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MEASURING MAJORITY TYRANNY: AXIOMATIC APPROACH

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Measuring majority tyranny: axiomatic approach*

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Abstract

We study voting rules with respect to how they allow or limit a majority to dominate minorities. For this purpose we propose a novel quantitative criterion for voting rules: *the qualified mutual majority criterion* (q, k)-MM. For a fixed total number of m candidates, a voting rule satisfies (q, k)-MM if whenever some k candidates receive top k ranks in an arbitrary order from a majority that consists of more than $q \in (0, 1)$ of voters, the voting rule selects one of these k candidates. The standard *majority criterion* is equivalent to $(1/2, 1)$-MM. The standard *mutual majority criterion* (MM) is equivalent to $(1/2, k)$-MM, where k is arbitrary. We find the bounds on the size of the majority q for several important voting rules, including the plurality rule, the plurality with runoff rule, Black's rule, Condorcet least reversal rule, Dodgson's rule, Simpson's rule, Young's rule and monotonic scoring rules; for most of these rules we show that the bound is tight.

Keywords. Majority tyranny, single winner elections, plurality voting rule, plurality with runoff, instant runoff voting, mutual majority criterion, voting rules

1 Introduction

In this paper for various voting rules we propose a simple way to quantitatively measure their robustness to majority tyranny, that is the extent to which each voting rule allows a majority to dominate minorities.

Majority tyranny has been a buzzword for centuries and can be traced back to the ancient Greek ochlocracy. A more modern yet classical reference are the works of James Madison:

> If a majority be united by a common interest, the rights of the minority will be insecure. (Federalist 51.)

Consider the following illustrative example. Let there be 5 candidates (Bernie, Donald, Hillary, John and Ted) and let the voters have one of the five rankings of the candidates as presented in Table 1, where the top row gives the share of these voters in the population.

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Table 1. Preference profile

share of voters	24%	25%	16%	17%	18%
1st candidate	Donald	Hillary	John	Bernie	Ted
2nd candidate	John	Bernie	Ted	John	Bernie
3rd candidate	Ted	John	Bernie	Ted	John
4th candidate	Bernie	Ted	Donald	Hillary	Donald
5th candidate	Hillary	Donald	Hillary	Donald	Hillary

Consider the voters in the last three columns in Table 1. These voters constitute a mutual majority of 51% as they prefer the same subset of candidates (Bernie, John and Ted) over all other candidates. Informally, by majority tyranny we mean that a voting rule chooses one of these three candidates regardless of the opinion and the voting strategy of others. To formalize, a voting rule satisfies the **mutual majority criterion** if whenever more than half of voters prefer each candidate from a group of some k candidates over each other candidate outside the group (in an arbitrary order), then the rule must select one of these k candidates.

The *mutual majority criterion* is a generalization of the innocuous and desirable **majority criterion**: if only one of the candidates (i.e. $k = 1$) is top-ranked by more than half of voters, then this one candidate is selected.

The previous literature has studied two specific types of majority tyranny. The first type deals with the rules that do not satisfy *the majority criterion* (scoring rules like Borda rule) and thus let the minorities veto the only top-ranked candidate of the majority [2, 31]. The second type deals with the rules that satisfy *the mutual majority criterion* and thus even a simple majority is enough to vote through one of its top-ranked candidates (a review of these results can be found in [40]). However, there was no systematic analysis of majority tyranny for the rules between these two extremes.

To fill up this gap we focus on the rules that satisfy *the majority criterion* and do not satisfy *the mutual majority criterion* and determine where exactly between these two axioms does each rule lie. To do that we introduce a quantitative criterion that we call **the qualified mutual majority criterion** and denote as (q, k)-MM: for a fixed total number of m candidates, it requires that if there is a group of k candidates that get top k positions by a qualified mutual majority of more than q of voters, then the rule must select one of these k candidates. Thus, *the majority criterion* is equivalent to (q, k)-MM with $k = 1$ and $q = 1/2$, and *the mutual majority criterion* is equivalent to (q, k)-MM with arbitrary k and a fixed quota $q = 1/2$.

Among such rules of our main interest are the plurality rule and the plurality with runoff rule. In the plurality rule the winner is determined by the highest number of top ranks. In the example above the plurality rule selects Hillary as she is top-ranked by 25% of voters which is higher than what other candidates get. In the plurality with runoff rule the two candidates with the highest number of top ranks proceed to the second round where the winner is determined by a simple majority. In the example above the plurality with runoff rule first selects Donald and Hillary into the second round, and then Donald wins by a simple majority.

Together with the instant-runoff rule[1] these rules are most widespread in political elections around the world.[2] We find that for the plurality rule for arbitrary number of preferred

[1] In the instant-runoff rule each voter submits a ballot with a rank-ordered list of the candidates. The candidate that gets the least number of the first positions in the ballots is eliminated, the candidates with the lowest number of the first positions keep being eliminated one by one until there is a candidate that receives more than half of first positions. In the example above the instant-runoff rule selects Ted: first it eliminates John, his first positions are transferred to Ted; then it eliminates Bernie, his first positions are transferred to Ted who wins the election by receiving more than half of first positions.

