HIGHER SCHOOL OF ECONOMICS

NATIONAL RESEARCH UNIVERSITY

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**ADDING ONE MORE INGREDIENT:
DECENTRALIZING OPTIMAL
ALLOCATIONS IN SEARCH
AND MATCHING MODELS WITH
COMPLEMENTARITIES IN LABOUR**

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This paper introduces complementarities in labour into the classical Mortensen – Pissarides – Diamond search model. Specifically, two workers are needed to perform the task. The assumption of Nash bargaining is maintained to provide an environment more conducive to the Hosios condition. In a continuous-time framework the necessity to keep one worker employed while searching for the second worker arises. We show this leads to a number of additional externalities that are impossible to correct for using Hosios-style conditions in a random search framework. Under directed search, we show such conditions are sufficient for optimality only in the rarest of cases.

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1   Introduction

The search and matching framework due to Diamond, Mortensen and Pissarides (e.g. Diamond, 1982, Mortensen, 1986 and Pissarides, 2000), DMP hereafter, has become the workhorse for modeling labour market frictions. It has an equally canonical optimality condition that minimizes frictions in the model, the Hosios condition (Hosios, 1990). However, while it is well understood that the Hosios condition is a coincidence of bargaining weights with the elasticity of matching, it is less appreciated that the Hosios condition only holds with linear technology.

To make this point,this paper employs the basic ingredients of the DMP framework (matching function, Beveridge Curve, Nash Bargaining). Our innovation is to assume that the production technology requires two workers to be present. Admittedly, this is an extreme form of complementarity, but the results hold for any number of workers larger than one[1]. The model of this paper is set in continuous time, preventing instantaneous occupation of all employment slots. Thus, there will be a phase between unemployment

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[1]This echoes the contributions on matching of Becker (1973) and Shimer and Smith (2000) in that multiple agents are needed for a match, but we abstract from sorting issues by assuming homogenous agents.

and productive employment. In turn this gives rise to additional thin-market and congestion externalities.

As Mangin and Julien (2018) demonstrate, the Hosios Condition (in a modified form) does work when the match surplus depends on the labour market tightness if the bargaining weight is equal to the elasticity of matching *plus* the elasticity of the surplus. However, there the production technology continues to be linear in individual workers and it turns out multiple-worker setups do not lend themselves to such a solution. An important paper introducing multiple workers is Stole and Zwiebel (1996) which extends the Nash Bargaining approach to firms with multiple workers and diminishing returns to scale. There, the firm bargains with each worker as marginal. They find that there is a motive to overextend on the part of firms since that way they can diminish the marginal contribution of the marginal worker and thus reduce wages. This means, as Mortensen (2009) demonstrates, there is a distortion the Hosios condition cannot correct for on top of thin-market and congestion externalities.

One may think with simple multilateral Nash-Bargaining and a simpler production function (linear for two workers), the Hosios Condition may be restored. This paper investigates this first in a random search framework,

much like the classical DMP model. It then takes cues from Moen (1997), Montgomery (1991) and Peters (1991) to let search be directed[2]. The results, in both cases, are that the Hosios condition is insufficient to ensure optimality. With random search it is impossible and in the directed search setup a knife-edge case.

The Hosios condition's lack of generality is not an arcane point. Many scholars make use of it in calibration exercises. Examples include: Albertini and Fairise (2013), Hornstein et al. (2005) and Shimer (2005). While these studies have other aims, so this not a diminishment of their contributions, it is important to point out assuming this condition is *with* loss of generality.

Section 2 solves for the decentralized case with random search and compares it to the social optimum. Section 3 treats directed search. Section 4 concludes.

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[2]The setup resembles that of Acemoglu and Shimer (1999) that also proposed a hybrid model of Bargaining and directed search (but no wage posting as in Menzio and Shi, 2009).

2 Random Search

2.1 The Model

Production can only commence once two workers are employed at the job. Together, and only together they produce y. Here however, agents are assumed to be homogenous, infinitely lived and risk neutral. This means the production function is essentially a Leontief-production function in that the first and the second worker are perfect complements. Let us discuss the Bellman-equations:

$$rV_U = \frac{m(V,U)}{U} \frac{v_L}{v_L + v_E L}(V_L - V_U) + \frac{m(V,U)}{U} \left(\frac{v_E L}{v_L + v_E L} \right) (V_E - V_U)$$

(2.1)

where $m(V,U)$ is the constant returns to scale, decreasing returns in both arguments, matching function. V is the total number of vacancies and U is the unemployed. r is the common discount rate. V_U is the net present value of being unemployed. Assuming no unemployment benefits its value is derived from potential employment and the search effort that is assumed to be costless for the worker. Either the worker gets the value V_L because he finds an employer who has not yet found another team member, with the

probability $m(V, U)/U$ times the probability of being an opening for a first slot $v_L/(v_L + v_E L)$ (v_L being the number of vacancies for the first slot and v_E the number of vacancies for the second slot), or he/she gets V_E because the employer has another worker and can start production, with probability $m(V, U)/U$ times the probability of this being an opening for a second slot $v_E L/(v_L + v_E L)$. V_L is:

$$rV_L = w_L + \frac{v_E}{v_L + v_E L} m(V, U)(V_{EL} - V_L) + s(V_U - V_L) \qquad (2.2)$$

