frac{1}{2} \leq v < \frac{2}{3}$, team C will just win against the loser, team B, if possible. So, when $e_b > 0$, the best response of team C here is to play $e_c < e_b$. When $e_b = 0$ and $e_a = 1$, team C gets $\frac{1}{3}$ in any case. So, $BR_c(1,0) = [0,1]$. When $e_c > 0$, every team gets a payoff of $\frac{1}{3}$; when $e_c = 0$, the payoffs are $(\frac{1}{3} + \frac{1}{4}v, \frac{1}{3} - \frac{1}{4}v, \frac{1}{3})$.

- If $v = \frac{2}{3}$, team C is indifferent. Then C can choose any $e_c \in [0, e_b] \cup [1 - e_a, 1] = [0,1]$. If $e_c < e_b$, then payoffs are $(\frac{2}{3}, 0, \frac{1}{3})$; if $e_c = e_b$, the payoff profile is $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$; and if $e_c > e_b$, then the payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

3. If $e_a + e_b = 1$ and $e_a < \frac{1}{2}$ (B defeats A), the case is symmetric with the previous one.

- If $v > \frac{2}{3}$, all strategies $e_c \leq 1 - e_a$ are the best responses for team C. The corresponding payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ if C chooses $e_c < 1 - e_a$, and $(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3})$ if C chooses $e_c = 1 - e_a$.

- If $\frac{1}{2} \leq v < \frac{2}{3}$, team C will win against the loser, team A, if possible. So, when $e_a > 0$, the best response of team C here is to play $e_c > 1 - e_a$. When $e_a = 0$, and thus $e_b = 1$, then in any case team C gets $\frac{1}{3}$. So, $BR_c(0,1) = [0,1]$. When $e_c > 0$, every team gets payoff of $\frac{1}{3}$, but when $e_c = 0$, the payoff profile is: $(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3})$.

- If $v = \frac{2}{3}$, team C is indifferent, then C can choose any $e_c \in [0,1]$. If $e_c < 1 - e_a$, then payoffs are $(0, v, 1 - v)$; if $e_c = 1 - e_a$, the payoff profile is $(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3})$, and if $e_c > 1 - e_a$, then the payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

4. If $e_a = e_b = \frac{1}{2}$, team C must choose one of two actions in each of the two states of the tie-break. Defeating the winner of Round 1 (say, team A) leads to payoffs of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Defeating the loser of Round 1 (say, team B) gives payoffs of $(v, 0, 1 - v)$. The best decision for team C here is similar to the previous case:

- If $\frac{1}{2} \leq v < \frac{2}{3}$, it is better for C to win against the loser of Round 1.

- If $v = \frac{2}{3}$, any action is equally good for C: to defeat the winner, to defeat the loser, or to split its effort equally between both games. Three symmetric actions with respect to teams A and B bring the payoffs $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. If C always prefers to defeat A, then the teams get $(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$, if C always prefers to defeat B, then the teams get $(\frac{1}{2}, \frac{1}{6}, \frac{1}{3})$. If C defeats the winner A and splits its effort in case B wins, then the expected payoffs are $(\frac{1}{4}, \frac{5}{12}, \frac{1}{3})$. The same payoffs appear if C defeats the loser A and splits its effort in case B wins. Similarly, the two opposite actions result in $(\frac{5}{12}, \frac{1}{4}, \frac{1}{3})$.

- If $v > \frac{2}{3}$, and A defeated B in a tie break, then all $e_c \in [\frac{1}{2}, 1]$ are equally good for team C. If C chooses $e_c \in (\frac{1}{2}, 1]$, then the payoff profile is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and if C exerts an effort level of $\frac{1}{2}$, the interim payoff profile is $(\frac{1}{3}+\frac{1}{4}v, \frac{1}{3}-\frac{1}{4}v, \frac{1}{3})$. The same works for the case in which B defeated A in a tie-break in Round 1. So, when C does play symmertic actions (with respect to a lottery outcome), the ex-ante payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. If C plays $e_c > \frac{1}{2}$ when A won in Round 1, but plays $e_c = \frac{1}{2}$ when B won in Round 1, then the ex-ante payoffs are $\left(\frac{1}{3}-\frac{1}{8}v, \frac{1}{3}+\frac{1}{8}v, \frac{1}{3}\right)$. Symmetrically, if C plays $e_c < \frac{1}{2}$ when B won in Round 1, but plays $e_c = \frac{1}{2}$ when A won in Round 1, then the ex-ante payoffs are $\left(\frac{1}{3}+\frac{1}{8}v, \frac{1}{3}-\frac{1}{8}v, \frac{1}{3}\right)$.