[2] A version of plurality with run-off is used for presidential elections in France and Russia. The US

candidates $k > 1$ the quota is $q = k/(k+1)$ of voters, and for the plurality with runoff rule the quota is $q = k/(k+2)$. Both quotas are tight: for each smaller quota we find a counterexample.

We further study other voting rules, which are popular in the literature, with respect to *the qualified mutual majority criterion*. We find bounds on q for Dodgson's rule (aka Lewis Carroll's rule), Young's rule, Condorcet least-reversal rule, Simpson's rule (aka maximin rule), and the convex median voting rule;[3] for most of them we show that the bounds are tight (except for Dodgson's rule). We show that unless we restrict the total number of candidates Black's rule fails the *qualified mutual majority criterion*: for each $q < 1$ and each $k > 1$ there exists a counter-example.

The summary of our results is presented in Table 2 below. For each of the listed rules we find a tight bound on the size q of the qualified mutual majority. If this bound equals 1/2 (as in the case of the instant-runoff rule), then *the mutual majority criterion* is satisfied.

Table 2. The minimal quota q such that (q, k)-MM is satisfied for any m candidates.

	Voting rule	q	$\sup_k q$	reference
1	Instant-runoff	$1/2$	$1/2$	[40]
2	Condorcet least reversal (even k)	$(5k-2)/(8k)$	$5/8$	Theorem 3.5
	Condorcet least reversal (odd k)	$(5k^2-2k+1)/(8k^2)$	$5/8$	Theorem 3.5
3	Convex median	$(3k-1)/(4k)$	$3/4$	Theorem 3.8
4	Plurality with runoff	$k/(k+2)$	1	Theorem 3.2
5-6	Simpson's	$(k-1)/k$	1	Theorem 3.3
5-6	Young's	$(k-1)/k$	1	Theorem 3.4
7	Plurality	$k/(k+1)$	1	Theorem 3.1
8	Black's	1	1	Theorem 3.7

Notes. The voting rules are ordered according to the minimal size of the qualified mutual majority $q(k)$ (for $k > 4$). For all these rules $q = 1/2$ whenever k=1. The instant-runoff rule satisfies the *mutual majority criterion* and therefore $q = 1/2$.

Of special interest is the inverse plurality rule: there each candidate gets an additional point from each voter unless this voter ranks him as the worst among all, and the candidate with the highest total number of points wins. When the number of the preferred candidates k is only by one smaller than the total number of candidates $m = k + 1$, then the tight quota for the mutual majority is less than one half and equals $1/m$. This effectively means that the mutual majority can be very small but still be able to select one of its preferred candidates, which is the same as to veto their least preferred candidate. Generalizing this result we find the tight quota q for any monotonic scoring rules, which incorporates the results for the plurality rule and for Borda rule.

The paper proceeds as follows. Section 2 presents the model and the necessary definitions. Section 3 presents the voting rules satisfying the majority criterion and the results. Section 4 presents the results (Theorems 4.1 and 4.2) on monotonic scoring rules. Section 5 concludes.

Presidential election system with primaries also resembles the plurality with runoff rule given the dominant positions of the two political parties. The instant runoff rule is currently used in parliamentary elections in Australia, presidential elections in India and Ireland. According to the Center of Voting and Democracy fairvote.org [16] the instant-runoff and plurality with runoff rules have the highest prospects for adoption in the US. In the UK a 2011 referendum proposing a switch from the plurality rule to the instant-runoff rule lost when almost 68% voted *No*.

[3] Let us briefly motivate the results about the convex median voting rule. Borda rule satisfies positional dominance of second order (WPD) but it fails the majority criterion (Maj). The convex median voting rule was proposed in [26] as a rule that satisfies both WPD and Maj. Our paper shows that this rule is much closer to satisfying the stronger criterion of mutual majority (or, equivalently 1/2-MM) as it satisfies 3/4-MM. According to the incompatibility result (e.g. in [26]) there is no rule that satisfies both WPD and 1/2-MM.

2 The model

This section introduces the standard voting problem and the main criteria for voting rules.

Consider a **voting problem** where $n \geq 1$ voters $I = \{1, \ldots, n\}$ select one winner among $m \geq 1$ candidates (alternatives) $A = \{a_1, \ldots, a_m\}$. Let $L(A)$ be the set of linear orders (complete, transitive and antisymmetric binary relations) on the set of candidates A.

Each voter $i \in I$ is endowed with a **preference relation** $\succ_i \in L(A)$. (Voter i prefers a to b when $a \succ_i b$.)

Preference relation $\succ_i$ corresponds to a unique ranking bijection $R_i : A \to \{1, \ldots, m\}$, where R_i^a is the relative rank that voter i gives to candidate a,

$$R_i^a = |\{b \in A : b \succ_i a\}| + 1, \quad a \in A, \quad i \in \{1, \ldots, n\}.$$

The collection of the individual preferences $\succ\, = (\succ_1, \ldots, \succ_n) \in L(A)^n$ as well as corresponding ranks $(R_1, \ldots, R_n)$ are referred to as the **preference profile**. (There exist $m!$ different linear orders and $(m!)^n$ different profiles.)