w_L is the wage rate of being in the loop. The probability of finding a second worker is $(v_E L/(v_L + v_E L))m(V, U)$ since once a worker is in a loop his probability of finding a second worker and obtaining V_{EL}, is the same as for the employer who posts v_E vacancies. The last term reflects that with Poisson-rate s the match is destroyed and the worker goes back unemployment while the employer goes back to not having any workers. Once a productive match has been made, workers get depending on when they joined:

$$rV_E = w_E + s(V_U - V_E) \qquad (2.3)$$

$$rV_{EL} = w_{EL} + s(V_U - V_{EL}) \qquad (2.4)$$

7

w_E and w_{EL} are the wages of being productively employed immediately and after having been in the loop. V_V is the value of a vacancy, $V_{\frac{1}{2}}$ the value of having one worker and J_E the value of having two workers.

$$rV_V = \frac{m(V,U)}{v_L + v_E L} V_{1/2} - k \tag{2.5}$$

$$rV_{1/2} = -w_L + v_E \frac{m(V,U)}{v_L + v_E L}(J_E - V_{1/2}) - v_E k - sV_{1/2} \tag{2.6}$$

$$rJ_E = y - w_E - w_{EL} - sJ_E \tag{2.7}$$

Employers post vacancies for a loop v_L until there is no profit from doing so. Therefore we have the free entry condition $V_V = 0$. Once an employer has obtained a first worker, v_E vacancies will be posted until there is no profit. The total number of vacancies posted at a time will be $v_L + v_E L$. We also need to define the flows between employment E, Unemployment U and being in the Loop L. The total number of workers is a large number N. The number of unemployed U and workers in the loop L evolve according to:

$$\dot{U} = s(L + 2E) - m(V,U) = 0 \tag{2.8}$$

$$\dot{L} = m(V,U)\frac{v_L}{v_L + v_E L} - sL - m(V,U)\frac{v_E L}{v_L + v_E L} = 0 \tag{2.9}$$

Together U and L define E, the number of workers in productive employment.

$$\dot{E} = 2m(V,U)\frac{v_E L}{v_L + v_E L} - s2E \qquad (2.10)$$

These equations are the equivalent of the Beveridge curve in the standard model. To stay as close as possible to the standard DMP model the rents will be divided according to the Nash Bargaining solution. Formally, when an employer meets a worker and has not another worker in the loop the wage must satisfy:

$$argmax(V_{1/2} - V_V)^\mu (V_L - V_U)^{1-\mu} \qquad (2.11)$$

When a second worker arrives the bargaining problem for three parties is:

$$argmax(J_E - V_{1/2})^{1-\phi_1-\phi_2}(V_E - V_U)^{\phi_1}(V_{EL-V_L})^{\phi_2} \qquad (2.12)$$

Examination of equation (2.5) yields that $V_{1/2} = k/q(\theta)$ where q(.) is $m(V,U)/(v_L + v_E L)$ and θ is $(v_L + v_E L)/U$. From (2.6) we get $(m(V,U)/(v_L + v_E L))(J_E - V_{1/2}) = k$, using $V_{1/2} = k/q(\theta)$ implies:

$$J_E = 2\frac{k}{q(\theta)} \qquad (2.13)$$

Using (2.6) and applying the previous results, we get:

$$w_L = -\frac{(s+r)k}{q(\theta)} \qquad (2.14)$$

This means the first employee that meets a given employer has to pay the employer to take him on and post vacancies. This is because in this stage no production takes place but still rent has to be shared. Since the utility from $V_{1/2}$ must be $k/q(\theta)$ but adding another vacancy yields 0 net returns by the free entry condition the looped employee must compensate for this. This is only acceptable to the employee because the probability of entering productive work is so much higher because the employer posts many v_E vacancies. Now, this result should of course not be taken literally. In reality a positive wage will be paid even for an employee in something of a waiting position, but this wage might be below the reservation wage. Since the employee expects a promotion to a much better job in the future. Using (2.14) and (2.11) and (2.12) we can solve for V_L given in (2.2). This yields:

$$rV_L = -\frac{(s+r)k}{q(\theta)} + \frac{v_E q(\theta)\phi_2 k}{(1-\phi_1-\phi_2)q(\theta)} - s\frac{\mu k}{(1-\mu)q(\theta)} \qquad (2.15)$$

Next, we plug V_L into the FOC for (2.11) and get:

$$rV_U = \frac{kv_E\phi_2}{1 - \phi_1 - \phi_2} - \frac{k(r+s)}{q(\theta)(1-\mu)} \qquad (2.16)$$

Using unemployment and the solutions for $V_{\frac{1}{2}}$ and J_E we have:

$$rV_U = \frac{\mu kv_L}{(1-\mu)U} + \frac{v_E Lk\phi_1}{(1 - \phi_1 - \phi_2)U} \qquad (2.17)$$

Together they yield the first equilibrium condition:

$$\frac{(r+s)}{q(\theta)(1-\mu)} = \frac{v_E\phi_2}{(1 - \phi_1 - \phi_2)} - \frac{\mu v_L}{(1-\mu)U} - \frac{v_E L\phi_1}{(1 - \phi_1 - \phi_2)U} \qquad (2.18)$$

