NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

Ovanes Petrosian

ABOUT THE LOOKING FORWARD APPROACH IN COOPERATIVE DIFFERENTIAL GAMES WITH TRANSFERABLE UTILITY

BASIC RESEARCH PROGRAM
WORKING PAPERS

SERIES: ECONOMICS
WP BRP 183/EC/2018

SERIES: *ECONOMICS*

Ovanes Petrosian★

About the Looking Forward Approach in Cooperative Differential Games with Transferable Utility*

This paper presents a complete description and the results of the Looking Forward Approach for cooperative differential games with transferable utility. The approach is used for constructing game theoretical models and defining solutions for conflict-controlled processes where information about the process updates dynamically or for differential games with dynamic updating. It is supposed that players lack certain information about the dynamical system and payoff function over the whole time interval on which the game is played. At each instant, information about the game structure updates, players receive new updated information about the dynamical system and payoff functions. A resource extraction game serves as an illustration in order to compare a cooperative trajectory, imputations, and the imputation distribution procedure in a game with the Looking Forward Approach and in the original game with a prescribed duration.

JEL classification: C71, C73.
Keywords: Differential Games, Cooperative Differential Games, Looking Forward Approach, Time Consistency, Strong Time Consistency.

1 Introduction

This research examines a cooperative differential game with transferable utility in which the game structure can change or update with time (time-dependent formulation) and it is assumed that the players do not have information about the change of the game structure over the full time interval, but they have certain information about the game structure over the truncated time interval. Under the information about the game structure we understand information about the dynamical system and payoff functions. The interpretation can be given as follows: players have certain information about the game structure, but the duration of this information is less than the length of the initial game. Evidently,

★ National Research University Higher School of Economics, St.Petersburg, Russia; e-mail: opetrosyan@hse.ru

* Support from the Basic Research Program of the National Research University Higher School of Economics is gratefully acknowledged.

this truncated information is valid only for a certain time and has to be updated. In order to define the best possible behavior for players in this type of cooperative differential game, a special approach is needed, which is called the Looking Forward Approach. This approach brings up the following points: how to define a cooperative trajectory, how to define a cooperative solution and allocate the cooperative payoff, and what properties the obtained solution will have. This paper answers these questions and gives an overview of the results corresponding to the Looking Forward Approach. It is demonstrated that the newly built arbitrary solution for a class of differential games with dynamic updating is not only time consistent (which is very rare in cooperative differential games), but is also strongly time consistent. Haurie analyzed the problem of the dynamic instability of Nash bargaining solutions in differential games [1]. The notion of the time consistency of differential game solutions was formalized mathematically by Petrosyan [14] who presented results related to the connection between solutions chosen by the players in truncated subgames and in the overall game. The notion of the characteristic function for the general game is introduced, with the help of this and the solution for the general game, it is proved that solutions chosen by the players in the truncated subgames correspond to the resulting solution based on the new characteristic function.

The concept of the Looking Forward Approach is new in game theory especially in cooperative differential games and gives a foundation for the further study of differential games with dynamic updating. There are currently almost no results in constructing approaches for modeling conflict-controlled processes where information about the process updates in real time. To get more information about the approach one may read the following papers: [7], [8], [9], [21], [22], [23], [25], [26]. In [21] the Looking Forward Approach was applied to a cooperative differential game with a finite horizon. The notion of a truncated subgame, the procedure for defining optimal strategies, a conditionally cooperative trajectory and solution concept, and the solution property of Δt-time consistency for a fixed information horizon were determined. [23] focuses on the study of the Looking Forward Approach with a stochastic forecast and dynamic adaptation when information about the conflicting process can change during the game. In [9] the Looking Forward Approach was applied to a cooperative differential game of pollution control. The paper studies the dependency of the resulting solution on the value of the information horizon, and the corresponding optimization problem was formulated and solved. In [22] the Looking Forward Approach was applied to a cooperative differential game with an infinite horizon. In [26] the Looking Forward Approach with a random horizon was presented, which is one of the variations of the Looking Forward Approach introduced in [21]. Papers [7] and [8] study cooperative differential games with an infinite horizon where information about the process updates dynamically, the focus of the papers is a profound formulation of Hamilton-Jacobi-Bellman equations for different types of forecasts and information structures. In [24] an imputation distribution procedure (IDP)-core was used as a cooperative solution and it was proved that the resulting solution is strongly time consistent. The last paper on the Looking Forward Approach [25] is devoted to studying the

Looking Forward Approach for cooperative differential games with nontransferable utility and the real life application of the Looking Forward Approach to economic simulations.

The characteristic function of a coalition is an essential concept in the theory of differential games. This function is defined in [27] as the total payoff for players from coalition S in Nash equilibrium in a game with the following set of players: coalition S (acting as one player) and players from the set $N \setminus S$. A computation of the Nash equilibrium fully described in [32] is necessary for this approach. A set of imputations or a solution for the game is determined by the characteristic function at the beginning of each subinterval. For any set of imputations the IDP first introduced by Petrosyan in [16] is analyzed. See recent publications on this topic in [15], [29], [30]. In order to determine a solution for the whole game combined partial solutions and their IDP on subintervals is required. The property of time consistency and strong time consistency introduced by Petrosyan in [12] and [14] are also examined for the proposed solution.

The Looking Forward Approach has similarities with the Model Predictive Control (MPC) theory worked out within the framework of numerical optimal control. We analyze [10], [11], [19], [33] to get recent results in this area. MPC is a method of control when the current control action is achieved by solving at each sampling instant a finite horizon open-loop optimal control problem using the current state of an object as the initial state. This type of control is able to cope with strict limitations on controls and states, which is an advantage over other methods. There is, therefore, a wide application in petro-chemical and related industries where key operating points are located close to the set of admissible states and controls. The main problem that is solved in MPC is the provision of movement along the target trajectory under the conditions of random perturbations and an unknown dynamical system. At each time step the optimal control problem is solved for defining controls which will lead the system to the target trajectory. The Looking Forward Approach on the other hand solves the problem of modeling player behavior when information about the process updates dynamically. This means that the Looking Forward Approach does not use the target trajectory, but answers the question of composing a trajectory which will be used by players, and the question of allocating the cooperative payoff along the composed trajectory.

To demonstrate the Looking Forward Approach we present an example of a cooperative resource extraction game with a finite horizon. The original example was introduced in [29], the problem of time consistency in this game was examined in [6]. We present both analytic and numerical solutions for specific parameters. The comparison between the original game and the game with the Looking Forward Approach is presented. In the final part of the example model we demonstrate the strong time consistency of the solution. The structure of the article is as follows. In section 1 the description of the original game is presented. In Section 2, the definition of a truncated subgame is presented. In Section 3 the solution of the truncated subgame is described, and the conditional-cooperative trajectory is constructed. In Section 4, based on the results in section 3, a solution is constructed in a game with dynamic updating, the theorem of strong Δt-time consistency is presented. Section 5 is devoted to the construction of the characteristic function in a game with dynamic updating. In Section 6, the connection between the solutions in truncated subgames and the resulting solution in the original game with dynamic updating is described and formalized mathematically. In Section 7, the Looking Forward Approach is applied to a three player cooperative resource extraction game. The results of a numerical simulation in Matlab are presented.

2 The Original Game

Consider an n-player differential game $\Gamma(x_0, T - t_0)$ with a finite horizon $T - t_0$, with an initial state $x_0 \in R^m$ and an initial time instant t_0 (t_0 and T are a fixed values). Denote the set of players by $N = \{1, \ldots, n\}$. At each time instant player $i \in N$ chooses a control or a strategy $u_i \in U_i \subset CompR^k$. The payoff function of the player $i \in N$ is defined by

$$K_i(x_0, T - t_0; u) = \int_{t_0}^{T} h_i(x(\tau), u(\tau)) e^{-r(\tau - t_0)} d\tau \tag{1}$$

subject to the dynamical system

$$\dot{x} = g(t, x, u), \;\; x(t_0) = x_0, \tag{2}$$

where $x(\tau) \in R^m$ is the trajectory (the solution) of the system (1) with the control input $u = (u_1, \ldots, u_n)$, $r \geq 0$, functions $h_i(x(\tau), u(\tau))$ and $g(t, x, u)$ are differentiable. $h_i(x(\tau), u(\tau))$ shows an instant payoff that is received by player $i \in N$ in state $x(\tau) \in R^m$ with control input $u = (u_1, \ldots, u_n)$, function $e^{-r(\tau - t_0)}$ defines the discount factor for the players' payoff. The solution of the system (1) determines the trajectory of the game.

When open-loop strategies are used, we require piece-wise continuity with a finite number of breaks. For feedback strategies we follow [32]. We require that for any n-tuple of strategies $u(t, x) = (u_1(t, x), \ldots, u_n(t, x))$ the solution of the Cauchy problem of (1) exists and is unique at the time interval $[t_0, T]$. For a more sophisticated definition of feedback strategies in zero-sum differential games see [20].

3 Truncated Subgame

As mentioned above, during time interval $[t_0 + j\Delta t, t_0 + (j + 1)\Delta t]$ players have full information about the dynamics of the game and the payoff function on time interval $[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, where $\overline{T}$ is a fixed value, namely the information horizon. At time instant $t = t_0 + (j + 1)\Delta t$ information about the game is being updated. At the next time interval $(t_0 + (j + 1)\Delta t, t_0 + (j + 2)\Delta t]$ players have full information about the game structure at time interval $(t_0 + (j+1)\Delta t, t_0 + (j+1)\Delta t + \overline{T}]$. For $j = 0$ we suppose that $\bar{t}_{j-1} = 0$.

To model this type of information structure we introduce the following definition (Fig. 1). Denote vector $x_{j,0} = x(t_0 + j\Delta t)$.

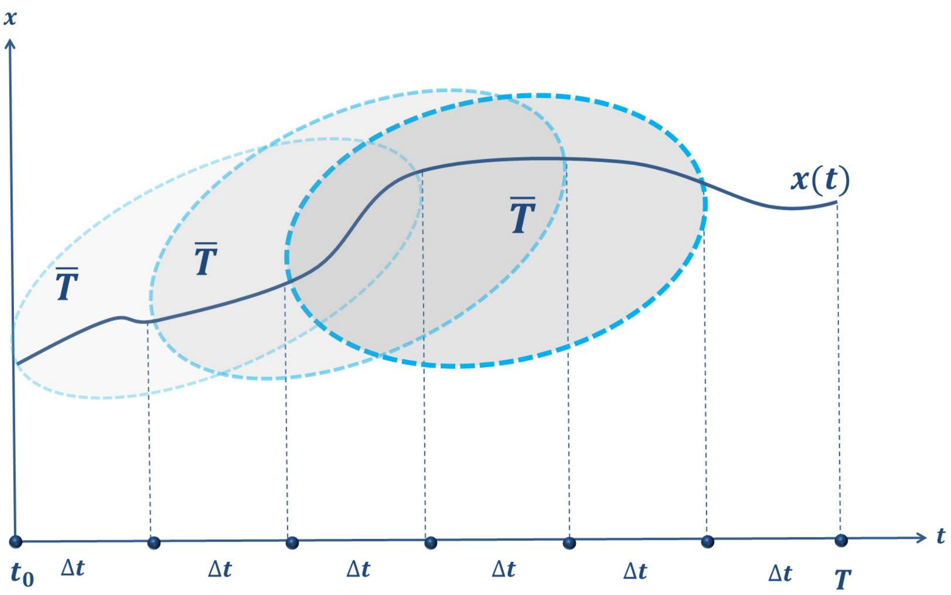

Figure 1: Each oval represents random truncated information, which is known to players during the time interval $[t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, $j = 0, \ldots, l$.