Table 3 provides an example of a preference profile for $n = 7$ voters over $m = 4$ candidates. Here voters are assumed to be anonymous which allows us to group voters with the same individual preferences. Each column represents some group of voters, the number of voters in the group is in the top row; the candidates are listed below (starting from the most preferred candidate) according to the preference of the group.

Table 3. Preference profile

2	2	2	1
a	b	c	c
b	a	d	d
c	c	a	b
d	d	b	a

Given a preference profile we determine function $h(a, b)$ as the number of voters that prefer candidate a over candidate b,

$$h(a, b) = |\{i : a \succ_i b, \quad 1 \leq i \leq n\}|, \quad a, b \in A, \quad a \neq b.$$

Matrix h with elements $h(a, b)$ is called a **tournament matrix**. (Note that $h(a, b) = n - h(b, a)$ for each $a \neq b$.)

Table 4 provides the tournament matrix for the preference profile from Table 3.

Table 4. Tournament matrix

	a	b	c	d
a		**4**	**4**	**4**
b	3		**4**	**4**
c	3	3		**7**
d	3	3	0	

Matrix h is called **transitive** if there exists a linear order $\succ_0 \in L(A)$ such that $h(a, b) \geq h(b, a)$ whenever $a \succ_0 b$. For example, the matrix in Table 4 is transitive.

We say that candidate a **(weakly) dominates** candidate b, or **(weakly) wins in pairwise majority comparison**, if $h(a, b) > n/2$ $(h(a, b) \geq n/2)$. For arbitrary disjoint subsets of candidates $A_1, A_2 \in A$ we say that A_1 (weakly) dominates A_2, if for each element $a \in A_1$ and each element $b \in A_2$ candidate a (weakly) dominates b.

For some subset $B \subseteq A$, a candidate is called a **Condorcet winner** [12],[4] if he/she dominates any other candidate in this subset. Thus, the set of Condorcet winners is

$$CW(B) = \{b \in B : h(b,a) > n/2 \quad \text{for each} \quad a \in B \setminus b\}, \quad B \subseteq A.$$

It is easy to see that the set of Condorcet winners CW is either a singleton or empty.

For some subset, a candidate is called a **weak Condorcet winner**, if he/she weakly dominates any other candidate in this subset.

Similarly, a **Condorcet loser** in some subset $B \subseteq A$ is a candidate that loses in pairwise comparisons to each candidate in this subset:

$$CL(B) = \{b \in B : h(b,a) < n/2 \quad \text{for each} \quad a \in B \setminus b\}, \quad B \subseteq A.$$

Let a **positional vector** of candidate a be vector $n(a) = (n_1(a), \ldots, n_m(a))$, where $n_l(a)$ is the number of voters for whom candidate a has rank l in individual preferences,

$$n_l(a) = |\{i : R_i^a = l, \quad 1 \le i \le n\}|, \quad a \in A, \quad l \in \{1, \ldots, m\}.$$

The definition implies that each positional vector has nonnegative elements, $n_l(a) \ge 0$ for each l, and the sum of elements is equal to the number of voters $\sum\limits_{l=1}^{m} n_l(a) = n$.

Candidate a is called a **majority winner**, if $n_1(a) > n/2$.

A collection of positional vectors for all candidates is called a **positional matrix** $(n(a_1), \ldots, n(a_m)) = n(\succ)$.

Table 5 provides the positional matrix for the preference profile in Table 3.

Table 5. Positional matrix

Rank	a	b	c	d
1	2	2	3	0
2	2	2	0	3
3	2	1	4	0
4	1	2	0	4

A mapping $C(B, \succ)$ that to each nonempty subset $B \subseteq A$ and each preference profile $\succ$ gives a choice set is called a **social choice rule**,[5]

$$C : 2^A \setminus \emptyset \times L(A)^n \to 2^A,$$

where $C(B, \succ) \subseteq B$ for any B; and $C(B, \succ) = C(B, \succ')$, if preference profiles $\succ, \succ'$ coincide on B.

A rule is called **universal** if $C(B, \succ) \neq \emptyset$ for any nonempty B and any profile $\succ$.

Let us define the criteria that are critical for the results of the paper and the social choice rules considered below.[6]

Majority (Maj) criterion. For each preference profile, if some candidate a is top-ranked by more than half of voters ($n_1(a) > n/2$), then the choice set is a singleton and coincides with this candidate.

[4] The collection [29] contains English translations of original works by Borda, Condorcet, Nanson, Dodgson and other early researches.

[5] Any social choice rule is a voting rule. There exist voting rules (for example, approval voting, range voting and majority judgement) that are not social choice rules.

[6] The "extremely desirable" criteria of universality, non imposition, anonymity, neutrality, unanimity are satisfied by all social choice rules considered in this paper [18, 38, 40, 45].

Mutual majority (MM) criterion.[7] For each preference profile, if more than half of voters give to some k candidates ($B = \{b_1, \dots, b_k\}$, $1 \le k < m$) top k ranks in an arbitrary order, then the choice set is included in B.