The equation represents the decision of an employer to post vacancies v_L taking the number of v_E as given. The LHS is $(s+r)$ times the total surplus in the first stage W_1 (to be discussed later). The first term on the RHS represents the gain the first employee gets once a second worker is found. The second term represents what the employer will have to give up in terms of rent to the first employee in the first stage divided by the number of unemployed so representing the chance of each unemployed to receive it once a match is made for a loop contract. The third represents what the

11

second employee receives. Using V_E and V_{EL}, the solution for V_U, V_L and the FOCs of Nash Bargaining, we get:

$$w_E = \frac{(s+r)\phi_1 k}{(1-\phi_1-\phi_2)q(\theta)} + \frac{\mu k v_L}{(1-\mu)U} + \frac{v_E L \phi_1 k}{(1-\phi_1-\phi_2)U} \qquad (2.19)$$

and

$$w_{EL} = \frac{(s+r)\phi_2 k}{(1-\phi_1-\phi_2)q(\theta)} - \frac{(s+r)k}{q(\theta)} + \frac{v_E \phi_2 k}{1-\phi_1-\phi_2} \qquad (2.20)$$

as well as:

$$\frac{(s+r)k}{q(\theta)} = y - \frac{(s+r)(\phi_1+\phi_2)k}{(1-\phi_1-\phi_2)q(\theta)} - \frac{\mu k v_L}{(1-\mu)U} - \frac{v_E L \phi_1 k}{(1-\phi_1-\phi_2)U} - \frac{v_E \phi_2 k}{(1-\phi_1-\phi_2)}$$

$$(2.21)$$

This condition is for an employer with a looped employee. It determines the optimal number of v_E for any given U, L and v_L. The LHS represents the gain of the employer from a match that results in immediate production. This is equal to the produce y minus the rent that needs to be given up to both employees which is the second term but also minus a measure of the respective reservation utilities of both employees represented in the third and fourth term. The last term on the RHS is the utility the looped employee got from being in the loop. With (2.18), (2.21) and the flow equations (2.8) and (2.9) we define equilibrium:

Definition 1 *With random matching for 2 workers a decentralized equilibrium consists of v_E, v_L, L and U with (2.18),(2.21), (2.8) and (2.9) satisfied.*

2.2 Optimality

To find the optimal allocation, we set up the Hamiltonian and use two of the flow equations as constraints, the evolution of looped employees and of employment.

$$\mathcal{H} = \left(\frac{y}{2E} - Vk\right)e^{-rt} + \lambda_1 e^{-rt}(-m(V,U)p - sL + m(V,U)(1-p)) + \lambda - 2e^{-rt}(2m(V,U)p - 2sE)$$

(2.22)

We have used E instead of U and expressed the division among v_L and v_E as the ratio $p = v_E L/(v_L + v_E L)$. The λ are costate variables. Using:

$$\frac{\partial \mathcal{H}}{\partial p} = 0 \tag{2.23}$$

$$\frac{\partial \mathcal{H}}{\partial v} = 0 \tag{2.24}$$

$$-\frac{\partial \mathcal{H}}{\partial L} = -\dot{\lambda}_1 + r\lambda_1 \tag{2.25}$$

$$-\frac{\partial \mathcal{H}}{\partial E} = -\dot{\lambda}_2 + r\lambda_2 \tag{2.26}$$

13

and solving for the steady state, we get ($\partial m/\partial x = \partial m(V,U)/\partial x$ for brevity)

$$-k + \lambda_1(-\frac{\partial m}{\partial v}p + \frac{\partial m}{\partial v}(1-p) + \lambda_2(2\frac{\partial m}{\partial v}p)) = 0 \qquad (2.27)$$

$$\lambda_1(-m(V,U) - m(V,U)) + \lambda_2(2m(V,U) = 0 \qquad (2.28)$$

which leads to:

$$\lambda_1 = \lambda_2 \qquad (2.29)$$

which is already a striking result. The surplus form the first mach must exactly equal the surplus from the final match. Using this, one can obtain:

$$\lambda_1 = \lambda_2 = \frac{k}{\frac{\partial m}{\partial v}} \qquad (2.30)$$

This optimality condition where a social planner would stop posting vacancies is in contrast to where the employers stop posting vacancies, i.e: $V_{1/2} = k/q(\theta)$ and $J_E = 2k/q(\theta)$. This shows in the market equilibrium we have the agents not taking into account the negative congestion externalities not only on the other agents like themselves but also on employers in the other stage of search, because each vacancy that gets posted decreases the chance of any given vacancy to attract a worker. They do not take into account the positive externality on the laborers in the economy by increasing

14

their chance of matching. Next:

$$- \dot{\lambda_1} + r\lambda_1 = -(\lambda_1(\frac{\partial m}{\partial U}p - \frac{\partial m}{\partial U}(1-p) - s)) + \lambda_2\frac{\partial m}{\partial U}p) \qquad (2.31)$$