5. If $0 < e_a + e_b < 1$, team C cannot do better than to win one game and to lose another one.

   - When $v > \frac{2}{3}$, the best decision is to defeat the winner of the first round and to get a payoff of $\frac{1}{3}$.

   - When $\frac{1}{2} \leq v < \frac{2}{3}$, then for team C it is best to win over the loser of Round 1 (if possible; i.e., a loser spends strictly more than 0 in Round 1), getting payoffs $(v, 0, 1-v)$ if A won Round 1 and $(0, v, 1-v)$ if B won Round 1. If $e_a = 0$ and $e_b \in (0, 1)$, then winning over B is better, since it gives $\frac{1}{3}$, but a tying effort against A will give only $\frac{1}{2}(1-v) < \frac{1}{3}$. The same applies symmetrically to the case in which $e_b = 0$ and $e_a \neq 0$.

   - When $v = \frac{2}{3}$, team C is indifferent between defeating the winner and beating the loser of Round 1. To defeat the winner of Round 1, team C can choose any effort $x > 1 - e_a$ if A won over B and any $x < e_b$ if B won over A. All teams get $\frac{1}{3}$. To beat the loser of Round 1, team C can choose any effort $x < e_b$ if A won over B and any $x > 1 - e_a$ if B won over A. So, the best action is to play $x \in [0, e_b) \cup (1 - e_a, 1]$.

6. If $e_a = e_b = 0$, then

   - If $v > \frac{2}{3}$, team C should spend all its effort in a game against the winner of Round 1. In this case, C gets a payoff of $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, whereas both other teams get an ex-ante payoff of $\frac{5}{12}$.

   - If $\frac{1}{2} \leq v < \frac{2}{3}$, it is better for team C to spend all its effort on the loser of Round 1, getting a payoff of $\frac{1}{2}(1-v)$. The two other teams will get ex-ante payoffs of $\frac{1}{2}(1-\frac{1}{2}(1-v)) = \frac{1+v}{4}$.

   - If $v = \frac{2}{3}$, C is indifferent between spending all its effort either in one or another game. $\square$

Proof of Theorem 1. Let $v > \frac{2}{3}$. We check all pairs (e_a^*, e_b^*) for the profitability of deviations assuming that C plays their best response.

- If $e_a^* + e_b^* > 1$, the corresponding profile cannot be an equilibrium because playing 0 for one of the teams is certainly better.

- If $e_a^* = 0, 0 < e_b^* < 1$ or $e_b^* = 0, 0 < e_a^* < 1$, all teams get $\frac{1}{3}$. However, a team that exerts a positive level of effort can gain by decreasing it to 0. Thus, the corresponding strategies do not constitute an equilibrium.
- If $0 < e_a^*, e_b^* < \frac{1}{2}$, the payoffs are $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Deviation is unprofitable for both teams A and B if and only if team C chooses to defeat the winner of the first game in points $(e_a^*, 1 - e_a^*)$ and $(1 - e_b^*, e_b^*)$. This leads to a set of equilibria $\mathcal{S}_1$.
- Analogously, if $0 < e_a^* < \frac{1}{2}$ and $\frac{1}{2} < e_b^* < 1 - e_a^*$, we obtain a set of equilibria $\mathcal{S}_2$, whilst case $0 < e_b^* < \frac{1}{2}$ and $\frac{1}{2} < e_a^* < 1 - e_b^*$ brings a set of equilibria $\mathcal{S}_3$.
- Suppose that $0 < e_a^* < \frac{1}{2}$ and $e_b^* = \frac{1}{2}$. Team B would not deviate if $e_c^*(e_a^*, 1 - e_a^*) < 1 - e_a^*$. It is necessary and sufficient for the existence of a profitable deviation for team A that team C responds to $(\frac{1}{2}, \frac{1}{2})$ in such a way that C defeats B if B is the winner of the first game, but equalizes the efforts of opponents if A won the first game. Thus, $\mathcal{S}_4$ is the set of equilibria in this case. Similarly, $\mathcal{S}_5$ is the set of equilibria if $0 < e_b^* < \frac{1}{2}$ and $e_a^* = \frac{1}{2}$.
- If $e_a^* + e_b^* = 1$, $0 < e_a^*, e_b^* < 1$, both teams A and B must receive $\frac{1}{3}$ in equilibrium because both can guarantee a payoff of $\frac{1}{3}$ by deviating. This is possible if and only if the strategy profile belongs to the set $\mathcal{S}_6 \cup \mathcal{S}_7 \cup \mathcal{S}_8$.
- All strategy profiles from the set $\mathcal{S}_9$ are SPE: neither A, nor B, can get more than $\frac{5}{12}$.
- If $e_a^* = 0$, $e_b^* = 1$ and $e_c^* \neq 1$, the payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and team B would prefer to decrease their effort to 0. If $e_a^* = 0$, $e_b^* = 1$ and $e_c^* = 1$, the payoffs are $(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3})$, and no team is better off by deviating. This leads to a set of equilibria $\mathcal{S}_{11}$. Case $e_b^* = 0$, $e_a^* = 1$ is symmetric and leads to a set of equilibria $\mathcal{S}_{10}$.□