Definition 1. Let $j = 0, \ldots, l$. A truncated subgame $\bar{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ is defined on the time interval $[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$. The dynamical system and the payoff function on the time interval $[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$ coincide with that of the game $\Gamma(x_0, T - t_0)$ on the same time interval. The payoff function of player $i \in N$ in the truncated subgame j is

$$K_i^j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}; u) = \int\limits_{t_0 + j\Delta t}^{t_0 + j\Delta t + \overline{T}} h_i \quad (x(\tau), u(\tau)) e^{-r(\tau - t_0)} d\tau \qquad (3)$$

subject to the dynamical system

$$\dot{x} = g(t, x, u), \;\; x(t_0 + j\Delta t) = x_{j,0} \qquad (4)$$

with the initial condition $x_{j,0}$ of the truncated subgame $\bar{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$.

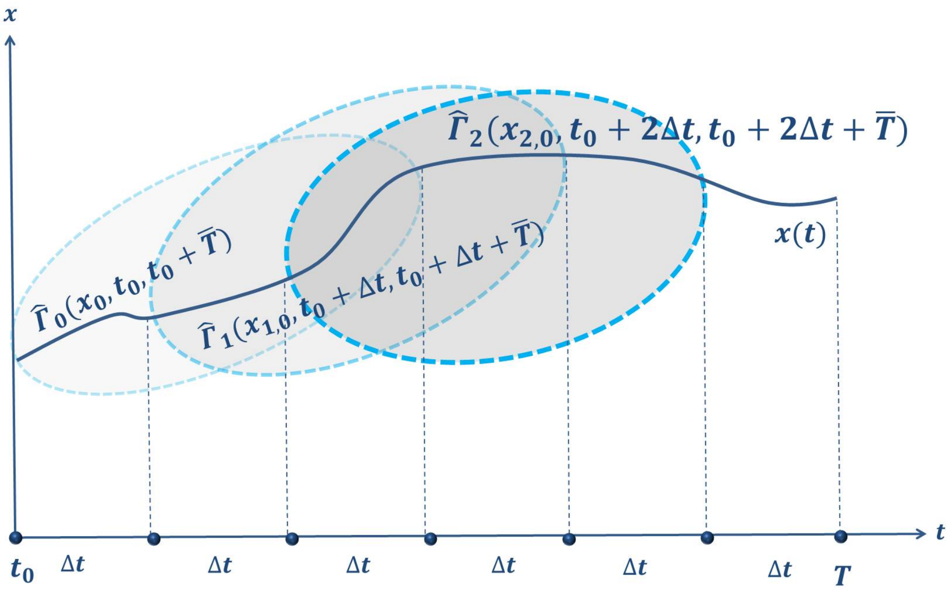

Figure 2: Behavior of players in the game with truncated information can be modeled using the truncated subgames $\bar{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$, $j = 0, \ldots, l$.

4 Solution of a Cooperative Truncated Subgame

Consider a truncated cooperative subgame $\bar{\Gamma}_j^c(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ defined on the time interval $[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$ with the initial condition $x(t_0 + j\Delta t) = x_{j,0}$. The total payoff of players to be maximized in this game is

$$\sum_{i \in N} K_i^j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}; u) = \sum_{i \in N} \int\limits_{t_0 + j\Delta t}^{t_0 + j\Delta t + \overline{T}} h_i \quad (x(\tau), u(\tau)) e^{-r(\tau - t_0)} d\tau \tag{5}$$

subject to

$$\dot{x} = g(t, x, u),\ \ x(t_0 + j\Delta t) = x_{j,0}. \tag{6}$$

This is an optimal control problem. Sufficient conditions for the solution and the optimal feedback are given by the following assertion [28]. Denote the maximum value of joint payoff (5) by the function $W^{(j\Delta t)}(t, x)$:

$$W^{(j\Delta t)}(t, x) = \max_{u \in U} \left\{ \sum_{i \in N} K_i^j(x, t, t_0 + j\Delta t + \overline{T}; u) \right\}, \tag{7}$$

where x, t are the initial state and time of the subgame of the truncated game respectively and $U = U_1 \times \ldots \times U_n$.

Theorem 1. *Assume there exists a continuously differential function* $W^{(j\Delta t)}(t,x) : [t_0 + j\Delta t, \overline{T}_j] \times R^m \to R$ *satisfying the partial differential equation*

$$-W_t^{(j\Delta t)}(t,x) = \max_{u \in U} \left\{ \sum_{i=1}^{n} h_i(t,x,u) e^{-r(\tau - t_0)} + W_x^{(j\Delta t)}(t,x) g(t,x,u) \right\}, \tag{8}$$

where $\lim\limits_{t \to T^-} W^{(j\Delta t)}(t,x) = 0$ *and the maximum in (8) is achieved under controls* $u_j^*(t,x)$. *Then* $u_j^*(t,x)$ *is optimal in the control problem defined by (5), (6).*

Theorem 1 requires that the function $W^{(j\Delta t)}$ be C^1. However, it is possible to assume continuity only considering viscosity-solutions using the Subbotin approach [2], [3]. But due to the shortage of space, it is not possible to properly introduce and define this solution in this paper. In the example model we define and get solution $W^{(j\Delta t)}$ from C^1.

4.1 Conditionally cooperative trajectory

During the game $\Gamma(x_0, T - t_0)$ players possess only truncated information about its structure. Obviously, this is not enough to construct optimal control and the corresponding trajectory for the game $\Gamma(x_0, T - t_0)$. As a cooperative trajectory in the game $\Gamma(x_0, T - t_0)$ we propose using a conditionally cooperative trajectory defined in the following way:

Definition 2. A conditionally cooperative trajectory $\{\hat{x}^*(t)\}_{t=t_0}^{T}$ is defined as a composition of cooperative trajectories $x_j^*(t)$ in the truncated cooperative subgames $\bar{\Gamma}_j^c(x_{j-1}^*(t_0 + j\Delta t), t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ defined on the successive time intervals $[t_0 + j\Delta t, t_0 + (j+1)\Delta t]$ (Fig.3):

$$\{\hat{x}^*(t)\}_{t_0}^{T} = \begin{cases} x_0^*(t), & t \in [t_0, t_0 + \Delta t), \\ \ldots, & \\ x_j^*(t), & t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t), \\ \ldots, & \\ x_l^*(t), & t \in [t_0 + l\Delta t, t_0 + (l+1)\Delta t], \end{cases} \tag{9}$$

On the time interval $[t_0 + j\Delta t, t_0 + (j+1)\Delta t]$ a conditionally cooperative trajectory coincides with the cooperative trajectory $x_j^*(t)$ in the truncated cooperative subgame $\bar{\Gamma}_j^c(x_{j-1}^*(t_0 + j\Delta t), t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$. In the position $x_j^*(t_0 + (j+1)\Delta t)$ at the time instant $t = t_0 + (j+1)\Delta t$ information about the game structure updates. On the time interval $(t_0 + (j+1)\Delta t, t_0 + (j+2)\Delta t]$ the trajectory $\hat{x}^*(t)$ coincides with the cooperative trajectory $x_{j+1}^*(t)$ in the truncated cooperative subgame $\bar{\Gamma}_{j+1}^c(x_j^*(t_0 + (j+1)\Delta t), t_0 + (j+1)\Delta t, t_0 + (j+1)\Delta t + \overline{T})$ which starts at the time instant $t = t_0 + (j+1)\Delta t$ in the position $x_j^*(t_0 + (j+1)\Delta t)$. For $j = 0$: $x_{j-1}^*(t_0 + j\Delta t) = x_0$.

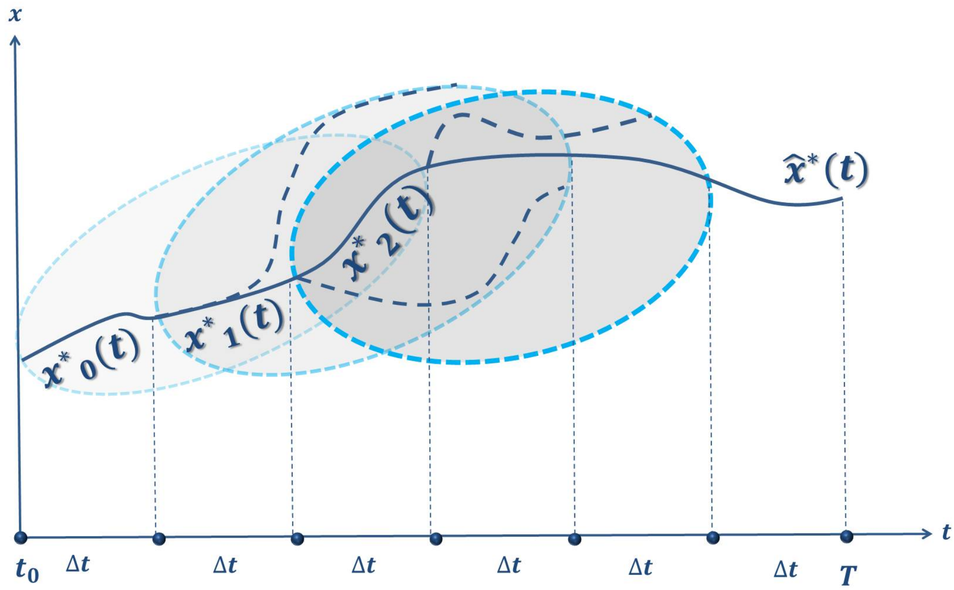

Figure 3: The solid line represents the conditionally cooperative trajectory $\{\hat{x}^*(t)\}_{t=t_0}^{T}$. Dashed lines represent parts of cooperative trajectories that are not used in the composition, i.e., each dashed trajectory is no longer optimal in the current random truncated subgame.

4.2 Characteristic Function

For each coalition $S \subset N$ and $j = 0, \ldots, l$ define the values of the characteristic function as in [27]:

$$V_j(S; \quad x^*_{j,0} \quad, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) = \begin{cases} \sum\limits_{i \in N} K_i^j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}; u^*_j), & S = N, \\ \tilde{V}_j(S, x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}), & S \subset N, \\ 0, & S = \emptyset, \end{cases} \tag{10}$$

where $\tilde{V}_j(S, x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ is defined as the total payoff for players from coalition S in Nash equilibrium $u_j^{NE} = (u_1^{NE,j}, \ldots, u_n^{NE,j})$ in the game with the following set of players: coalition S (acting as one player) and players from the set $N \setminus S$, i.e. in the game with $|N \setminus S| + 1$ players.