Using the previous result and assuming we are in the steady state, we get:

$$(r-s) = \frac{\partial m}{\partial U} \qquad (2.32)$$

This means that at the optimum the value of an additional match per additional unemployed in the matching function must be equal to $r - s$. This means that $r > s$. Essentially, at the optimum we need to take the risk of losing the match in the looped stage into account. This means that more unemployed are needed to increase the number of matches to make up for the potential premature loss of matches. The FOC for E is:

$$- \dot{\lambda_2} + r\lambda_2 = \frac{y}{2} + \lambda_1(p\frac{\partial m}{\partial U} - (1-p)\frac{\partial m}{\partial U}) + \lambda_2(2p\frac{\partial m}{\partial U} - s) \qquad (2.33)$$

Which combining previous results leads to:

$$\lambda_2 = \frac{y}{2}(2r + s) = \lambda_1 \qquad (2.34)$$

15

and:

$$\frac{\partial m}{\partial V} = \frac{2k(2r + s)}{y} \qquad (2.35)$$

Definition 2 *The optimal allocation of the random search model for teams is characterized by p, V, U and L such that (2.34) and (2.35) as well as (2.8) and (2.9) are satisfied. Also $r > s$ for the from the relationship (2.32) is needed.*

2.3 Implementing the Optimal Allocation

The next step is to see if there are values for the parameters of the model such that the competitive equilibrium coincides with the optimal allocation. Define:

$$W_1 = V_{1/2} + V_L - V_U \qquad (2.36)$$

$$W_2 = J_E + V_{EL} + V_E - V_{1/2} - V_L - V_U \qquad (2.37)$$

as total surplus generated in the first and second stage respectively. They imply:

$$(r + s)W_1 = \frac{v_E \phi_2 k}{1 - \phi_1 - \phi_2} - \frac{v_L \mu k}{U(1 - \mu)} - \frac{v_E L \phi_1 k}{U(1 - \phi_1 - \phi_2)} \qquad (2.38)$$

and

$$(r + s)W_2 = y - \frac{v_E \phi_2 k}{1 - \phi_1 - \phi_2} - \frac{v_L \mu k}{U(1 - \mu)} - \frac{v_E L \phi_1 k}{U(1 - \phi_1 - \phi_2)} \qquad (2.39)$$

From the bargaining solution we get that $k = (1 - \mu)q(\theta)W_1$. Plugging this into equation (2.34), we get:

$$\lambda_1 = \frac{(1 - \mu)q(\theta)W_1}{\frac{\partial m}{\partial v}} \qquad (2.40)$$

Substituting this into (2.24) yields:

$$2(r + s)\frac{(1 - \mu)W_1}{\eta_v} = y - \frac{(1 - \mu)W_1}{\eta_v}(\frac{\partial m}{\partial U} + s)2 \qquad (2.41)$$

17

From this we subtract (2.42) to obtain:

$$(r+s)\frac{2(1-\mu)}{\eta_v - 1} = \frac{y}{W_1} - \frac{2r(1-\mu)q(\theta)}{\frac{\partial m}{\partial v}} - \frac{v_E\phi_2(1-\mu)q(\theta)}{(1-\phi_1-\phi_2)} + \qquad (2.42)$$

$$+\frac{v_L\mu q(\theta)}{U} + \frac{v_E L\phi_1 q(\theta)(1-\mu)}{U(1-\phi_1-\phi_2)} \qquad (2.43)$$

using $\partial m/\partial U = r - s$ which must hold at the optimal allocation to make

sure the decisions of the workers are optimal and rearranging we get:

$$\frac{(2r+s)2(1-\mu)}{\eta_v} - (r+s) = \frac{y}{W_1} - \frac{v_E\phi_2(1-\mu)U}{(1-\phi_1-\phi_2)V\eta_U} + \frac{v_L\mu}{V\eta_U} + \frac{v_E L\phi_1(1-\mu)}{V(1-\phi_1-\phi_2)\eta_U}$$

$$(2.44)$$

Recognizing the last three terms on the RHS can be re-expressed using (2.19):

$$\frac{-U(r+s)}{\eta_U V q(\theta)} = -1 \qquad (2.45)$$

Which is with $r - s = \frac{\partial m}{\partial U}$ equal to -1. So we get:

$$(1-\mu) = \frac{\lambda_1 \eta_U}{W_1} \qquad (2.46)$$

If $\lambda_1 = W_1$, this is exactly the Hosios condition. It needs to be adapted for

the fact that the first matching does not yield an immediate production so if

for instance W_1 is smaller than λ_1 the bargaining weight for the first match

18

for the employer must be larger than in the standard case to compensate for the fact that this match does not yield an immediate benefit in terms of production. Since $k = (1 - \mu)q(\theta)W_1 = (1 - \phi_1 - \phi_2)q(\theta)W_2$ a similar analysis can be made for W_2 and λ_2. It yields after similar rearrangement as above:

$$\frac{(2r + s)(1 - \mu)}{\eta_v} = \frac{(r + s)2(1 - \mu)}{(1 - \phi_1 - \phi_2)} - r - s \tag{2.47}$$

Rearranging we get:

$$\eta_v = \frac{(1 - \phi_1 - \phi_2)(1 - \mu)(2r + s)2)}{(\phi_1 + \phi_2 - \mu)(r + s)} \tag{2.48}$$