For any fixed quota $q \in (0, 1)$ and any fixed number of preferred candidates k among the total of m candidates, we define the next criteria.

(q, k)-MM criterion.[8] For each preference profile, if a share of voters higher than q gives to some k candidates ($B = \{b_1, \dots, b_k\}$, $1 \le k < m$) top k ranks in an arbitrary order, then the choice set is included in B.

We say that a rule satisfies **q-MM** criterion, if it satisfies (q, k)-MM criterion for each k.

For universal rules, it is also apparent from the definitions that MM implies Maj; for any k and any $q' \ge q$, (q, k)-MM implies (q', k)-MM; Maj is equivalent to $(1/2, 1)$-MM; MM is equivalent to $1/2$-MM.[9]

3 Voting rules satisfying the majority criterion

This section considers the classic social choice rules that satisfy *the majority criterion* (thus, they satisfy (q, k)-MM with $k = 1$ and any $q \ge 1/2$) but do not satisfy *the mutual majority criterion*.[10] In case of only two candidates each rule satisfying the *majority criterion* coincides with the **simple majority rule** where the winner is the candidate that gets at least half of votes.[11] In what follows we consider the case of $m > 2$ candidates.

In the **plurality (Pl)** voting rule the candidate that receives the highest number of top positions is declared to be the winner,

$$\mathrm{Pl}(A, \succ) = \{a \in A : n_1(a) \ge n_1(b) \quad \text{for each} \quad b \in A \setminus a\}.$$

Theorem 3.1. For each $m > k \ge 1$, plurality voting rule satisfies (q, k)-MM criterion if and only if $q \ge k/(k + 1)$.

Proof.

Let $m \ge 3$, and let more than $nk/(k+1)$ voters give candidates from some subset $B \subsetneq A$ ($m > |B| = k \ge 1$) top k positions. Then all together candidates in B receive strictly more than $nk/(k + 1)$ top positions, while candidates from $A \setminus B$ all together receive strictly less than $n/(k+1)$ top positions. Therefore, at least one of the candidates in B receives strictly more than $n/(k+1)$ top positions, and each candidate from $A \setminus B$ receives strictly less than $n/(k+1)$ top positions. Therefore, the plurality voting rule can only select a candidate from set B.

For any smaller quota $q < k/(k + 1)$ we can always find the following counterexample. Let the total number of voters be $n = k + 1$ and let k voters give candidates from set B top k positions such that each of these candidates gets the top position exactly once. Let some

[7] MM is implied by a more general axiom for multi-winner voting called Droop-Proportionality for Solid Coalitions [43].

[8] (q, k)-MM is even more general than the concept q-PSC formalized in [1] if the latter is applied to single-winner elections. The *weak mutual majority criterion* defined in [26] turns as a particular case of $q = k/(k + 1)$.

[9] Also one can see that unanimity criterion is equivalent to $(1 - \varepsilon, 1)$-MM with infinitely small $\varepsilon > 0$.

[10] For completeness of results, we should mention well-studied voting rules that satisfy *the mutual majority criterion*: Nanson's [29, 30], Baldwin's [4], single transferable vote [23], Coombs [13], sequential majority comparison [45], maximal likelihood [25], ranked pairs [41], beat paths [33], median voting rule [6], Bucklin's [40], majoritarian compromise [35], q-approval fallback bargaining [9], and those tournament solutions which are refinements of the top cycle [21, 34]. For their formal definitions and properties, we also advise [10, 18, 38, 40, 45].

[11] In case of only $m = 2$ candidates the simple majority rule is the most natural as it satisfies a number of other important axioms according to May's Theorem [28].

voter give the top position to some other candidate $a \notin B$. Then the plurality voting rule selects all candidates from the set $B \cup a$. ■

The **plurality with runoff (RV)** voting rule proceeds in two rounds: first the two candidates with the highest number of top positions are determined, then the winner is chosen between the two using simple majority rule.

Theorem 3.2. For each $m - 1 = k \geq 1$, plurality with runoff satisfies (q, k)-MM criterion if and only if $q \geq 1/2$; for each $m - 1 > k > 1$, the rule satisfies the criterion if and only if $q \geq k/(k + 2)$.

Proof.

In case $m = 3$ the *mutual majority criterion* holds ($q = 1/2, k = 1, 2$).

In case $k = m - 1$, and $q = 1/2$, in the second round there is at least one candidate from the supported k candidates, and this candidate wins.

Let $m > 3$, and let more than $nk/(k+2)$ voters give candidates from some subset $B \subsetneq A$ ($m - 1 > |B| = k > 1$) top k positions. Then all together candidates in B receive strictly more than $nk/(k+2)$ top positions, while candidates from $A \setminus B$ all together receive strictly less than $2n/(k + 2)$ top positions. Therefore, at least one of the candidates in B and at most one of the candidates in $A \setminus B$ receive strictly more than $n/(k + 2)$ of top positions. Thus, in the second round there is at least one candidate from set B. Even if the second candidate is from $A \setminus B$, this second candidate loses to the candidate from B by simple majority rule. Hence, the winner is from B.