An imputation $\xi_j(x^*_{j,0}, t_0 + j\Delta t)$ for each truncated cooperative subgame $\bar{\Gamma}^c_j(x^*_{j,0}, t_0 +$

$j\Delta t, t_0 + j\Delta t + \overline{T})$ is defined as an arbitrary vector which satisfies the conditions

$$\xi_i^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) \geq V_j(\{i\}, x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}),\ i \in N,$$
$$\sum_{i\in N} \xi_i^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) = V_j(N, x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}).$$

Denote the set of all possible imputations for each truncated subgame by $E_j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$. Suppose that for each truncated subgame a non-empty solution is defined:

$$W_j(x_{j,0}^*, t, t_0 + j\Delta t + \overline{T}) \subset E_j(x_{j,0}^*, t, t_0 + j\Delta t + \overline{T}) \quad (11)$$

This can be a Core, an NM solution, a Nucleus, or a Shapley value.

5 Solution Concept in an Original Game with Dynamic Updating

It is logical to assume that the distribution of the total payoff between players in the game $\Gamma(x_0, T - t_0)$ along the conditionally cooperative trajectory $\{\hat{x}^*(t)\}_{t=t_0}^T$ is defined as a combination of the imputations at time intervals $[t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, $j = 0, \ldots, l$. This construction will be called the new concept of solution.

The combination of the family of sets $W_j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ does not allow us to obtain a solution in the game $\Gamma(x_0, T - t_0)$ directly. For each $j = 0, \ldots, l$ the solution in a truncated subgame is $\hat{\Gamma}_v^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ is defined for the time interval $[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$. But information about the game is updated with step Δt, and the use of such a solution in the time interval $[t_0 + j\Delta t, t_0 + (j+1)\Delta t]$ is not possible. A necessary part of the solution can be obtained by using the imputation distribution procedure for each truncated subgame. The IDP also provides the time consistency property of the new solution concept and the ability to determine solutions within the time interval $[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$.

In order to construct a solution in the game $\Gamma(x_0, T - t_0)$, you need to define the IDP for all truncated subgames $\hat{\Gamma}_v^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$, $j = 0, \ldots, l$. We denote the family of subgames $\hat{\Gamma}_v^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ along the cooperative trajectory $x_j^*(t)$ by $\hat{\Gamma}_v^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$, where $t \in (t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$ is the initial moment of the subgame. The characteristic function along $x_j^*(t)$ in the family of subgames $\hat{\Gamma}_v^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$ is also defined as in (10). We denote by $E_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$ the set of imputations in the subgame $\hat{\Gamma}_v^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$.

Suppose that in each truncated subgame $\hat{\Gamma}_v^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ the solution $W_j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) \neq \emptyset$ along the cooperative trajectory $x_j^*(t)$ is selected. Also, suppose that for any truncated subgame $\hat{\Gamma}_v^j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ in the starting position $x^*_{j,0}$ the imputation

$$\xi_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) \in W_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$$

and the corresponding IDP are selected

$$\beta_j(t, x^*_j) = [\beta^j_1(t, x^*_j), \ldots, \beta^j_n(t, x^*_j)],\ t \in (t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}],$$

which guarantees the time consistency of the selected imputation [16]:

$$\xi_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) = \int\limits_{t_0+j\Delta t}^{t_0+j\Delta t+\overline{T}} \beta_j(t, x^*_j) e^{-r(\tau - t_0)} dt. \tag{12}$$

The IDP $\beta_j(t, x^*_j)$ can be obtained by differentiating the imputation $\xi_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$, the corresponding theorem is presented in [6]:

Theorem 2. *If the function $\xi_j(x^*_j, t, t_0 + j\Delta t + \overline{T})$ is continuously differentiable for t and x^*_j, then*

$$\beta_j(t, x^*_j) = -\xi^j_t(x^*_j, t, t_0 + j\Delta t + \overline{T}) - \\ - \xi^j_{x^*_j}(x^*_j, t, t_0 + j\Delta t + \overline{T}) f\left(x^*_j, u^{*j}_1(\tau), \ldots, u^{*j}_n(\tau)\right). \tag{13}$$

The new solution concept in the game $\Gamma(x_0, T - t_0)$ consists of a combination of solutions $W_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ (corresponding to the IDP) in truncated subgames $\hat{\Gamma}^j_v(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$, $j = 0, \ldots, l$. Suppose that for each imputation $\xi_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) \in W_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ there exists $\beta_j(t, x^*_j)$. Define the resulting IDP for the whole game $\Gamma(x_0, T - t_0)$ as follows:

Definition 3. *The resulting IDP $\hat{\beta}(t, \hat{x}^*)$ is defined for each set $\xi_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) \in W_j(x^*_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ using the corresponding $\beta_j(t, x^*_j)$ as follows:*

$$\hat{\beta}(t, \hat{x}^*) = \begin{cases} \beta_0(t, x^*_0),\ t \in [t_0, t_0\Delta t], \\ \ldots \\ \beta_j(t, x^*_j),\ t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t], \\ \ldots \\ \beta_l(t, x^*_l),\ t \in [t_0 + l\Delta t, t_0 + (l+1)\Delta t]. \end{cases} \tag{14}$$

Using the resulting IDP $\hat{\beta}(t, \hat{x}^*)$ we define the following vector:

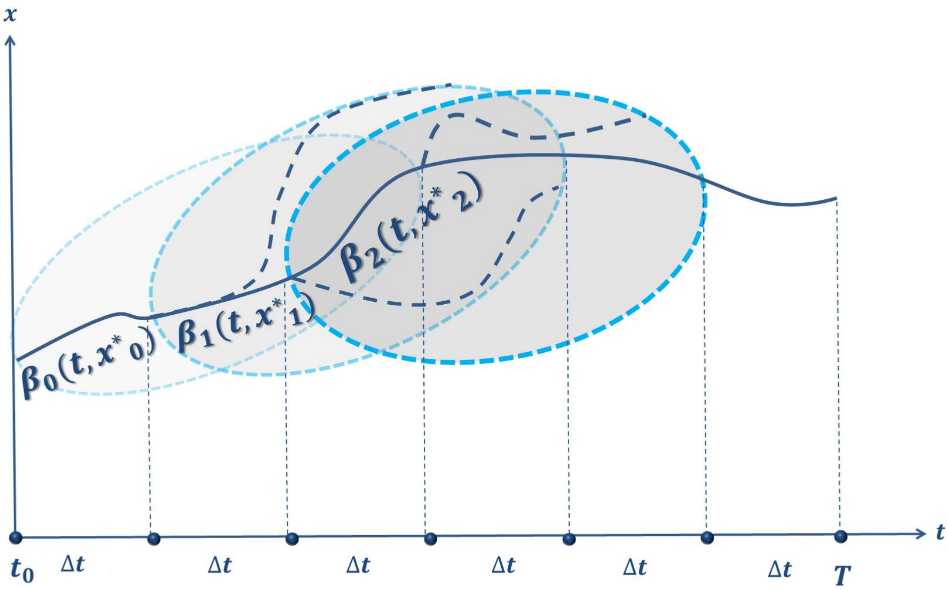

Figure 4: The combination of the $\beta_j(t, x_j^*)$ IDP is defined for each $\xi_j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}) \in W_j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$, $j = 0, \ldots, l$ determines the distribution of the total payoff among players using $\hat{\beta}(t, \hat{x}^*)$.

Definition 4 *The resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ is the vector defined by the resulting IDP $\hat{\beta}(t, \hat{x}^*)$ in the following way, let $t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$:*

$$\hat{\xi}(\hat{x}^*(t), T - t) = \int_t^T \hat{\beta}(\tau, \hat{x}^*(\tau)) e^{-r(\tau - t_0)} d\tau = \int_t^{t_0 + j\Delta t} \beta_j(\tau, x_j^*(\tau)) e^{-r(\tau - t_0)} d\tau + \sum_{m=j+1}^{l} \left[\int_{t_0 + m\Delta t}^{t_0 + (m+1)\Delta t} \beta_m(\tau, x_m^*(\tau)) e^{-r(\tau - t_0)} d\tau \right], \quad (15)$$

in particular:

$$\hat{\xi}(x_0, T - t_0) = \int_{t_0}^{T} \hat{\beta}(\tau, \hat{x}^*(\tau)) e^{-r(\tau - t_0)} d\tau.$$

We introduce the concept of the resulting solution in the game $\Gamma(x_0, T - t_0)$ with dynamic updating:

Definition 5 *The resulting solution $\hat{W}(\hat{x}^*(t), T - t)$ is the set of the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$, (15) for all possible resulting IDP $\hat{\beta}(t, \hat{x}^*)$ (14).*

In [23], it was proved that using the resulting imputation $\hat{\xi}(\hat{x}^*(t), T-t)$ and the corresponding resulting solution $\hat{W}(\hat{x}^*(t), T-t)$ you can allocate actual total payoffs among players:

Assertion 1. *Any resulting imputation $\hat{\xi}(x_0, T-t_0) \in \hat{W}(x_0, T-t_0)$ and the corresponding resulting IDP $\hat{\beta}(t, \hat{x}^*(t))$ allocates total player payoffs (5) along the conditionally cooperative trajectory $\hat{x}^*(t)$ in a game with the prescribed duration $\Gamma(x_0, T-t_0)$:*

$$\sum_{i=1}^{n} \int_{t_0}^{t} \hat{\beta}_i(\tau, \hat{x}^*(\tau)) e^{-r(\tau-t_0)} d\tau = \sum_{i=1}^{n} \int_{t_0}^{t} h_i(\hat{x}^*(\tau), \hat{u}^*(\tau)) e^{-r(\tau-t_0)} d\tau. \tag{16}$$

The resulting solution $\hat{W}(x_0, T-t_0)$ is time consistent by construction. In [21] it was proved that it also has the property of strong time consistency:

Definition 6 *The solution $W(x_0, T-t_0)$ is called strongly Δt-time consistent if for any $j = 0, \ldots, l$ and every $\xi(x_0, T-t_0) \in W(x_0, T-t_0)$ the corresponding IDP $\beta(t, x^*)$ satisfies the condition*

$$\int_{t_0}^{t_0+j\Delta t} \beta(\tau, x^*(\tau)) e^{-r(\tau-t_0)} d\tau \oplus W(x^*_{j,0}, T-t_0+j\Delta t) \subset W(x_0, T-t_0) \tag{17}$$

in which $a \oplus A = \{a + a' : a' \in A\}$.