Using both of these results gives:

$$W_1 = \frac{(1 - \phi_1 - \phi_2)y}{(\phi_1 + \phi_2)(r + s)}, W_2 = \frac{(1 - \mu)y}{(\phi_1 + \phi_2)(r + s)} \tag{2.49}$$

Now, if the competitive equilibrium is going to be equal to the optimal equilibrium we need $W_1 = W_2 = \lambda_1 = \lambda_2$ since not only the marginal decisions must be optimal but also the levels of activity. This leads to:

$$\phi_1 + \phi_2 = \mu \tag{2.50}$$

19

but then the denominator of (2.47) is not defined, establishing the following:

Result 1 *In a random matching framework with ex-post bargaining and the necessity of two workers it is impossible equalize the decentralized equilibrium from definition 1 with the optimal allocation given in definition 2.*

This means we cannot implement the optimal allocation by matching the Hosios condition. The result echoes Acemoglu and Shimer(1999) in Proposition 2. There decentralization was impossible due to ex ante investment. Wages need to optimize three decision margins, the first entry and how many vacancies to open once the employer has found a first employee as well as the decisions of the laborers. This cannot be achieved because we can adapt the bargaining weights to optimize the entry decisions at the margin, but not at optimal levels.

The result that optimality is infeasible with random search is because the externalities are along more dimensions than parameters can adapt for. For workers there are externalities they exert on others when entering the labour market. First, they exert a negative externality on other unemployed due to the crowding effect. Second, they exert a positive externality on employers posting v_L vacancies. Thirdly they exert a positive externality on

employers posting v_E vacancies and on attached workers. The same for employers. A v_L posting employer exerts a positive externality on the other side of the market on the unemployed and at the same time a negative externality on other v_L-employers and the v_E -employers and attached workers as well. A v_E-employer exerts the same negative externalities. Table 1 summarizes:

Table 1	Unemployed	Firm V_L	Firm $V_{1/2}$
Unemployed	Congestion externality(-)	Thin market externality(+)	Thin market externality(+)
Firm V_L	Thin market externality(+)	Congestion externality(-)	Congestion externality(-)
Firm $V_{1/2}$	Thin market externality(+)	Congestion externality(-)	Congestion externality(-)

Each of the three decision makers is exerting three externalities upon others. But the bargaining weights can only adapt on two dimensions. We can for example adapt the bargaining weights such that the v_L posting employers take into account externalities they exert on unemployed and on other employers seeking workers for an initial match and the v_E posting employers can internalize the externalities on unemployed and other employers searching for a second worker. This would yield the standard Hosios conditions (Hosios, 1990). However, if we internalize these externalities, we cannot in-

ternalize the negative externalities a v_L and v_E employers inflict on each other.

3 Directed Search

3.1 The Modified Model

To see why we cannot decentralize the optimal allocation in the framework of completely random search it is instructive to look at what happens if we allow for some degree of directed search. To this end the previously presented model will be modified to allow unemployed agents to choose for what kind of slot to apply for. This is not the kind of directed search model that is dealt with in the bulk of the literature, where wages are posted, or entry fees are posted to join a club (Moen, 1997). Rather, this is a hybrid model, to put it in the words of Acemoglu and Shimer(1999), and represents only a minimal departure from the standard Mortensen-Pissarides Framework. We still have ex post bargaining.This sidesteps the issue of commitment and makes the resulting model more comparable to the one of random search.

The important addition is that workers can now choose the probability

which market segment to join. This probability is going to be defined as $a \in [0, 1]$, where a is the probability of applying for a job with a completely unmatched employer. We can interpret this as an individual probability or as the proportion of workers applying for one kind of job or the other. This leads to modified conditions for the value of unemployment:

$$rV_{UL} \quad = \quad 0 + \frac{m_L(v_L, aU)(V_L - V_{UL})}{aU} \tag{3.1}$$

$$rV_{UE} \quad = \quad 0 + \frac{m_E(v_{EL}, (1-a)U)(V_E - V_{UE})}{(1-a)U} \tag{3.2}$$

For an equilibrium these need to be equal otherwise arbitrage will take place. m_L and m_E are the respective matching functions for the market for looped slots and for second slots. In m_L only v_L and the proportion aU feature and in m_E only v_{EL} and $(1-a)U$.

$$rV_{UL} = rV_{UE} = rV_U \tag{3.3}$$

This further changes the equations for V_L, $V_{1/2}$ and V_V

$$rV_L = w_L + \frac{v_E m_E(v_E L, (1-a)U)(V_{EL} - V_L)}{v_E L} + s(V_U - V_L) \qquad (3.4)$$

$$rV_{1/2} = -w_L + \frac{v_E m_E(v_E L, (1-a)U)(J_E - V_{1/2})}{v_E L} + s(V_V - V_{1/2}) - (3.5)$$

$$rV_V = \frac{m_L(v_L, aU)(V_{1/2} - V_V)}{v_L} - sV_V - k \qquad (3.6)$$

The Bellman-equations for V_E, V_{EL} and J_E are reproduced for convenience:

$$rV_E = w_E + s(V_{UE} - V_E) \qquad (3.7)$$

$$rV_{EL} = w_{EL} + s(V_{UL} - V_{EL}) \qquad (3.8)$$

$$rJ_E = y - w_E - w_{EL} - sJ_E \qquad (3.9)$$

Again V_V yields:

$$V_{1/2} = \frac{k}{q_L(\theta_L)} \qquad (3.10)$$

$m_L/v_L = q_L(\theta_L)$ and $\theta_L = v_L/aU$ and $q_E(\theta_E)$ and θ_E are defined analogously. Also:

$$J_E = \frac{k}{q_E(\theta_E)} + \frac{k}{q_L(\theta_L)} \qquad (3.11)$$

Already we can see a difference to the random search variant. The value of a half filled vacancy $V_{1/2}$ is determined by the probability of finding an