For any smaller quota $q < k/(k + 2)$ we can always find the following counterexample. Let the total number of voters be $n = (k+2)n' + 2$ and let kn' voters give k candidates from set B top k positions such that each candidate in B gets the top position exactly n' times. Consider the other $2 \cdot (n' + 1)$ voters and two other candidates $a_1, a_2 \notin B$. Let $n' + 1$ voters top-rank candidate a_1 and the other $n' + 1$ voters top-rank candidate a_2. Then candidates a_1 and a_2 make it to the second round.

If we set $n' > 2q/(k - kq - 2q)$ then set B is supported by more than qn voters. ■

According to **Simpson's rule** (also known as maximin voting rule) [36, 44] each candidate receives a score equal to the minimal number of votes that this candidate gets compared to any other candidate,

$$Si(a) = \min_{b \in A \setminus \{a\}} h(a, b).$$

The winner is the candidate with the highest score.

Theorem 3.3. For each $m > k > 1$, Simpson's rule satisfies (q, k)-MM criterion if and only if $q \geq (k - 1)/k$.[12]

Proof.

In case $m = 3$ the *mutual majority criterion* holds ($q = 1/2, k = 1, 2$).

Let $m > 3$, and more than $n(k - 1)/k$ voters top-rank $k \geq 2$ candidates, denote this subset of candidates as $B = \{b_1, \ldots, b_k\}$. It is easy to see that each candidate in $A \setminus B$ gets less than n/k of Simpson's scores (a candidate from $A \setminus B$ gets the highest score when it is top-ranked by all voters that do not top-rank B).

Denote the number of the first positions of some candidate $b \in B$ among all other candidates in B as $n_1(b, B)$:

$$n_1(b, B) = |\{i : b \succ_i b' \quad \text{for each} \quad b' \in B \setminus b\}|. \tag{1}$$

[12] This tight bound $q = (k - 1)/k$ for Simpson's rule coincides with the tight bound of q-majority equilibrium [22, 27], and with the minimal quota, that guarantees acyclicity of preferences [14, 17, 42].

Since the total number of first positions is fixed $n_1(b_1, B) + \ldots + n_1(b_k, B) = n$, there is a candidate $b \in B$ with the number of top positions weakly higher than the average: $n_1(b, B) \geq n/k$.

Hence, there is a candidate that receives not less than n/k of scores, and each candidate from $A \setminus B$ gets less than n/k scores and cannot be the winner.

To see that the bound $q = (k-1)/k$ is tight consider the following counterexample in Table 6: each candidate $b \in B$ receives exactly qn/k first positions, qn/k second positions and so on from the qualified mutual majority of qn voters, while all voters outside of the qualified mutual majority top-rank some other candidate a_1 and also prefer all candidates in $A \setminus B$ over candidates in B.

Table 6. Preference profile

$\frac{qn}{k}$	...	$\frac{qn}{k}$	$\frac{(1-q)n}{k}$	...	$\frac{(1-q)n}{k}$
b_1	...	b_k	a_1	...	a_1
b_2	...	b_1	...	...	...
...	...	...	a_{m-k}	...	a_{m-k}
b_k	...	b_{k-1}	b_1	...	b_k
a_1	...	a_1	...	...	...
...	...	...	b_{k-1}	...	b_{k-2}
a_{m-k}	...	a_{m-k}	b_k	...	b_{k-1}

Notes. The qualified mutual majority of qn voters give exactly qn/k first, second and so on positions to each candidate $b_i \in B$, all preferences over remaining alternatives $A \setminus B$ are the same. The other $(1-q)n$ voters prefer each candidate in $A \setminus B$ over each candidate in B, and have identical relative ordering of candidates within these two sets. This type of cyclical preferences over B is known as a Condorcet k-tuple.

For each $k>1$ we can set $n=k^2$ and $q=(k-1)/k$. Then set B is supported by $n(k-1)/k$ voters, while each candidate from the set $B \cup a_1$ gets the same Simpson's score. ■

By **Young's rule** [11, 44] the winner is the candidate that needs the least number of voters (integer or not) to be removed for this candidate to become a Condorcet winner.

Theorem 3.4. For each $m > k > 1$, Young's rule satisfies (q, k)-MM criterion if and only if $q \geq (k-1)/k$.

Proof.

In case $m = 3$ the *mutual majority criterion* holds ($q = 1/2, k = 1, 2$).

Let $m > 3$, and let more than $n(k-1)/k$ voters top-rank $k \geq 2$ candidates, denote this subset of candidates as $B = \{b_1, \ldots, b_k\}$. For each candidate from $A \setminus B$ to make him a Condorcet winner, we need to remove strictly more than $n(k-2)/k$ voters.

Consider some candidate $b \in B$ with a higher than average number of top positions $n_1(b, B) \geq n/k$ (as defined in equation (1)). For b to win, at most $n(k-2)/k + \varepsilon$ voters have to be removed.