Theorem 3. *An arbitrary resulting solution $\hat{W}(x_0, T-t_0)$ is strongly Δt-time consistent in the game $\Gamma(x_0, T-t_0)$ with the prescribed duration.*

Under an arbitrary resulting solution we understand any resulting solution which is constructed using solutions $W_j(x^*_{j,0}, t_0+j\Delta t, t_0+j\Delta t+\overline{T})$, $j = 0, \ldots, l$ as an arbitrary subset of the set of imputations $E_j(x^*_{j,0}, t_0+j\Delta t, t_0+j\Delta t+\overline{T})$. Solutions in each truncated subgame can differ, i.e. in the first truncated subgame players can choose a Core, in the second truncated subgame players can choose a Shapley value etc.

6 The Construction of the Characteristic Function in a Game with Dynamic Updating

As the characteristic function in a differential game $\Gamma(x_0, T-t_0)$ with dynamic updating, the resulting characteristic function is used.

Definition 7 *The resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T-t)$ in the game $\Gamma(\hat{x}^*(t), T-t)$ with dynamic updating the function calculated using the values of the characteristic functions $V_j(S; x^*_j(t), t, t_0+j\Delta t+\overline{T})$ in every truncated subgame $\hat{\Gamma}^j_v(x^*_j(t), t, t_0+j\Delta t+\overline{T})$ along conditionally cooperative trajectory $\hat{x}^*(t)$ for $j = 0, \ldots, l$, $\forall t \in [t_0+j\Delta t, t_0+$*

$j\Delta t + \overline{T}]$. *Let* $t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, *then:*

$$\overline{V}(S; \hat{x}^*(t), T - t) = \sum_{m=j+1}^{l} \Big[V_m(S; x^*_{m,0}, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \\ - V_m(S; x^*_{m,1}, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T})\Big] + \\ + \Big[V_j(S; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) - V_j(S; x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T})\Big], \quad (18)$$

where $x^*_{j,0} = \hat{x}^*(t_0 + j\Delta t)$ $x^*_{j,1} = \hat{x}^*(t_0 + (j+1)\Delta t)$.

In the following theorem it is shown that the resulting imputation $\hat{\xi}(x_0, T - t_0)$ can be used as an imputation in the game $\Gamma(x_0, T - t_0)$ with $\overline{V}(S; x_0, T - t_0)$ used as a characteristic function.

Theorem 4. *The resulting imputation* $\hat{\xi}(x_0, T - t_0)$ *is the imputation in the game* $\Gamma(x_0, T - t_0)$ *with dynamic updating, if for* $\forall t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, $j = 0, \ldots, l$ *the following condition is satisfied:*

$$\xi^j_i(x^*_j(t), t, t_0 + j\Delta t + \overline{T}) - V_j(\{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) \geq \\ \geq \xi^j_i(x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) - V_j(\{i\}; x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}). \quad (19)$$

Proof. First let us show that for $\forall t \in [t_0, T]$ the following conditions are fulfilled:

$$\sum_{i=1}^{n} \hat{\xi}_i(\hat{x}^*(t), T - t) = \overline{V}(N; \hat{x}^*(t), T - t), \quad (20)$$

$$\hat{\xi}_i(\hat{x}^*(t), t) \geq \overline{V}(\{i\}; \hat{x}^*(t), T - t). \quad (21)$$

According to the definition of $\hat{\xi}(\hat{x}^*(t), T - t)$ and $\overline{V}(S; \hat{x}^*(t), T - t)$ left part of (20) can be rewritten as:

$$\sum_{i=1}^{n} \hat{\xi}_i(\hat{x}^*(t), T-t) = \sum_{i=1}^{n} \int_{t}^{T} \hat{\beta}(\tau, \hat{x}^*(\tau))d\tau =$$

$$= \sum_{i=1}^{n} \left[\sum_{m=j+1}^{l} \left[\int_{m\Delta t}^{(m+1)\Delta t} \beta_m(\tau, x_m^*(\tau))d\tau \right] + \left[\int_{t}^{j\Delta t} \beta_j(\tau, x_m^*(\tau))d\tau \right] \right] =$$

$$= \sum_{i=1}^{n} \left[\sum_{m=j+1}^{l} \left[\xi_i^m(x_{m,0}^*, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \right.\right.$$
$$\left. - \xi_i^m(x_{m,1}^*, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \right] +$$
$$\left. + \left[\xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - \xi_i^j(x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) \right] \right]. \quad (22)$$

Since the condition (23) in (22) is satisfied then (20) is correct.

$$\sum_{i=1}^{n} \xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = V_j(N; x_j^*(t), t, t_0 + j\Delta t + \overline{T}), \ j = 0, \dots, l, \quad (23)$$

Now lets prove (21) by substituting the expression $\hat{\xi}_i(\hat{x}^*(t), T-t)$ and $\overline{V}(\{i\}; \hat{x}^*(t), T-t)$ for left part of (21). For right part of (21) we substitute $\overline{V}(\{i\}; x_0, T-t_0)$ (18):

$$\sum_{m=j+1}^{l} \left[\xi_i^m(x_{m,0}^*, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \right.$$
$$\left. - \xi_i^m(x_{m,1}^*, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \right] +$$
$$+ \left[\xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - \xi_i^j(x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) \right] \geq$$
$$\geq \sum_{m=j+1}^{l} \left[V_m(\{i\}; x_{m,0}^*, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \right.$$
$$\left. - V_m(\{i\}; x_{m,1}^*, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \right] +$$
$$+ \left[V_j(\{i\}; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - V_j(\{i\}; x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) \right] \quad (24)$$

(24) is fulfilled for $\forall t \in [t_0, T]$, if for $\forall m = 0, \dots, l$ is fulfilled

$$\xi_i^m(x^*_{m,0}, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \xi_i^m(x^*_{m,1}, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \geq$$
$$V_m(\{i\}; x^*_{m,0}, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - V_m(\{i\}; x^*_{m,1}, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \quad (25)$$

and for $\forall t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, $m = 0, \dots, l$ is fulfilled

$$\xi_i^j(x^*_j(t), t, t_0 + j\Delta t + \overline{T}) - \xi_i^j(x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) \geq$$
$$V_j(\{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) - V_j(\{i\}; x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}). \quad (26)$$

The fulfillment of condition (26) for $\forall t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, $m = 0, \dots, l$ implies the fulfillment of condition (25). Rewrite (26):

$$\xi_i^j(x^*_j(t), t, t_0 + j\Delta t + \overline{T}) - V_j(\{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) \geq$$
$$\geq \xi_i^j(x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) - V_j(\{i\}; x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}). \quad (27)$$

The condition (27) means that in every truncated subgame changing the values of the characteristic function and imputation according to the time happens evenly in reference to each other. The theorem is proved.

In this paragraph, the notion of the characteristic function $\overline{V}(S; x_0, T - t_0)$ in the game $\Gamma(x_0, T - t_0)$ with dynamic updating is introduced. It is shown that the resulting imputation $\hat{\xi}(x_0, T - t_0)$ is an imputation in the classical meaning fulfilled with individually rational conditions. Nevertheless condition (19) is not fulfilled $\forall t \in [t_0, T]$.

7 The Relationship of Solutions in Truncated Subgames and the Resulting Solutions

In this paragraph it is shown that if players choose the imputation $\xi_j(x^*_j(t), t, t_0 + j\Delta t + \overline{T}) \in E_j(x^*_j(t), t, t_0 + j\Delta t + \overline{T})$ based on $V_j(S; x^*_j(t), t, t_0 + j\Delta t + \overline{T})$, $j = 0, \dots, l$ in every truncated subgame by the same rule, then the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ corresponds to the imputation chosen by the same rule using the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$. Further prove of it for the number of optimality principals.

First of all show, that if in every truncated subgame $\hat{\Gamma}_j(x^*_j, t, t_0 + j\Delta t + \overline{T})$ players choose a Shapley value $Sh_j(x^*_j(t), t, t_0 + j\Delta t + \overline{T})$ as an imputation, then the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ (15) coincides with the Shapley value $\hat{Sh}(x^*_j(t), t, t_0 + j\Delta t + \overline{T})$, calculated using the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$ (18).

Theorem 5. *Suppose that in every truncated subgame* $\hat{\Gamma}_j(x_j^*, t, t_0 + j\Delta t + \overline{T})$*:*

$$\xi_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = Sh_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}),$$

where $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \dots, l$. *Then the resulting imputation* $\hat{\xi}(\hat{x}^*(t), T - t)$ *coincides with* $\hat{Sh}(\hat{x}^*(t), T - t)$*:*

$$\hat{\xi}(\hat{x}^*(t), T - t) = \hat{Sh}(\hat{x}^*(t), T - t), \ \forall t \in [t, T],$$

where $\hat{Sh}(\hat{x}^*(t), T - t)$ *is the Shapley value calculated using the resulting characteristic function* $\overline{V}(S; \hat{x}^*(t), T - t)$ *(18).*

Proof.

In this case the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ is calculated by following formulas (14), (15) using the Shapley values $Sh_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$ in every truncated subgame $\hat{\Gamma}_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$:

$$\begin{aligned}\hat{\xi}(\hat{x}^*(t), T - t) = \sum_{m=j+1}^{l} \Big[& Sh_m(x_{m,0}^*, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \\ & - Sh_m(x_{m,1}^*, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \Big] + \\ & + \left[Sh_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - Sh_j(x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) \right], \quad (28)\end{aligned}$$

where $Sh_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$, $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \dots, l$ is calculated by following formula

$$\begin{aligned}Sh_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = \sum_{\substack{S \subset N \\ i \in S}} & \frac{(|N| - |S|)!(|S| - 1)!}{|N|!} \cdot \\ & \cdot \Big(V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - V_j(S \backslash \{i\}; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \Big). \quad (29)\end{aligned}$$

$\hat{Sh}(\hat{x}^*(t), T - t)$ is calculated using the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$. Substitute the expression for $\overline{V}(S; \hat{x}^*(t), T - t)$ in the formula for $\hat{Sh}(\hat{x}^*(t), T - t)$. let $t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, then:

$$\hat{Sh}(\hat{x}^*(t), T-t) = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|N|-|S|)!(|S|-1)!}{|N|!} \cdot$$

$$\cdot \left(\sum_{m=j+1}^{l} \left[\left[V_m(S; x^*_{m,0}, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \right.\right.\right.$$

$$\left. - V_m(S; x^*_{m,1}, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \right] -$$

$$- \left[V_m(S \backslash \{i\}; x^*_{m,0}, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \right.$$

$$\left.\left. - V_m(S \backslash \{i\}; x^*_{m,1}, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T}) \right] \right] +$$

$$+ \left[\left[V_j(S; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) - \right.\right.$$

$$\left. - V_j(S; x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) \right] - \left[V_j(S \backslash \{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) - \right.$$

$$\left.\left.\left. - V_j(S \backslash \{i\}; x^*_{j,1}, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) \right] \right] \right). \quad (30)$$

After substitution of (29) in (28) we obtain (30). Theorem is proved.