24

applicant conditional on being in the market for initial job offers while that of J_E is determined by the sum of two terms related to the applicant finding rates in both markets. We can also see that if the markets were to be merged again, these expressions reduce to the expressions for the random search model. Similar to the result above we have:

$$w_L = \frac{-(r+s)k}{q_L(\theta_L)} \tag{3.12}$$

The looped worker pays the employer (or accepts a wage below reservation wage) to induce the employer to search for a second employee. Using the first order conditions from Nash-Bargaining, after rearrangements we get that:

$$V_{UE} = \frac{\theta_E k \phi_1}{(1 - \phi_1 - \phi_2)r} \tag{3.13}$$

Substituting this into the equation for V_E we get:

$$w_E = \frac{(r + s + p(\theta_E))k\phi_1}{(1 - \phi_1 - \phi_2)} \frac{1}{q(\theta_E)} \tag{3.14}$$

The weights for the first stage of bargaining do not appear as the directed search isolates the bargaining processes at the different stages and the influence of the negotiation weight for the already looped labourer only features

in as much as it decreases the pie to be shared. For w_{EL} we have:

$$w_{EL} = \frac{(s+r)\phi_2 k}{(1 - \phi_1 - \phi_2)q_E(\theta_E)} + \frac{v_E k \phi_2}{(1 - \phi_1 - \phi_2)} - \frac{(r+s)K}{q_L(\theta_L)} \qquad (3.15)$$

Continuing to solve the model we get the first novel condition:

$$\frac{\phi_1 v_E L}{(1 - \phi_1 - \phi_2)(1 - a)} = \frac{\mu v_L}{(1 - \mu)a} \qquad (3.16)$$

This condition determines the a chosen by the labourers for any given combination of L, v_L and v_E. If for example the number of v_L increases the RHS increases, so a must increase to equalize the condition again. The probability of being matched conditional of applying for a v_L vacancy is now increased for any given number of unemployed in that market. This means that more workers will apply for a newly opened production site and try to become the first matched worker. The next condition is equivalent to (2.18):

$$\frac{(r+s)}{(1 - \mu)q_L(\theta_L)} = \frac{v_E k \phi_2}{(1 - \phi_1 - \phi_2)} - \mu \theta_L k (1 - \mu) \qquad (3.17)$$

The LHS is similar to the random search analogue , but the RHS shows some differences. We do not have a term for the bargaining strength of the worker should he be the second to join a match, since in this set up the worker has

opted to not apply for such a match and hence has 0 probability of finding such a slot. For subsequent analysis (3.17) is described in the $v_L - U$ space as an increasing and concave function. In the $v_L - L$ space (3.17) is always satisfied since L does not feature. This is because the decision to post a v_L vacancy is independent of the other market. The condition analogous to (2.19) is:

$$\frac{(r+s)k}{q_E(\theta_E)} = y - \frac{\theta_E k \phi_1}{(1-\phi_1-\phi_2)} - \frac{(s+r)(\phi_1+\phi_2)k}{(1-\phi_1-\phi_2)q_E(\theta_E)} - \frac{v_E k \phi_2}{(1-\phi_1-\phi_2)} \quad (3.18)$$

Again, the different matching stages are disentangled. What happens in the other market matters in so far as it alters the number of unemployed. The reservation utilities of the workers in the match still need to be taken into account and the LHS still represents the gain the employer makes from the match. The second term on the RHS represents the reservation utility of the second worker to join the match and the last the reservation utility of the first worker to join. The third term on the RHS represents the rent that needs to be given to both workers. Finally, we define the Beveridge Curves:

$$\dot{U} = s(2N - L - 2U) - m_E(v_E L, (1-a)U) - m_L(v_L, aU) \quad (3.19)$$

$$\dot{L} = m_L(v_L, aU)sL - m_E(v_E L, (1-a)U) \qquad (3.20)$$

$$\dot{E} = 2m_E(v_E, (1-a)U) - 2s(N-L-U) \qquad (3.21)$$

As before, one of the above conditions is redundant.