The example from Table 6 shows that the bound $(k-1)/k$ is tight. ■

According to **Condorcet least-reversal rule** (the simplified Dodgson's rule) [40] the winner is, informally, the candidate $a \in A$ that needs the least number of reversals in pairwise comparisons in order to become the a Condorcet winner. Formally, the winner d minimizes the following sum of losing margins compared to each other candidate c:

$$p_d^{CLR} = \sum_{c \in A \setminus d} \max\left\{\frac{n}{2} - h(d, c), 0\right\}. \tag{2}$$

Theorem 3.5. For each $m > k \geq 2$ and for each even k, Condorcet least-reversal rule satisfies (q, k)-MM criterion if and only if $q \geq (5k-2)/(8k)$; for each $m > k \geq 1$ and for each odd k, the rule satisfies the criterion if and only if $q \geq (5k^2 - 2k + 1)/(8k^2)$.

Proof.

Again we use the preference profile in Table 6. Let's first show that it is the worst possible profile for each candidate $b \in B$ to win by the Condorcet least-reversal rule, i.e. it has the maximum minimum score p_b^{CLR} among all candidates $b \in B$. To maximize the minimum score p^{CLR} for candidates in B we can maximize the scores (2) for the subset B separately: $\sum\limits_{c \in B \setminus b}$. This is true, because the other part $\sum\limits_{c \in A \setminus B}$ is zero whenever $q \geq 1/2$.

According to Proposition 5 in [32], each tournament matrix with k candidates has unique representation as the sum of its transitive matrix and its Condorcet k-tuple matrix (Table 7).

Table 7. Condorcet k-tuple profile and tournament matrix

$\frac{n}{k}$	$\frac{n}{k}$	$\dots$	$\frac{n}{k}$
b_1	b_2	$\dots$	b_k
b_2	b_3	$\dots$	b_1
$\dots$	$\dots$	$\dots$	$\dots$
b_k	b_1	$\dots$	b_{k-1}

	b_1	b_2	$\dots$	b_k
b_1		$n(k-1)/k$	$\dots$	n/k
b_2	n/k		$\dots$	$2n/k$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
b_k	$n(k-1)/k$	$n(k-2)/k$	$\dots$	

Thus, the maximal element of the transitive matrix gets not more total scores p^{CLR} than in the k-tuple matrix only. Hence, the profile in Table 6 qualifies as the worst case.

Next we find the bound for the profile in Table 6. Each candidate $a \in A \setminus B$ gets at least $p_a^{CLR} \geq nk(2q-1)/2$ points.

Candidate b_1 gets $n/k, 2n/k, \dots, (k-1)n/k$ pairwise majority wins against candidates $b_k, \dots, b_2$ correspondingly. For even k the score for each $b \in B$ is $p_b^{CLR} = n(k-2)/8$, for odd k the score is $p_b^{CLR} = n(k-1)^2/(8k)$. Setting these scores equal to the score $p_{a_1}^{CLR} = nk(2q-1)/2$ received by a_1 we get the tight bounds. ∎

The classic **Dodgson's** [11, 15, 29] winner is determined as the candidate that needs the least upgrades by one position in individual preferences that makes him a Condorcet winner. To satisfy homogeneity property such upgrades are allowed to perform for non integer amount of voters.

Theorem 3.6. For $m > k \geq 1$, Dodgson's rule (with non integer amount of upgrades) satisfies (q,k)-MM criterion with $q \geq k/(k+1)$; the rule fails the criterion with $q < (5k-2)/(8k)$ in case of even $k \geq 2$, and $q < (5k^2-2k+1)/(8k^2)$ in case of odd $k \geq 1$.

Proof.

Let more than $k/(k+1)$ portion of voters give candidates from some subset $B \subsetneq A$ top k positions. Then each candidate $a \in A \setminus B$ gets less than $n/(k+1)$ votes in pairwise comparison against each candidate in set B. Upgrading candidate a by one position in the preference profile adds not more than one vote in a pairwise comparison against each candidate in set B. Therefore candidate a needs more than $k(\frac{n}{2} - \frac{n}{k+1})$ upgrades to become a Condorcet winner. A candidate in B that gets more than $n/(k+1)$ top positions needs not more than $(k-1)(\frac{n}{2} - \frac{n}{k+1})$ upgrades in the preference profile in order to become a Condorcet winner. Since $(k-1)(\frac{n}{2} - \frac{n}{k+1}) < k(\frac{n}{2} - \frac{n}{k+1})$, Dodgson's rule selects from set B.

The second statement follows from the calculations for profile in Table 6. ∎

By **Borda rule** [8, 29] the first best candidate in an individual preference gets $m-1$ points, the second best candidate gets $m-2$, ..., the last gets 0 points.

The total Borda score can be calculated using positional vector $n(a)$ as follows:

$$Bo(a) = \sum_{i=1}^{m} n_i(a)(m-i), \quad a \in A. \tag{3}$$

The candidate with the highest total score wins. The score can also be calculated using the tournament matrix:

$$Bo(a) = \sum_{b \in A \setminus \{a\}} h(a, b), \quad a \in A. \tag{4}$$

The previous equation readily shows that the Borda score of a Condorcet loser is always lower than the average score of all candidates (half of the maximally reachable points), $Bo(a) < n(m-1)/2$ whenever $a \in CL(A)$. Similarly, the Borda score of a Condorcet winner is always higher than the average score of all candidates, $Bo(a) > n(m-1)/2$ whenever $a \in CW(A)$.

