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VULNERABILITY OF VOTING PARADOXES AS A CRITERIA FOR VOTING PROCEDURE SELECTION

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VULNERABILITY OF VOTING PARADOXES AS A CRITERIA FOR VOTING PROCEDURE SELECTION[2]

Correct aggregation of individual preferences into collective one is central problem of nowadays Social Choice theory. After the Arrow's and Gibbard–Satterthwaite impossibility theorems it became clear that our desire to justify an electoral procedure is doomed to failure. At the same time a lot of scholars continued exploring different properties of existing voting rules and constructing the new ones. Contemporary research in this area explore two main properties of aggregation procedures — their degree of manipulability and computational complexity of manipulation. Quantitative evaluations of these properties tend to be main criteria of voting procedure selection. But last decades it turned out that another threat for theory of voting is incompatibilities and unexpected outcomes of different kind, usually called paradoxes. This article provides complete systematization of voting paradoxes known for today. We also presented an attempt to formulate a complete proof of the (in)stability of seven most common used voting rules to paradoxes of any type, which had not been undertaken before. Our results show that different voting procedures are qualitatively different in the sense of vulnerability to voting paradoxes which makes reasonable to propose additional criteria of voting procedure selection and opens the gate for further quantitative research.

Keywords: Social choice, elections, voting behavior, paradoxes, voting procedures
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1. Introduction

Voting is necessary and inseparable component of any democratic system. The institution of voting provides people an opportunity to collect individual opinions into collective one. Democracy and institution of voting appeared at the very beginning of the formation of modern civilizations - in antiquity. Voting is the only tool for citizens to control the decisions taken by the authorities. Such a control can be direct (in the case of referendum) or indirect (in the case of representative's elections). For quite a long time, people believed that voting was, a priori, way to find decisions that would correspond to a common will or public interest. Until the second half of the 18th century, voting was understood as plurality or majority voting when the largest group of citizens should decide what decision is good for the society. So, we can hardly find any discussions about alternative voting procedures before 18'th century. But gradually people began to understand that voting system can be used by power to manipulate the results of elections and legitimize them.

The first significant theoretical results in the theory of voting were obtained by great 18th century French mathematicians: Jean-Charles de Borda and Marquis de Condorcet. Borda and Condorcet both faced with difficulties of aggregating results of elections with more than two candidates. Condorcet found out that sometimes it's rationally impossible to aggregate individuals' preferences into consistent collective decision. This result is known as Condorcet paradox or Paradox of Voting. The work of Borda and Condorcet on the mathematical analyses of elections and voting procedures established a new theoretical sphere of knowledge - Social Choice Theory.

This result showed that majority rule can fail some important properties of good electoral procedure and lead to the paradoxes. It reflected in a huge activity of different theorists who invented new voting procedures to avoid paradoxes. This situation was stable for more than 150 years and a lot of amazing and complicated voting rules were described, despite the fact that none of them could avoid paradoxes or other weaknesses. But in 1951 another great result in the theory of voting was obtained - Arrow's impossibility theorem or Arrow's paradox. Great American economist Kenneth Arrow showed that any voting procedure for more than two candidates which satisfies some very intuitive properties will be 'dictatorial' (See: Arrow 1951, 1963). The similar result was reflected in Gibbard–Satterthwaite theorem [Gibbard 1973, Satterthwaite 1975] that states that any voting rule for more than 2 candidates is dictatorial or manipulated e.g. susceptible to tactical voting. These results showed that all our attempts to design fair and invulnerable to any manipulations voting procedure are doomed to failure.

Since the problem of manipulability turned out to be unavoidable, justification of voting procedures splitted up into two research directions. The first direction is to accept the fact of manipulability and try to compare different procedures by their degree of manipulability — if we cannot design any procedure invulnerable to manipulations, the only way for us is to find the less vulnerable procedures. Last two decades were very effective, mathematicians and economists, in particular A.D. Taylor [Taylor, 2005], G. Pritchard and M. C. Wilson

[Pritchard, Wilson 2007] and F. Aleskerov [Aleskerov et.al, 2015] and others provided important results which show that degree of manipulability of different procedures is significantly different. Another well-developed direction is analysing of computational complexity of manipulation under different voting procedures. Results obtained in this field show that voting procedures differ in the sense of complexity of manipulation, and sometimes manipulation is NP-hard problem, i.e. in some voting procedures it's impossible to calculate the possibility of strategic voting in a reasonable amount of time [7, 16, 23]. Thus, degree of manipulation and computational complexity become main criteria of voting procedure selection. And it makes sense, but manipulation or tactic voting is not the only difficulty in the theory of voting.

Another crucial problem is various paradoxes that occur in different voting rules. As it was said before, two main events in the history of social choice theory are Condorcet paradox and Arrow's paradox. But since the Condorcet times a lot of other paradoxes of different kind were discovered and they pose a threat to our theory. At the same time, this topic is represented in academic literature very poor. The first fundamental work in this field appears only at the end of 20th century. The monograph "Voting Paradoxes and How to Deal with Them" by H. Nurmi [Nurmi, 1999] is the first attempt to systematize voting paradoxes in some proper way. But Nurmi provides only general description of voting paradoxes without exploring vulnerability of voting procedures to these paradoxes. Another fundamental works concerning voting paradoxes are "Voting Paradoxes and Group Coherence" and "Elections, Voting Rules and Paradoxical Outcomes" by W. V. Gehrlein and D. Lepelley [Gehrlein and Lepelley, 2011, 2017] appear last decade. These works provide very important results on the probabilistic evaluations of particular paradoxes occurrence in some voting procedures, impact of group coherence and analysis of Condorcet efficiency of voting procedures. But this research considers only a few most known paradoxes — paradoxes of incompatibility and particular paradoxes of monotonicity without observing full list of procedures vulnerable to them.