Definition 3 *With directed search, the equilibrium is defined by the a, v_E, v_L, U and L such that (3.16), (3.17),(3.18),(3.19) and (3.20) are satisfied.*

3.2 Optimality in Directed Search

Again, to derive the optimal allocation we have to adapt the procedure from section 2 for the now separate labour markets. The Hamiltonian is:

$$\mathcal{H} = (\frac{y}{2E} - k(v_L + v_E L)e^{-rt} + \lambda_1 e^{-rt}(-m_E(v_E L, (1-a)U)) - sL + m_L(v_L, aU)) +$$

$$(3.22)$$

$$\lambda_2 e^{-rt}(2m_E(v_E L, (1-a)U - 2 - sE)$$

28

The first order conditions are:

$$\frac{\partial \mathcal{H}}{\partial v_E} = 0 \tag{3.23}$$

$$\frac{\partial \mathcal{H}}{\partial v_L} = 0 \tag{3.24}$$

$$\frac{\partial \mathcal{H}}{\partial a} = 0 \tag{3.25}$$

$$\frac{\partial \mathcal{H}}{\partial L} = -\dot{\lambda}_1 + r\lambda_1 \tag{3.26}$$

$$\frac{\partial \mathcal{H}}{\partial E} = -\dot{\lambda}_2 + r\lambda_2 \tag{3.27}$$

These FOC's can be rearranged to yield:

$$\lambda_1 = \frac{k}{\frac{\partial m_L(v_L, aU)}{\partial v}} \tag{3.28}$$

$$\lambda_2 = \frac{k}{\frac{2\partial m_L(v_L, aU)}{\partial v}} + \frac{k}{\frac{2\partial m_E(v_E, aU)}{\partial v}} \tag{3.29}$$

$\partial m_i / \partial v$ is the derivative of the matching function w.r.t. vacancies, either v_L or v_{EL}. The link between the first and second stage is loosened. In the first market the marginal value of an additional vacancy matters and in the second market the marginal impact of a vacancy and the value of an initial match matter, since with each productive match we are destroying a looped relationship. The new condition for a allows arbitrage of the returns of joining the labour market for initial openings or half filled openings. Combining

29

the FOCs for v_E and v_L with that for a we obtain:

$$\frac{\frac{\partial m_E}{\partial v}}{\frac{\partial m_E}{\partial U}} = \frac{\frac{\partial m_L}{\partial v}}{\frac{\partial m_L}{\partial U}} \tag{3.30}$$

This condition means the relative impact of vacancies and unemployed must be equal across the two markets. a will be adapted so that this condition holds for the optimum. Rearrangement of the FOCs for L and E yields:

$$(r - s)\lambda_1 = kv_E + \lambda_1 \left(\frac{\partial m_L}{\partial U} - \frac{\partial m_L}{\partial V} v_E \right) \tag{3.31}$$

$$(r + 2s)\lambda_2 = \frac{y}{2} + \lambda_2 \left(-\frac{\frac{2\partial m_E}{\partial U} \frac{\partial m_L}{\partial V}}{\frac{\partial m_L}{\partial V} + \frac{\partial m_E}{\partial V}} \right) \tag{3.32}$$

The first of these conditions together with (3.28) implies:

$$r - s = \frac{\partial m_L}{\partial U} \tag{3.33}$$

which means more unemployed must join the market for loops since the risk of losing a looped worker before production needs to be compensated. Here we can clearly see that this condition, which also featured in the previous section, is relevant to the first market but not for the second market. There the risk of separation is already fully taken into account by discounting the

produce, but in the first market there is the additional risk of losing the match before it becomes productive. These two equations are stating the value of the loop must develop taking into account the marginal impact of unemployed and vacancies of the first market. However, v_E is crucial in determining the value of joining a loop for a worker. The value of the first match must incorporate the loss of one unemployed in the matching function, one less vacancy competing for the unemployed that now the search process for the second stage has started and the cost of posting v_E vacancies per looped worker.

The condition (3.32) means, the second slot the value needs to evolve while taking into account the additional production per worker, the marginal impact of an additional unemployed on the matching function for the second stage and the marginal impact of an additional vacancy on both markets, since when a productive match is made not only is a worker subtracted from the stock of unemployed and thus no longer available, but there is also one less looped worker and less competition for unemployed workers. This double loss of income must be taken into account in the case of a separation, hence s features twice.

Definition 4 *In the directed search framework, the optimal allocation is*

31

characterized by values of v_E, v_L, a, U and L such that (3.30),(3.31),(3.32),

(3.19) and (3.20) are satisfied. Again, $r > s$ is required.

3.3 Implementing Optimality

Can we decentralize the optimal allocation? Table 2 contrasts the the optimal and the decentralized solution:

$W_1 = \dfrac{k}{(1-\mu)\, q_L\,(\theta_L)}$	$\lambda_1 = \dfrac{k}{\partial m_L(v_L, aU)/\partial v}$
$(r+s)\,W_1 = kv_E + \dfrac{\varphi_2\, kv_E}{(1-\varphi_1-\varphi_2)} - \dfrac{\mu kv_L}{(1-\mu)aU}$	$(r-s)\,\lambda_1 = kv_E + \lambda_1 \left(\dfrac{\partial m_L}{\partial U} - \dfrac{\partial m_L}{\partial V}\, v_E \right)$
$W_2 = [\dfrac{k}{q_E\,(\theta_E)} + \dfrac{k}{q_L\,(\theta_L)}]/(\,1 - \varphi_1 - \varphi_2)$	$\lambda_2 = \dfrac{k}{2\,\partial m L(v_L, aU)/\partial v} + \dfrac{k}{2\,\partial mE(v_E L, aU)/\partial v}$
$(r+s)\,W_2 = y - kv_E - \dfrac{\varphi_2\, kv_E}{(1-\varphi_1-\varphi_2)} - \dfrac{\theta_E\, k\varphi_1}{(1-\varphi_1-\varphi_2)}$	$(r+2s)\,\lambda_2 = \dfrac{y}{2} + \lambda_2\,(- \dfrac{2\,\partial m_E/\partial U \times \partial m_L/\partial V}{\partial m_L/\partial V + \partial m_E/\partial V}\,)$