Black's rule [7] selects a Condorcet winner. If a Condorcet winner does not exist, then the candidate with the highest Borda score (4) is selected.

Theorem 3.7. For each $m > k > 1$, Black's rule satisfies (q, k)-MM criterion if and only if $q \geq (2m - k - 1)/(2m)$.[13]

Proof.

In case $m = 3$ the *mutual majority criterion* holds ($q = 1/2, k = 1, 2$).

Let $m \geq 3$, and let more than qn voters give candidates from some subset $B \subsetneq A$ ($m > |B| = k > 1$) top k positions. Then one can find the tight bound for the quota $q = q(k, m)$ using the following equation:

$$(1-q)(m-1) + q(m-k-1) = q\frac{m-1+m-k}{2} + (1-q)\frac{k-1}{2},$$

where the left part is the maximal Borda score for any $a \notin B$, and the right part is the minimal maximal Borda score for any $b \in B$.

The example from Table 6 shows that the bound $(2m - k - 1)/(2m)$ is tight. ■

Based on truncated Borda scores the **convex median voting rule (CM)** is defined in the following way [26]. First for some positional vector $n(a)$ and some real number $t \in (0, +\infty)$ define the truncated Borda score [19] as

$$B_t(a) = t \cdot n_1(a) + (t-1)n_2(a) + \ldots + (t - \lfloor t \rfloor)n_{\lfloor t \rfloor + 1}(a), \quad t \in (0, +\infty),$$

where formally put $n_i(a) = 0$ for $i > m$. The definition implies that $B_{m-1}(a) = Bo(a)$.

For each candidate a define the score of convex median using the following formula:

$$\mathrm{CM}(a) = \begin{cases} m - 1 - \frac{1}{2}\max\left\{t \in [1, 2(m-1)] : \frac{B_t(a)}{t} \leq \frac{n}{2}\right\}, & n_1(a) \leq \frac{n}{2}, \\ m - 2 + \frac{n_1(a)}{n}, & n_1(a) > \frac{n}{2}, \end{cases}$$

The winner is the candidate with the highest value of the convex median; ties are broken using Borda scores (3).

Theorem 3.8. For each $m > 2k$, the convex median voting rule satisfies (q, k)-MM criterion if and only if $q \geq (3k - 1)/(4k)$; for each $m = k + 1$ – if and only if $q \geq 1/2$; for each $2k \geq m > k + 1$, the tight bound q satisfies to the inequality $\frac{1}{2} < q < \frac{3k-1}{4k}$ and to the equation

$$4k(m - k - 1)q^2 + (5k^2 + 5k - 2mk - m^2 + m)q + m(m - 1 - 2k) = 0. \tag{5}$$

Proof.

Let qn voters give candidates from some subset $B = \{b_1, \ldots, b_k\}$ top k positions. Then each candidate $a \notin B$ gets the following truncated Borda score:

[13] In the theorem, we actually find the tight bound of quota for Borda rule. In particular case $k = 1$, this quota equals $q = (m - 1)/m$, and also was calculated in [2, 31].

$$\frac{B_{2kq}(a)}{2kq} \le (1-q)n + \frac{(2kq-k)qn}{2kq} = \frac{n}{2}.$$

Let $m > 2k$ and $q > (3k-1)/(4k)$. It is sufficient to show that for some $b \in B$ its truncated Borda score is higher: $B_{2kq}(b)/(2kq) > n/2$. For a contradiction assume the opposite:

$$\frac{B_{2kq}(b)}{2kq} \le \frac{n}{2} \quad \text{for each} \quad b \in B.$$

Then

$$\frac{(2kq)n_1(b) + \ldots + (2kq-k+1)n_k(b)}{2kq} \le \frac{n}{2} \quad \text{for each} \quad b \in B,$$

whence, after summing up k inequalities, we get:

$$\frac{qnk(4kq-k+1)}{4kq} < \frac{nk}{2}.$$

The latter inequality contradicts the assumption $q > (3k-1)/(4k)$.

To show that the bound is tight we again use the preference profile from Table 6.

Similarly we find a tight bound for the case $2k \ge m \ge k+1$:

$$\min_{\succ} \max_{b \in B} \frac{B_{2kq}(b)}{2kq} = \frac{\frac{qn}{k}\frac{(4kq-k+1)}{2}k}{2kq} + \frac{\frac{(1-q)n}{k}\frac{(4kq-m)}{2}(2k-m+1)}{2kq} = \frac{n}{2},$$

which leads to equation (5) and also to a special case $m = k+1, q = 1/2$. ■

4 Monotonic scoring rules

In a scoring rule each of m candidates is assigned a score from $s_1, \ldots, s_m$ for a corresponding position in a voter's individual preference and then the scores are summed up over all voters. In the paper we consider monotonic scoring rules in which $s_1 > s_m$ and $s_1 \ge s_2 \ge \ldots \ge s_m$.