The same result can be obtained for the proportional solution. Suppose that the characteristic function $V(\{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T})$ is differentiable along the cooperative trajectory $x^*_j(t)$. Define the proportional solution using its IDP $\beta^{Prop}_j(t)$ in the following way:

$$\beta^{Prop,j}_i(t) = \frac{-\frac{d}{dt} V_j(\{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T})}{\sum\limits_{i \in N} -\frac{d}{dt} V_j(\{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T})} \cdot$$

$$\left(-\frac{d}{dt} V_j(N; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) \right), \; i \in N. \quad (31)$$

The corresponding imputation, obtained by the direct integration of $\beta^{Prop}_j(t)$ using the formula (12) and is denoted by $Prop_j(x^*_j(t), t, t_0 + j\Delta t + \overline{T})$.

Prove that in every truncated subgame $\hat{\Gamma}_j(x^*_j(t), t, t_0 + j\Delta t + \overline{T})$ players choose the proportional solution $Prop_j(x^*_j(t), t, t_0 + j\Delta t + \overline{T})$ (31), then the resulting imputation, defined by the formula $\hat{\xi}(\hat{x}^*(t), T-t)$ (15) coincides with the proportional solution $\hat{Prop}(\hat{x}^*(t), T-t)$ (31), calculated using the characteristic function $\overline{V}(S; \hat{x}^*(t), T-t)$ (18).

Theorem 6. *Suppose that in every truncated subgame* $\hat{\Gamma}_j(x_j^*, t, t_0 + j\Delta t + \overline{T})$

$$\xi_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = Prop_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}),$$

where $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \ldots, l$. *Then the resulting vector* $\hat{\xi}(\hat{x}^*(t), T - t)$ *will coincide with* $\hat{Prop}(\hat{x}^*(t), T - t)$ *(31):*

$$\hat{\xi}(\hat{x}^*(t), T - t) = \hat{Prop}(\hat{x}^*(t), T - t), \ \forall t \in [t, T],$$

where $\hat{Prop}(\hat{x}^*(t), T - t)$ *is the proportional solution calculated using the resulting characteristic function* $\overline{V}(S; \hat{x}^*(t), T - t)$ *(18).*

Proof. In this case the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ is calculated using the formula (14) for the IDP using the combination of the values of the IDP for the proportional solution (31) in every truncated subgame on the interval $[t_0 + j\Delta t, t_0 + (j+1)\Delta t]$. $i \in N$.

Show that the formula for $\hat{Prop}(\hat{x}^*(t), T - t)$ or for the IDP $\hat{\beta}^{Prop}(t)$ (31), where we use $\overline{V}(S; \hat{x}^*(t), T - t)$ (18) as a characteristic function leads to the right part of (31). Substitute the expression for $\overline{V}(S; \hat{x}^*(t), T - t)$ (18) in (31). Consider one of the addends. Let $t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$:

$$-\frac{d}{dt}\left(\overline{V}(\{i\}; \hat{x}^*(t), T - t)\right) = -\frac{d}{dt}\Bigg(\sum_{k=j+1}^{l}\Big[V_k(\{i\}; x_{k,0}^*, t_0 + k\Delta t, t_0 + k\Delta t + \overline{T}) -$$
$$- V_k(\{i\}; x_{k,0}^*, t_0 + (k+1)\Delta t, t_0 + k\Delta t + \overline{T})\Big] +$$
$$+ \Big[V_j(\{i\}; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - V_j(\{i\}; x_{j,0}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T})\Big]\Bigg). \quad (32)$$

From (32) we can see, that for $t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, $j = 0, \ldots, l$ under the derivative sign there is only one addend, depending on t, therefore

$$-\frac{d}{dt}\Big(\overline{V}(\{i\}; \hat{x}^*(t), T - t)\Big) = -\frac{d}{dt}\Big(V_j(\{i\}; x_j^*(t), t, t_0 + j\Delta t + \overline{T})\Big). \quad (33)$$

Substitute (33) and the formula for $\overline{V}(N; \hat{x}^*(t), T - t)$ (31). It is easy to see, that in this case the right hand side of IDP $\hat{\beta}^{Prop}(t)$ and (31) are equal. Theorem is proved.

Show, that if in every truncated subgame $\hat{\Gamma}_j(x_j^*, t, t_0 + j\Delta t + \overline{T})$ players choose the Core $C_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$ as the optimality principle, then the resulting solution, every element of which is $\hat{\xi}(\hat{x}^*(t), T - t)$ (15) is a Core, calculated using the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$ (18).

Theorem 7. *Suppose that in every truncated subgame* $\hat{\Gamma}_j(x_j^*, t, t_0 + j\Delta t + \overline{T})$*:*

$$W_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = C_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}),$$

where $\forall t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \dots, l$, *then for every* $\xi_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \in C_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$, *for which following condition is satisfied*

$$\sum_{i \in S} \xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \geq$$
$$\geq \sum_{i \in S} \xi_i^j(x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) - V_j(S; x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}), \tag{34}$$

the following is satisfied

$$\hat{\xi}(\hat{x}^*(t), T - t) \in \hat{C}(\hat{x}^*(t), T - t), \;\forall t \in [t, T],$$

where $\hat{C}(\hat{x}^*(t), T - t)$ *is the Core, calculated using the resulting characteristic function* $\overline{V}(S; \hat{x}^*(t), T - t)$ *(18).*

Proof. The following statements should be proven:

1. If players in every truncated subgame choose the imputation $\xi_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \in C_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$ calculated using $V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T})$, $j = 0, \dots, l$, then the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ belongs to the Core $\hat{C}(\hat{x}^*(t), T - t)$, calculated using the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$.

2. The Core $\hat{C}(\hat{x}^*(t), T - t)$ shouldn't contain an imputation $\hat{\xi}(\hat{x}^*(t), T - t)$, for which it is impossible to find a set of imputations in truncated subgames $\xi_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \in C_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$.

Prove the first statement, that if the set of imputations $\xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$, satisfies the system of inequalities:

$$\sum_{i \in S} \xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \geq V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}), \; S \subset N,$$

for each $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \dots, l$, $i = 1, \dots, n$, then the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ satisfies the system of inequalities:

$$\sum_{i \in S} \hat{\xi}_i(\hat{x}^*(t), T - t) \geq \overline{V}(S; \hat{x}^*(t), T - t), \;\forall t \in [t_0, T], \; S \subset N. \tag{35}$$

Substitute the expression $\hat{\xi}(\hat{x}^*(t), T - t)$ for the left hand side of (35). For the right hand side of (35) substitute (18). As in theorem 6. show that for every $S \subset N$, $t \in$

$[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$ the fulfilment of (35) leads to the fulfilment of (34). The first statement is proved.

Further prove the second statement, that in the set of imputations $\xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$, $i = 1, \ldots, n$, $j = 0, \ldots, l$, satisfying the system of inequalities $\forall j = 0, \ldots, l$, $S \subset N$:

$$\sum_{m=j+1}^{l} \Big[\xi_i^m(x_m^*(t), t, t_0 + m\Delta t + \overline{T}) - \\ - \xi_i^m(x_{m,1}^*, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T})\Big] + \\ + \Big[\xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - \xi_i^j(x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T})\Big] \geq \\ \geq \sum_{m=j+1}^{l} \Big[V_m(S; x_{m,0}^*, t_0 + m\Delta t, t_0 + m\Delta t + \overline{T}) - \\ - V_m(S; x_{m,1}^*, t_0 + (m+1)\Delta t, t_0 + m\Delta t + \overline{T})\Big] + \\ + \Big[V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - V_j(S; x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T})\Big], \quad (36)$$

there exists at least one set of $\xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$, $i = 1, \ldots, n$, $j = 0, \ldots, l$ satisfying:

$$\sum_{i \in S} \xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \geq \\ \geq V_j(S; x_j^*, t, t_0 + j\Delta t + \overline{T}), \ \forall j = 0, \ldots, l, \ S \subset N. \quad (37)$$

Proof from the opposite. Suppose, that for the imputations satisfying (34) and (37) the inequality is not satisfied (36). Show that for $\forall j = 0, \ldots, l$ the following condition is satisfied:

$$\sum_{i \in S} \xi_i^j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \geq \\ \geq \sum_{i \in S} \xi_i^j(x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}) - V_j(S; x_{j,1}^*, t_0 + (j+1)\Delta t, t_0 + j\Delta t + \overline{T}), \quad (38)$$

then from (37) the sign of the right and left hand sides are always positive, and using (34) it follows that (38) is satisfied. The theorem is proved.

The same results can be obtained for an IDP-core, a new cooperative solution presented in [24]. Introduce the following notation:

$$U_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = -\frac{d}{dt} V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}), \quad (39)$$

where $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$ and $S \subseteq N$.

Define $B_j(t)$ as a set of integrable vector functions $\beta_j(t)$ satisfying the following system of inequalities:

$$B_j(t, x_j^*) = \Big\{\beta_j(t) = (\beta_1^j(t), \ldots, \beta_n^j(t)) : \sum_{i \in S} \beta_i^j(t) \geq U_j(S, x_j^*(t), t, t_0 + j\Delta t + \overline{T}), \sum_{i \in N} \beta_i^j = U_j(N, x_j^*(t), t, t_0 + j\Delta t + \overline{T}), \forall S \subset N\Big\}. \quad (40)$$

Suppose that $B_j(t, x_j^*) \neq \emptyset$, $\forall t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \ldots, l$. Then using the set $B_j(t, x_j^*)$ it is possible to define the following set of vectors:

Definition 8 *The set of all $\xi_j(x_j^*(t), t)$ for some integrable selectors $\beta_j(t, x_j^*) \in B_j(t, x_j^*)$ we shall call an IDP-core and denote it as $\overline{C}_j(x_j^*(t), t_0 + j\Delta t + \overline{T})$, where*

$$\overline{C}_j(x_j^*(t), t) = \big\{\xi_j(x_j^*(t), t),\ t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]\big\} \quad (41)$$

and for $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$

$$\xi_j(x_j^*(t), t) = \int_t^T \beta_j(\tau, x_j^*) d\tau. \quad (42)$$

In [24] it was also proved that an IDP-core is strongly time consistent.

Show, that if in every truncated subgame $\hat{\Gamma}_j(x_j^*, t, t_0 + j\Delta t + \overline{T})$ players choose an IDP-core $\overline{C}_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$ (41) as an optimality principle, then the resulting solution, every element of which is defined by formula (15), is an IDP-core, calculated using the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$ (18).

Theorem 8. *Suppose in every truncated subgame $\hat{\Gamma}_j(x_j^*, t, t_0 + j\Delta t + \overline{T})$*

$$W_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = \overline{C}_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \neq \emptyset,$$

where $\forall t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \ldots, l$, then

$$\hat{W}(\hat{x}^*(t), T - t) = \hat{\overline{C}}(\hat{x}^*(t), T - t),\ \forall t \in [t, T],$$

where $\hat{\overline{C}}(\hat{x}^*(t), T - t)$ is the IDP-core, calculated using the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$ (18).