Proof of Theorem 2. We check all pairs (e_a^*, e_b^*) for the profitability of deviations, assuming that C plays their best response.

- If $2 > e_a^* + e_b^* > 1$, the corresponding profile cannot be an equilibrium because playing 0 for one of the teams is certainly better since it allows it to get at least $\frac{1}{3}$.
- If $e_a^* = e_b^* = 1$, neither team can profitably deviate if and only if team C ties the winners' efforts at $(1, 0)$ and $(0, 1)$. This brings out equilibria from the set $\mathcal{S}_{12}$.
- Any strategy profile such that $0 < e_a^*, e_b^* < 1$ or any strategy profile such that $e_a^* = 1, 0 < e_b^* < 1$ or $e_b^* = 1, 0 < e_a^* < 1$ can not be an equilibrium because each of the teams A and B is able to guarantee a payoff of $\frac{1}{3}$ by decreasing their efforts to 0.

- If $e_a^* = 0, 0 < e_b^* < 1$ or $e_b^* = 0, 0 < e_a^* < 1$, all teams get $\frac{1}{3}$. However, a team that exerts a positive level of effort can gain by decreasing its effort to 0. Thus, the corresponding strategies do not constitute an equilibrium.

- If $e_a^* = 0$, $e_b^* = 1$ and $e_c^* \neq 1$, the payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and team B would prefer to decrease their effort to 0. If $e_a^* = 0$, $e_b^* = 1$ and $e_c^* = 1$, the payoffs are $(\frac{1}{3} - \frac{1}{4}v, \frac{1}{3} + \frac{1}{4}v, \frac{1}{3})$, and team A is better off by exerting all its effort in the first game against B, changing its own payoff to $\frac{1-v}{2}$. Case $e_b^* = 0$, $e_a^* = 1$ is symmetric. There are no equilibria here.

- It is easy to check that the strategy profiles from the set $\mathcal{S}_9$ are SPE: neither A, nor B, can enlarge their payoffs.

- Other profiles that include $e_a^* = e_b^* = 0$ are not equilibria since either A would prefer $e_a = 1$ instead of $e_a^* = 0$, or B would prefer $e_b = 1$ instead of $e_b^* = 0$.□

Proof of Theorem 3. An internal point of the square $PMNO$ could be an equilibrium only if all teams get a payoff of $\frac{1}{3}$: teams A and B can guarantee this payoff by decreaing their efforts in the first game to 0.

- If $e_a^* = e_b^* = 1$, then no team has an incentive to deviate, if and only if team C ties the winners' efforts at $(1, 0)$ and $(0, 1)$.

- If $2 > e_a^* + e_b^* > 1$, these points do not satisfy the necessary condition. Hence, this is not an equilibrium.

- If $0 < e_a^* + e_b^* \leq 1$, $(e_a^*, e_b^*) \notin \{(0, 1), (1, 0)\}$, there exists an equilibrium in which team C always defeats the winner of the first round (if possible). The payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

- If $e_a^* = e_b^* = 0$, all payoffs can occur in equilibrium if team C defeats the winner of the first round in points P and N.

- Suppose that $e_a^* = 0$, $e_b^* = 1$. If $e_c^* \neq 1$, the payoffs are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and if $e_c^* = 1$, the payoffs are $(\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$. In both cases, team A can not deviate profitably. If $e_c^* = 1$, team B can not deviate. If $e_c^* \neq 1$, team B would not prefer to decrease their effort if the payoffs at point $(0, 0)$ are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. This brings two types of equilibria. Case $e_b^* = 0$, $e_a^* = 1$ is symmetric.□

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