The next theorem generalizes Theorem 3.1 for the plurality rule ($s_1 = 1, s_2 = \ldots s_m = 0$) and Theorem 3.7 for Borda rule ($s_i = m - i,\ i = 1, \ldots, m$). In particular case $k = 1$, the quota (6) equals $q = (s_1 - s_m)/(s_1 - s_m + s_1 - s_2)$, and also was calculated in [2, 31].

Theorem 4.1. For each $m > k \ge 1$, a monotonic scoring rule satisfies (q,k)-MM criterion if and only if the quota q satisfies the next inequality

$$q \ge \frac{s_1 - \frac{1}{k}\sum_{i=1}^{k} s_{m-i+1}}{s_1 - \frac{1}{k}\sum_{i=1}^{k} s_{m-i+1} + \frac{1}{k}\sum_{i=1}^{k} s_i - s_{k+1}}. \tag{6}$$

Proof.

Let $m \ge 3$, and let more than qn voters give candidates from some subset $B \subsetneq A$ ($m > |B| = k \ge 1$) top k positions. Then one can find the tight bound for the quota $q = q(k, s_1, \ldots, s_m)$ using the following equation:

$$(1-q)\cdot s_1 + q \cdot s_{k+1} = q \cdot \frac{s_1 + \ldots + s_k}{k} + (1-q)\cdot\frac{s_m + \ldots + s_{m-k+1}}{k},$$

where the left part is the maximal total score for any $a \notin B$, and the right part is the minimal maximal total score for any $b \in B$.

The example from Table 6 shows that the bound (6) is tight. ■

Inverse plurality rule is a monotonic scoring rule with the scores $s_1 = \ldots = s_{m-1} = 1, s_m = 0$. Theorem 4.1 directly implies the next statement.

Theorem 4.2. For each $m - 1 > k \geq 1$ and for each $q < 1$, the inverse plurality rule fails (q, k)-MM criterion; for each $m - 1 = k \geq 1$, the rule satisfies (q, k)-MM criterion if and only if $q \geq 1/m$.

Our analysis confirms the idea of [3] that among monotonic scoring rules the plurality rule respects the majority most and the inverse plurality rule respects the minority most.

5 Conclusions

We introduced and studied the quantitative property which we call the *qualified mutual majority criterion.* We focused on the widespread voting rules such as the plurality rule, the plurality with runoff rule, and the instant runoff rule. The instant-runoff rule respects the majority extremely well as it satisfies the *mutual majority criterion.* We show that plurality with runoff does it slightly better than the plurality rule as it has a smaller tight bound. According to this criterion, all other voting rules considered here are located between the quota of the instant-runoff rule and the quota of the plurality rule.

Besides majority tyranny our results have two secondary interpretations. The first interpretation is about the easiness with which a mutual majority that happens to be smaller than the required quota q can modify their preferences for the top k candidates in order to make sure that one of these candidates wins. At the extreme, this mutual majority can top-rank one of these k candidates and thus always vote him through. This is the same as saying that the mutual majority agrees to eliminate $k-1$ candidates among their k preferred candidates. Yet, if this type of coordination is difficult, then they might agree to eliminate fewer candidates. For instance, in the example above presented in Table 1 with $k = 3$ preferred candidates, if the plurality rule is used, the mutual majority has to have at least $3/(3+1) = 75\%$ to guarantee that one of the three candidates wins regardless of the opinion of other voters. But if the mutual majority agrees to top-rank only two candidates, then they need only $2/(2 + 1) = 67\%$ to guarantee that one of these two candidates wins. This marginal decrease in q as k decreases corresponds to the incentives of the mutual majority to coordinate on the smaller set of preferred candidates.

The second interpretation is related to incentives to participate. If the mutual majority is large enough, then others can only influence which of the top-ranked k candidates is selected, but they cannot effectively veto any of these candidates. When voting is voluntary, the rules that have higher quota q give stronger incentives for minorities to participate since their votes are more likely to be pivotal. This somewhat opposes the common opinion of the social choice literature that the instant-runoff rule promotes participation. Indeed, when minorities face a larger group of voters with a single preferred candidate, then the instant-runoff allows the minorities to transfer their votes instead of wasting them. However, if minorities face a mutual majority with more than one preferred candidate, then the instant-runoff works in favor of this majority [39, 45].

One specific open question arises from the incomplete result regarding Dodgson's rule: in contrast to other results, Theorem 3.6 does not specify the tight bound on the size of the qualified mutual majority. The value of the tight bound seems to be a hard question, as Dodgson's rule is known to be difficult to work with [5, 11, 24]. It is not easy to check whether the profile in Table 6 gives the worst case for each candidate in group of mutually supported candidates B and at the same time the best case for some other candidate in $A \setminus B$.

A more general open question is the analysis of the aggregative properties of voting rules in practically-relevant scenarios. In this paper the main results are based on the worst-case analysis. Future research can make use of more realistic scenarios inspired by theories of individual decision-making, empirical results and experiments on voting.

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