Proof. The resulting solution $\hat{W}(\hat{x}^*(t), T - t)$ consists of $\hat{\xi}(\hat{x}^*(t), T - t)$, each of which is defined by the IDP set of imputations

$\xi_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T}) \in \overline{C}_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$, $j = 0, \dots, l$ by formula (14). According to the definition of an IDP-core, each imputation from the IDP-core satisfies the following system of inequalities (40):

$$\sum_{i \in S} \beta_i^j(t, \hat{x}^*(t)) \geq -\frac{d}{dt}\Big(V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}\Big),$$
$$\sum_{i \in N} \beta_i^j(t, \hat{x}^*(t)) = -\frac{d}{dt}\Big(V_j(N; x_j^*(t), t, t_0 + j\Delta t + \overline{T})\Big), \forall S \subset N.$$

Thus, the resulting solution $\hat{W}(\hat{x}^*(t), T - t)$ is defined by (40) for $t \in [t_0 + j\Delta t, t_0 + (j + 1)\Delta t]$, $j = 0, \dots, l$.

Write out the expression for $\hat{\overline{C}}(\hat{x}^*(t), T - t)$ with the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$ (18). Show, that it leads to (40). Consider separately one of the constraints in it and substitute the expression for $\overline{V}(S; \hat{x}^*(t), T - t)$ (18), let $t \in [t_0 + j\Delta t, t_0 + (j + 1)\Delta t]$:

$$-\frac{d}{dt}\Big(\overline{V}(S; \hat{x}^*(t), T - t)\Big) = -\frac{d}{dt}\Big(\sum_{k=j+1}^{l} \Big[V_k(S; x_{k,0}^*, t_0 + k\Delta t, t_0 + k\Delta t + \overline{T}) -$$
$$- V_k(S; x_{k,0}^*, t_0 + (k + 1)\Delta t, t_0 + k\Delta t + \overline{T})\Big] +$$
$$+ \Big[V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) - V_j(S; x_{j,0}^*, t_0 + (j + 1)\Delta t, t_0 + j\Delta t + \overline{T})\Big]\Big). \quad (43)$$

From (43) it follows that for $t \in [t_0 + j\Delta t, t_0 + (j + 1)\Delta t]$, $j = 0, \dots, l$ under the desired sign there is only one addend depending on t, so

$$-\frac{d}{dt}\Big(\overline{V}(S; \hat{x}^*(t), T - t)\Big) = -\frac{d}{dt}\Big(V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T})\Big). \quad (44)$$

Substitute (44) in the formula for $\hat{\overline{C}}(\hat{x}^*(t), T - t)$:

$$\sum_{i \in S} \hat{\beta}_i(t, \hat{x}^*(t)) \geq -\frac{d}{dt}\Big(V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T})\Big),$$
$$\sum_{i \in N} \hat{\beta}_i(t, \hat{x}^*(t)) = -\frac{d}{dt}\Big(V_j(N; \hat{x}^*(t), t, t_0 + j\Delta t + \overline{T})\Big), \forall S \subset N.$$

Thus, the IDP-core $\hat{\overline{C}}(\hat{x}^*(t), T - t)$ with the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$, coincides with the resulting solution $\hat{W}(\hat{x}^*(t), T - t)$, calculated using the combination of solutions $\overline{C}_j(x_j^*(t), t, t_0 + j\Delta t + \overline{T})$ in the truncated subgames. The theorem is proved.

In this paragraph it is shown, that if players in every truncated subgame $\hat{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ choose a proportional solution, a Shapley value, an imputation from Core or an imputation from an IDP-core as an optimality principle, then the resulting imputation is also a proportional solution, a Shapley value, an imputation from Core or an imputation from an IDP-core in the game $\Gamma(x_0, T - t_0)$ with dynamic updating.

The theorems proved in this section give the approach for directly calculating the resulting solution.

8 The Cooperative Limited Resource Extraction Game with Dynamic Updating

Consider the resource extraction game defined on a closed time interval. The solution of the two person game in the classical form is presented in [29]. The problem of time consistency was studied by Yeung [6]. In this example, a game of limited resource extraction with dynamic updating for three persons is presented. A Core is used as an optimality principle. The characteristic function is calculated as in [27]. In the last part of the example the property of strong time consistency is discussed.

8.1 Initial Game

The following dynamical system describes the change in the stock of resources $x(t) \in X \subset R$:

$$\dot{x} = a\sqrt{x(t)} - bx(t) - \sum_{i=1}^{3} u_i, \ x(t_0) = x_0,$$

where u_i is the player's production level $i = \overline{1,3}$. The payoff of player i:

$$K_i(x_0, t_0; u) = \int_{t_0}^{T} h_i(x(\tau), u(\tau))d\tau,$$

here

$$h_i(x(\tau), u(\tau)) = \sqrt{u_i(\tau)} - \frac{c_i}{\sqrt{x(\tau)}} u_i(\tau), \ i = \overline{1,3},$$

where c_i is a constant, $c_i \neq c_k$, $\forall i \neq k = \overline{1,3}$.

8.2 Truncated Subgame

The initial game $\Gamma(x_0, T - t_0)$ is defined on the time interval $[t_0, T]$. Suppose that for any $t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t]$, $j = 0, \ldots, l$, players have truncated information about the game. This includes information about the dynamical system and the payoff function on the time interval $[t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$. This model is constructed using the truncated subgame $\hat{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$. The dynamical system and the initial data have the following form:

$$\dot{x} = a\sqrt{x(t)} - bx(t) - \sum_{i=1}^{3} u_i, \;\; x(t_0 + j\Delta t) = x_{j,0}. \tag{45}$$

The payoff function of player i is:

$$K_i^j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}; u) = \int\limits_{t_0+j\Delta t}^{t_0+j\Delta t+\overline{T}} h_i(x(\tau), u(\tau))d\tau.$$

Consider the case when players agree to cooperate in the truncated subgame $\hat{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$. Then players act to maximize the total payoffs.

8.3 Cooperative Trajectory

Suppose that the maximum joint payoff of players in each truncated subgame $\hat{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ has the following form [29]:

$$W^j(t, x) = A^j(t)\sqrt{x} + C^j(t), \tag{46}$$

where functions $A^j(t)$, $C^j(t)$ satisfy the system of differential equations:

$$\begin{aligned} \dot{A}^j(t) &= \frac{b}{2}A^j(t) - \sum_{i=1}^{3}\left[\frac{1}{4\left[c_i + \frac{A^j(t)}{2}\right]}\right], \\ \dot{C}^j(t) &= -\frac{a}{2}A^j(t), \\ A^j(t_0 + j\Delta t + \overline{T}) &= 0, \;\; C^j(t_0 + j\Delta t + \overline{T}) = 0. \end{aligned}$$

The cooperative trajectory $x_j^*(t)$ in each truncated subgame can be calculated as follows [29]:

$$x_j^*(t) = \varpi_j^2(t_0 + j\Delta t, t)\left[\sqrt{x_{j,0}^*} + \frac{1}{2}a \cdot \int\limits_{t_0+j\Delta t}^{t} \varpi_j(t_0 + j\Delta t, \tau)^{-1}d\tau\right]^2,$$

where $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$ and

$$\varpi_j(t_0 + j\Delta t, t) = \exp\int\limits_{t_0+j\Delta t}^{t} -\left[\frac{1}{2}b + \sum_{i=1}^{3}\left[\frac{1}{4\left[c_i + \frac{A^j(\tau)}{2}\right]^2}\right]\right]d\tau.$$

The initial position for the cooperative trajectory in each truncated subgame is defined from the previous truncated subgame: $x^*_{0,0} = x_0$ and $x^*_{j,0} = x^*_{j-1}(t_0 + j\Delta t)$ for $1 \le j \le l$. The conditionally cooperative trajectory $\hat{x}^*(t)$ is defined as follows:

$$\hat{x}^*_j(t) = x^*_j(t),\ t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t],\ j = 0, \dots, l.$$

8.4 Characteristic Function

In order to allocate the cooperative payoff between players in each truncated subgame, it is necessary to determine the values of the characteristic function $V_j(S; x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ for each coalition $S \subset N$. In accordance with the formula (10), the maximum total payoff for players $W_j(t_0+j\Delta t, x_{j,0})$ (46) corresponds to the value of the characteristic function $V_j(N; x_{j,0}, t_0+j\Delta t, t_0+j\Delta t+\overline{T}))$ of the coalition $S = N$ in the truncated subgame $\hat{\Gamma}^j_v(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$:

$$V_j(N; x^*_j(t), t, t_0 + j\Delta t + \overline{T}) = W_j(t, x^*_j(t)), \tag{47}$$

where $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \dots, l$. Next, we need to determine the values of the characteristic function for the following coalitions:

$$\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}.$$

For each coalition $\{i\}$, $i = \overline{1,3}$, we need to determine the Nash equilibrium in the truncated subgame $\hat{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ and as a result $V_j(\{i\}; x^*_j(t), t, t_0 + j\Delta t + \overline{T})$.

8.5 One Player Coalitions

The Nash equilibrium in the truncated subgame $\hat{\Gamma}_j(x_{j,0}, t_0+j\Delta t, t_0+j\Delta t+\overline{T})$ is determined by the following strategies of players:

$$u^j_i(t, x) = \frac{x}{4[c_i + A^j_i(t)/2]^2},\ i = \overline{1,3},$$

where functions $A^j_i(t)$ for $i = \overline{1,3}$ are defined by the system of differential equations:

$$\begin{aligned}
\dot{A}^j_i(t) &= A^j_i(t)\left[\frac{b}{2} + \sum_{k \ne i} \frac{1}{8(c_k + A^j_k(t)/2)^2}\right] - \frac{1}{4(c_i + A^j_i(t)/2)}, \\
\dot{C}^j_i(t) &= -\frac{a}{2} A^j_i(t), \\
A^j_i(t_0 + j\Delta t + \overline{T}) &= 0,\ C^j_i(t_0 + j\Delta t + \overline{T}) = 0.
\end{aligned}$$

The corresponding payoff for players $i = \overline{1,3}$ in the Nash equilibrium is determined by the function:

$$V^j_i(t, x) = A^j_i(t)\sqrt{x} + C^j_i(t),\ i = \overline{1,3}.$$

The value of the characteristic function for coalitions consisting of one player $S = \{i\}$, $i \in N$ is calculated as follows:

$$V_j(\{i\}; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = V_i^j(t, x_j^*(t)), \tag{48}$$

where $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \ldots, l$.

8.6 Two Player Coalitions

In accordance with the formula (10), the characteristic function $V_j(S; x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ for coalitions consisting of two players $S = \{1,2\}, \{1,3\}, \{2,3\}$ is defined as the best reply of coalition S against the strategies from the Nash equilibrium $u_j^{NE} = (u_1^{NE,j}, u_2^{NE,j}, u_3^{NE,j})$ in the truncated subgame $\hat{\Gamma}_j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$, used by players from N/S. In our case, this means that players from coalition S act as one player and maximize their total payoff. Using this approach, we determine the equilibrium between the two players: the combined player (coalition S), and a player not included in the coalition S (coalition N/S).