This version of the Hosios condition equalizes shadow value and welfare:

$$(1 - \mu) = \eta_{Lv} \tag{3.34}$$

$$(1 - \phi_1 - \phi_2) = 2\eta_{Lv} \tag{3.35}$$

$$(1 - \phi_1 - \phi_2) = 2\eta_{Ev} \tag{3.36}$$

What remains is to see if the values of ϕ_1 and ϕ_2 can be adapted so the dynamic equations also coincide. The same methodology is employed as in the previous section. This time no contradiction emerges, rather two conditions the parameters ϕ_1 and ϕ_2 must satisfy. We can derive:

$$\phi_1 = \frac{\frac{(1-2s+4r)\partial m_E}{(r-s)\partial U} - v_E q_E}{\frac{m_E}{U(1-a)} - v_E q_E} \qquad (3.37)$$

This value is within $[0, 1 - 2\eta_{Ev}]$. For this to be positive $1/U(1-a) > 1/L$ and the numerator must be positive or $1/U(1-a) < 1/L$ and the numerator negative. A worker entering the market for second slots must receive a compensation proportional to the marginal input he/she has, scaled by discounting and the probability of separation $(1 - 2s + 4r)/(r - s)$. For ϕ_2 we have:

$$\phi_2 = \frac{L\phi_1}{U(1-a)2\eta_{Lv}} + \frac{(r+s)\partial m_L}{v_E \partial v} - 1 = 1 - 2\eta_{Ev} - \phi_1 \qquad (3.38)$$

For this to even be possible the first two terms must together exceed 1. One way to think of this is to realize that the following (necessary, not sufficient) condition needs to be true:

$$1 - 2\eta_{Ev} - \phi_1 - \phi_2 \geq 0 \qquad (3.39)$$

33

Which is already more stringent than the Hosios condition. This establishes the following result:

Result 2 *In a directed matching framework with ex-post bargaining and the necessity of two workers it is possible equalize the decentralized equilibrium from definition 3 with the optimal allocation given in definition 4, but only if equations (3.34)-(3.38) hold, conditions much stronger than the Hosios Condition.*

With this knife-edge result, while there is a positive probability of being possible, it remains dubious that the Hosios Condition can achieve optimality. It is in principle possible because now there is an additional degree of freedom to adapt for the externalities, discussed in section 2. The separation of the effects of v_L and v_E on the other markets matching function is crucial as now we do not have the negative externality of one kind of employer upon the other. Table 3 gives externalities left for the bargaining weights to correct for.

34

Table 3	Unemployed U_L	Unemployed U_E	Firm V_L	Firm $V_{1/2}$
Unemployed U_L	Congestion externality(-)	/	Thin market externality(+)	/
Unemployed U_E	/	Congestion externality(-)	/	Thin market externality(+)
Firm V_L	Thin market externality(+)	/	Congestion externality(-)	/
Firm $V_{1/2}$	/	Thin market externality(+)	/	Congestion externality(-)

The workers are internalizing the negative externality they exert to a larger degree than previously through their choice of a. This removes some externalities and directs the effects of the remaining ones, the congestion externalities among the same kind of employers and the thin market externalities from employers to only the workers in their designated market and the negative externalities among the workers, can be corrected with the appropriate choice of bargaining weights. This is what adding another stage in the hiring process amounted to. In the random search framework externalities were added and in the directed search version they were partially removed.

4 Conclusion

This paper demonstrates that the Hosios condition generally does not hold in a multi-worker setup. With random search even if we allow for simple Nash Bargaining and not for Stole and Zwiebel type bargaining it cannot hold. It may hold in a knife edge case with directed search, but this puts one more coincidence on top of another. Thus, once we are interested in modeling more complex labour markets it becomes not only unlikely, but in most cases impossible to implement optimality. This illustrates in very stark terms just how special a case the Hosios condition really is. Not only do the bargaining weights need to almost magically coincide with the elasticities of the matching function with respect to vacancies and unemployed, the production function cannot be of any different from a linear AL style. What the models also illustrate is that if the phenomenon of hiring workers specifically for teamwork is sufficiently important, not only the number of openings in any labour market will be of interest, but also the nature of each opening because this might give us an indication if the filling of that vacancy increases or decreases labour market tightness.

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Работа вводит комплементарность в классическую поисковую модель рынка труда Мортенсена – Писсаридеса – Дайамонда. В частности, двое рабочих требуются для выполнения производственной задачи. Предпосылка торга по Нэшу сохраняется для формирования среды, благоприятной для выполнения условия Хосиоса. В модели с непрерывным временем возникает необходимость держать одного работника во время поиска второго. Мы демонстрируем, что это ведет к ряду дополнительных экстерналий, которые невозможно скорректировать для использования условий аналогичных условию Хосиоса в модели случайного поиска. В модели направленного поиска мы показываем, что эти условия являются достаточными для оптимальности только в исключительных случаях.

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(на английском языке)