Consider the formula for $V_j(S; x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ in the case of $S = \{1, 2\}$. The formula for the remaining coalitions can be obtained by the same principle:

$$\begin{aligned} V_{\{1,2\}}^j(t, x) &= A_{\{1,2\}}^j(t)\sqrt{x} + C_{\{1,2\}}^j(t), \\ V_3^j(t, x) &= A_3^j(t)\sqrt{x} + C_3^j(t), \end{aligned}$$

where functions $A_{\{1,2\}}^j(t)$, $A_3^j(t)$, $C_{\{1,2\}}^j(t)$, $C_3^j(t)$ satisfy the system of differential equations:

$$\begin{aligned} \dot{A}_{\{1,2\}}^j(t) &= A_{\{1,2\}}^j(t)\left[\frac{b}{2} + \frac{1}{8(c_3 + A_3^j(t)/2)^2}\right] - \sum_{k \in S} \frac{1}{4(c_k + A_{\{1,2\}}^j(t)/2)}, \\ \dot{A}_3^j(t) &= A_3^j(t)\left[\frac{b}{2} + \sum_{k \in S} \frac{1}{8(c_k + A_{\{1,2\}}^j(t)/2)^2}\right] - \frac{1}{4(c_3 + A_3^j(t)/2)}, \\ \dot{C}_{\{1,2\}}^j(t) &= -\frac{a}{2}A_{\{1,2\}}^j(t),\ \dot{C}_3^j(t) = -\frac{a}{2}A_3^j(t) \end{aligned}$$

with the initial condition $A_{\{1,2\}}^j(t_0 + j\Delta t + \overline{T}) = 0$, $A_3^j(t_0 + j\Delta t + \overline{T}) = 0$, $C_{\{1,2\}}^j(t_0 + j\Delta t + \overline{T}) = 0$, $C_3^j(t_0 + j\Delta t + \overline{T}) = 0$.

The value of the characteristic function of coalition $S = \{1, 2\}$ is calculated as follows:

$$V_j(\{1,2\}; x_j^*(t), t, t_0 + j\Delta t + \overline{T}) = V_{\{1,2\}}^j(t, x_j^*(t)), \tag{49}$$

where $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \ldots, l$.

8.7 The Concept of the Solution

Suppose that in each cooperative truncated subgame $\hat{\Gamma}_v^j(x_{j,0}, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ the players use a Core as the principle of optimality. This means that players in each truncated subgame choose the imputation $\xi_j(x_j^*, t, t_0 + j\Delta t + \overline{T}) \in C_j(x_j^*(t), T - t)$ according to the following rule:

$$\sum_{i \in S} \xi_i^j(x_j^*, t, t_0 + j\Delta t + \overline{T}) \geq V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}), \ S \subset N,$$

for any $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \ldots, l$. The resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ for any set of distributions in truncated subgames $\xi_j(x_j^*, t, t_0 + j\Delta t + \overline{T}) \in C_j(x_j^*(t), T - t)$, $t \in [t_0 + j\Delta t, t_0 + j\Delta t + \overline{T}]$, $j = 0, \ldots, l$ can be calculated by the formula (15). We denote by $\hat{C}(\hat{x}^*(t), T - t)$ the set of imputations $\hat{\xi}(\hat{x}^*(t), T - t)$ constructed using (14) and (15).

Using the results obtained in Sections 4 and 5, the solution $\hat{C}(\hat{x}^*(t), T - t)$ can be constructed according to the following rule:

$$\sum_{i \in S} \hat{\xi}(\hat{x}^*(t), T - t) \geq \overline{V}(S; \hat{x}^*(t), T - t), \ S \subset N, \tag{50}$$

where $\overline{V}(S; \hat{x}^*(t), T - t)$ is calculated using the formula (18).

Further, using an example of a particular deviation from $\hat{C}(\hat{x}^*(t), T - t)$, we show that the solution constructed is strongly Δt-time consistent in the game $\Gamma(x_0, T - t_0)$.

8.8 Numerical Simulation

Consider a numerical example of the resource extraction game defined on the time interval $T - t_0 = 4$, in which information about the game is known on the time interval with the duration $\overline{T} = 2$ and is updated every $\Delta t = 1$ time interval. The following parameters for the equation of motion $a = 5$, $b = 0.3$ for the payoff function $c_1 = 0.15$, $c_2 = 0.65$, $c_3 = 0.45$ and for the initial conditions $t_0 = 0$, $x_0 = 250$ are fixed.

Figure 5. shows the optimal strategies for the first player in the game with dynamic updating (solid line) and optimal strategies in the original game [29] (dotted line) with a prescribed duration.

The conditionally cooperative trajectory $\hat{x}^*(t)$ is constructed using the cooperative trajectories in truncated subgames $\hat{\Gamma}_j(x_{j,0}^*, t_0 + j\Delta t, t_0 + j\Delta t + \overline{T})$ with the dynamical system (45). In Figure 6. the comparison of the conditionally cooperative trajectory $\hat{x}^*(t)$ (solid line) in the game with dynamic updating and the cooperative trajectory $x^*(t)$ (dashed line) in the original game $\Gamma(x_0, T - t_0)$ [29] is displayed. For limited information, resource development occurs faster, because players are guided by a reduced time interval. The abscissa axis in Figure 6. determines the time t, the ordinate axis determines stock of resource x.

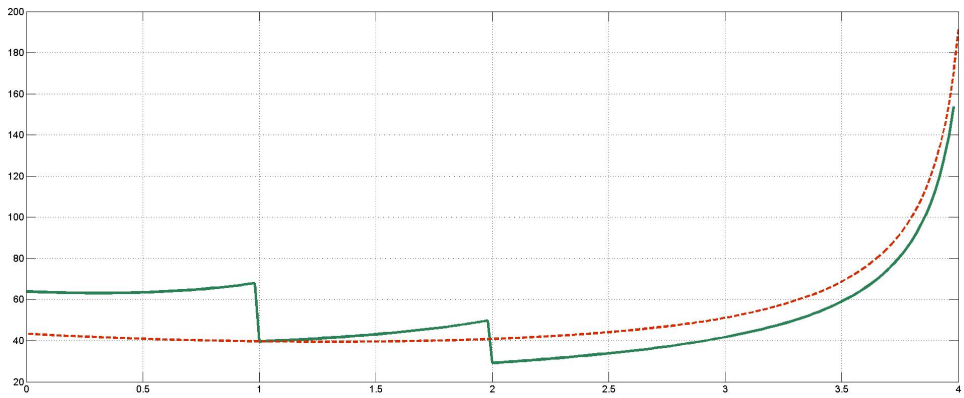

Figure 5: Optimal strategy for player 1 in the game with dynamic updating (solid line) and optimal strategies in the original game [29] (dotted line) with prescribed duration.

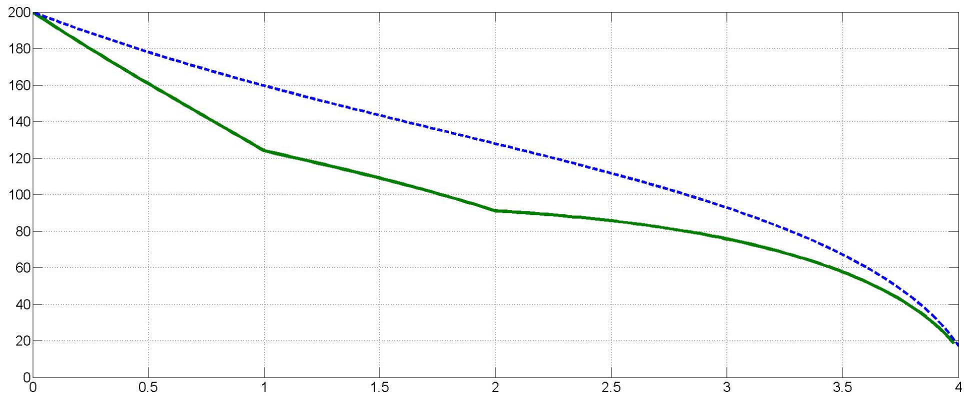

Figure 6: Conditionally cooperative trajectory $\hat{x}^*(t)$ (solid line) in a game with dynamic updating and cooperative trajectory $x^*(t)$ (dashed line) in the initial game [29].

Based on the values of the characteristic functions

$$V_j(S; x_j^*(t), t, t_0 + j\Delta t + \overline{T}),\ t \in [t_0 + j\Delta t, t_0 + (j+1)\Delta t],\ S \subset N,\ i = 0, \dots, l,$$

calculated using (47), (48), (49), the expression for the resulting characteristic function $\overline{V}(S; \hat{x}^*(t), T - t)$ (18), $t \in [t_0, T]$ is obtained. Using (50) $\hat{C}(\hat{x}^*(t), T - t)$ is constructed in the game with dynamic updating $\Gamma(x_0, T - t_0)$ (see Fig.9.).

Demonstrate the property of the strong Δt-time consistency of the solution $\hat{C}(\hat{x}^*(t), T - t)$. Suppose that at the beginning of the game $\Gamma(x_0, T - t_0)$ the players agree to use the proportional solution $\hat{Prop}(\hat{x}^*(t), T - t)$ (31) (further it is shown for the given parameters $\hat{Prop}(\hat{x}^*(t), T - t) \in \hat{C}(\hat{x}^*(t), T - t)$). Now suppose that at some point in time $t_{br} = t_0 + m\Delta t \in [t_0, T]$ players decided to chose another imputation from

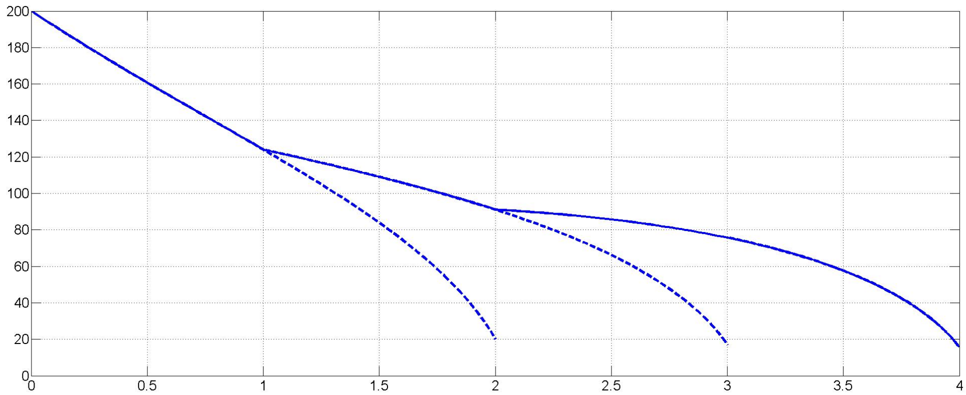

Figure 7: Conditionally cooperative trajectory $\hat{x}^*(t)$ (solid line) in a game with dynamic updating and the corresponding cooperative trajectories in truncated subgames (dashed lines).

$\hat{C}(\hat{x}^*(t_{br}), T - t_{br})$ instead of the proportional solution, for example, a Shapley value $\hat{Sh}(\hat{x}^*(t), T - t)$, $t \in [t_{br}, T]$ (29). The IDP for the proportional solution and the Shapley value are calculated using the formula (13).

Suppose that $m = 2$, then the resulting IDP (14) for the combined solution has the following form:

$$\hat{\beta}(t, \hat{x}^*) = \begin{cases} \hat{\beta}^{Prop}(t, \hat{x}^*), & t \in [t_0, t_{br}], \\ \hat{\beta}^{Sh}(t, \hat{x}^*), & t \in (t_{br}, T]. \end{cases} \tag{51}$$

In Figure 8. the resulting IDP for the proportional solution $\hat{\beta}^{Prop}(t, \hat{x}^*)$ (solid line) and $\hat{\beta}(t, \hat{x}^*)$ for the combined solution (51) (dashed line) are displayed.

In order to obtain the imputation (15) corresponding to the combined solution $\hat{\beta}(t, \hat{x}^*)$ (51) it is integrated by t. Denote the result of the integration by $\hat{\xi}(\hat{x}^*(t), T - t)$. In accordance with the resulting imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ players allocate a joint payoff in the game $\Gamma(x_0, T - t_0)$ with dynamic updating in the following way:

$$\hat{\xi}(\hat{x}^*(t), T - t) = (12.3, 30.2, 16.8).$$

In Figure 9. one can observe that the imputation corresponding to the combined solution $\hat{\xi}(\hat{x}^*(t), T - t)$ (dashed line) belongs to $\hat{C}(\hat{x}^*(t), T - t)$ (the selected region) for all $t \in [t_0, T]$. This shows the property of strong Δt-time consistency of $\hat{C}(\hat{x}^*(t), T - t)$, since the imputation $\hat{\xi}(\hat{x}^*(t), T - t)$ was constructed by the deviation of players from the proportional solution $\hat{Prop}(\hat{x}^*(t), T - t)$ (solid line) at the instant $t_{br} = t_0 + j\Delta t$ in favor of the Shapley value $\hat{Sh}(\hat{x}^*(t_{br}), T - t_{br})$.

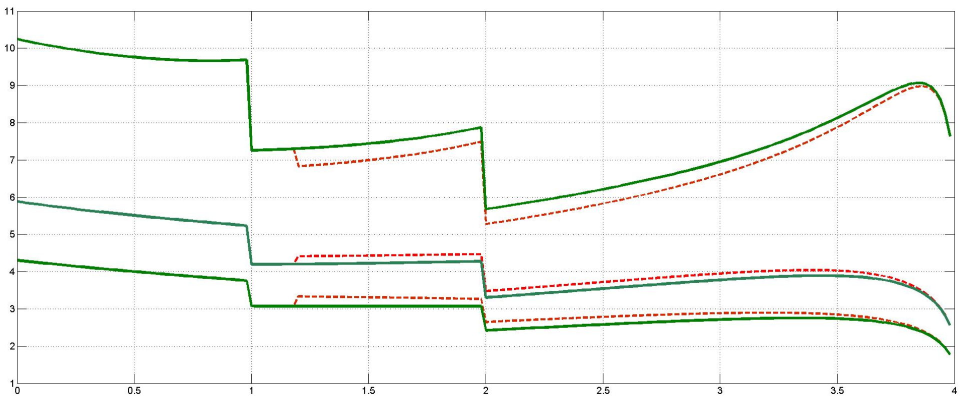

Figure 8: IDP $\hat{\beta}^{Prop}(t, \hat{x}^*)$ for the proportional solution (solid line), IPD $\hat{\beta}(t, \hat{x}^*)$ for the combined solution (51) (dashed line).

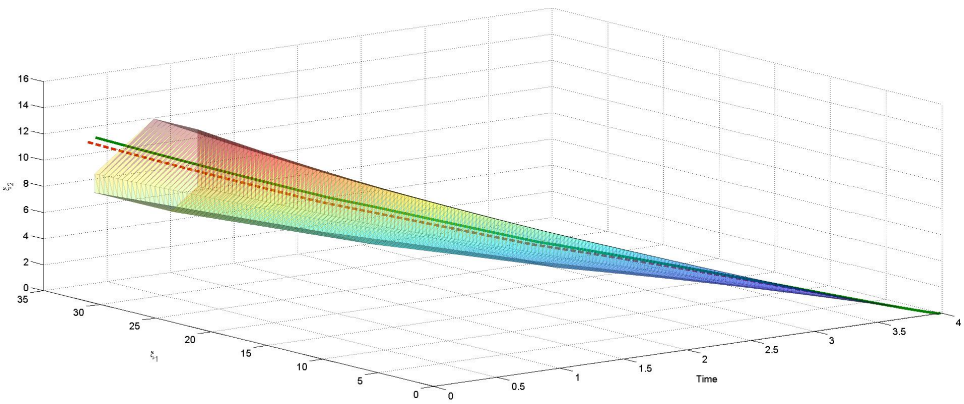

Figure 9: Axis: ξ_1, ξ_3, t. ξ_2 can be calculated using normalization condition.

References

[1] A. Haurie, A note on nonzero-sum differential games with bargaining solutions, Journal of Optimization Theory and Applications, 18, 31–39 (1976).

[2] A. I. Subbotin, Generalization of the main equation of differential game theory, Journal of Optimization Theory and Applications, 43, 103–133 (1984).

[3] A. I. Subbotin, Generalized Solutions of First Order PDEs, Basel, Birkhauser (1995).

[4] C. J. Cassandras, S. Lafortune, Introduction to discrete event systems, New York, Springer (2008).

[5] D. B. Gillies, Solutions to general non-zero-sum games, Contributions to the Theory of Games, 4, 47–85 (1959).

[6] D. W. K. Yeung, L. A. Petrosyan, Subgame-consistent Economic Optimization, New York, Springer (2012).

[7] D. Yeung, O. Petrosyan, Cooperative Stochastic Differential Games with Information Adaptation, International Conference on Communication and Electronic Information Engineering (CEIE 2016) (2017) (DOI: 10.2991/ceie-16.2017.47).

[8] D. Yeung, O. Petrosian, Infinite Horizon Dynamic Games: A New Approach via Information Updating, International Game Theory Review, 19, 1–23 (2017).

[9] E. V. Gromova, O. L. Petrosian, Control of information horizon for cooperative differential game of pollution control, 2016 International Conference Stability and Oscillations of Nonlinear Control Systems (Pyatnitskiy's Conference), (2016) (DOI:10.1109/STAB.2016.7541187).

[10] G. C. Goodwin, M. M. Seron, J. A. Dona, Constrained Control and Estimation: An Optimisation Approach, Springer, New York (2005).

[11] J. B. Rawlings, D. Q. Mayne, Model Predictive Control: Theory and Design, Madison, Nob Hill Publishing (2009).

[12] L. A. Petrosjan, Strongly time-consistent differential optimality principles, Vestnik St. Petersburg Univ. Math., 26, 40–46 (1993).

[13] L. A. Petrosjan, E. V. Shevkoplyas, Cooperative solutions for games with random duration, Game Theory Appl., 9, 125-139 (2003).

[14] L. A. Petrosyan, Time-consistency of solutions in multi-player differential games, Vestnik Leningrad State University, 4, 46–52 (1977).

[15] L. A. Petrosyan, D. W. K. Yeung, Dynamically stable solutions in randomly-furcating differential games, Trans. Steklov Inst. Math, 253, 208–220 (2006).

[16] L. A. Petrosyan, N. N. Danilov, Stability of solutions in non-zero sum differential games with transferable payoffs, Vestnik Leningrad State University, 1, 52–59 (1979).

[17] L. S. , A Value for n-person Games, Contributions to the Theory of Games, 2, 307–317 (1953).

[18] L. S. Shapley, Notes on the N-Person Game III: Some Variants of the von-Neumann-Morgenstern Definition of Solution, Rand Corporation research memorandum RM-, 817, 1–12 (1952).

[19] L. Wang, Model Predictive Control System Design and Implementation Using MATLAB, Springer, New York (2005).

[20] N. N. Krasovskii, A. N. Kotel'nikova, On a differential interception game, Proceedings of the Steklov Institute of Mathematics, 268, 161–206 (2010).

[21] O. L. Petrosian, Looking Forward Approach in Cooperative Differential Games, International Game Theory Review, 18, 1–14 (2016) (DOI:10.1142/S0219198916400077).

[22] O. L. Petrosian, Looking Forward Approach in Cooperative Differential Games with infinite-horizon, Vestnik S.-Petersburg Univ. Ser. 10. Prikl. Mat. Inform. Prots. Upr., 4, 18–30 (2016).

[23] O. L. Petrosian, A. E. Barabanov, Looking Forward Approach in Cooperative Differential Games with Uncertain-Stochastic Dynamics, Journal of Optimization Theory and Applications, 172, 328–347 (2017).

[24] O. L. Petrosian, E. V. Gromova, S. V. Pogozhev, Strong Time-consistent Subset of Core in Cooperative Differential Games with Finite Time Horizon, Mat. Teor. Igr Pril., 8 (4), 79–106 (2016).

[25] O. L. Petrosian, M. A. Nastych, D. A. Volf, Differential game of oil market with moving informational horizon and non-transferable utility, 2017 Constructive Nonsmooth Analysis and Related Topics (dedicated to the memory of V.F. Demyanov) (CNSA), 1-4 (2017) (DOI: 10.1109/CNSA.2017.7974002).

[26] O. L. Petrosian, S. V. Pogozhev, Looking Forward Approach with Random Horizon in Cooperative Differential Games, Automatica, (2017) (to be published).

[27] P. Chander, H. Tulkens, A core-theoretic solution for the design of cooperative agreements on transfrontier pollution, International Tax and Public Finance, 2(2), 279–293 (1995).

[28] R. Bellman, Dynamic Programming, Princeton, Princeton University Press (1957).

[29] S. Jorgensen, D. W. K. Yeung, Inter- and intragenerational renewable resource extraction, Annals of Operations Research, 88, 275–289 (1999).

[30] S. Jorgensen, G. Martin-Herran, G. Zaccour, Agreeability and Time Consistency in Linear-State Differential Games, Journal of Optimization Theory and Applications, 119, 49–63 (2003).

[31] S. Yu. Kostyunin, E. V. Shevkoplyas, On simplification of integral payoff in differential games with random duration, Vestnik S.-Petersburg Univ. Ser. 10. Prikl. Mat. Inform. Prots. Upr., 4, 47-56 (2011).

[32] T. Basar, G. J. Olsder, Dynamic Noncooperative Game Theory, London, Academic Press (1995).

[33] W. H. Kwon, S. H. Han, Receding Horizon Control: Model Predictive Control for State Models, Springer, New York (2005).