The main goals of this paper are to provide correct and complete list of voting paradoxes known for today, updating H. Nurmi's work and presented an attempt to formulate a complete proof of the (in)vulnerability of seven most common used voting rules to paradoxes of any type, which had not been undertaken before. This result can become a foundation for further research and gives us a hope to prove in(vulnerability) of all known procedures to all known paradoxes (which is very tricky mathematical task). Perspective results in this field must help us to judge about different voting rules' properties, obtain another criteria for voting system selection and rethink the possibility of fair aggregation of our individual preferences into the general will.

2. Main Voting Rules

When we talk about voting, it usually refers to political elections as some general voting, in which thousands and millions of people take part. In such elections the voter is usually required to mark the best candidate in the ballot and the winner of the election is determined on this basis. The universal rule of almost any political system is the acceptance of the winner by the majority of votes. However, when discussing electoral system design, some questions often arise[Volsky, 2016]:

— Should relative or absolute majority decide?

— Should we count the majority of the total number of voters or of those who participated in the voting?

— How to count abstentions (include "against all" in the ballot or not)?

These questions have been asked over centuries by Pliny the Younger in the I century and was raised again in 18'th century by Condorcet [15, 41]. Generally, voting procedures of this type are very simplified, because of their low computational complexity and high level of transparency. Unfortunately, it is always a kind of trade-off between fairness and complexity. More complicated, but more optimal procedures require too many resources and trust to electoral system. That's why most common used voting procedures today are quite simple like plurality rule or plurality with run-off. The simpler the procedure, the less likely mistakes in counting of votes are.

Another situation can be found in small group voting. Examples of such groups are various committees, commissions, boards of directors, groups of shareholders, etc. Examples of such groups are various committees, commissions, boards of directors, groups of shareholders, etc. In such groups we can get much more information about their members' preferences about alternatives. Such a detailed information about voters and their attitude to alternatives allows to use various procedures of aggregating individual preferences into a collective decision.

There are a lot of such procedures that differ in their properties, ways of aggregating and input parameters. This is the main classification of different voting rules and procedures [3]:

1. Positional procedures that take into account information about position of candidates in voters' ordering
    - 1.1. Simple Majority Rule
    - 1.2. Plurality Rule
    - 1.3. Plurality with Run-off
    - 1.4. Approval Voting
    - 1.5. Condorcet method
    - 1.6. Single Transferable Vote
    - 1.7. Inverse Plurality rule
    - 1.8. Instant-runoff voting (Ware's method)

[3] Classification of voting procedures proposed by Fuad Aleskerov and Vladimir Volsky [1, 2, 50, 55]

1.9. Coombs' method
1.10. Threshold procedure
2. Positional procedures that use the sum of the candidates' ranks voters' orderings
2.1. Borda's rule
2.2. Nanson's rule
2.3. Baldwin's rule
3. Procedures based on pairwise comparison of candidates
3.1. Direct procedures by the majority graph
3.1.1. Condorcet winner selection
3.1.2. Black's procedure
3.1.3. Amendment procedure
3.1.4. Minimal dominant subset procedure
3.1.5. Minimum non-dominated subsets procedure
3.1.6. Von Neumann–Morgenstern procedure
3.2. Procedures using a majority graph and an auxiliary scale on it
3.2.1. Copeland's method
3.2.2. Dodgson's rule
3.2.3. Young's rule
3.2.4. Procedures using the concept of the tournament's own vector
3.3. Procedures using a majority graph and auxiliary binary relation on it
3.3.1. Fishburn's voting procedure
3.3.2. Coverage relation procedure
3.3.3. Richelson's rule
3.4. Pareto procedures
3.4.1. Pareto set selection procedure
3.4.2. q-Pareto procedure
4. Approximation procedures
4.1. Kemeny's rule
4.2. Procedure for approximate triangulation of the majority tournament matrix
5. Simpson's rule

Despite the size of this list, it may not cover all existing voting rules for today, because new procedures are constantly being proposed by the scientific community and it's extremely difficult to describe them all. We will not describe all of these procedures in this text, but only most popular, although each one is important.

Before discussing certain voting rules we need to come closer to some fundamental notions in Social Choice Theory. These basic definitions and concepts are also essential to understand the nature of voting paradoxes and other obstacles of voting rule design. It's not difficult to see that any voting would be impossible without voters and their possibility to choose. The voting must result in a choice of several (at least one) winning alternatives. In the theory of voting the alternatives are often regarded as given. We will denote set of alternatives as

$A = \{a_1, a_2, a_3, ..., a_n\}$.

The other fundamental notion is voter's opinion about alternatives i.e. voter's preferences. Each voter has her preferences over all alternatives. The Social Choise theory deals with procedures for aggregating the voter preferences into collective decisions. If the alternatives are just finite sets, the voter opinions or preferences can be expressed in the form of rankings. The preference rankings show which alternative each voter ranks first, which second and so on. Let $A \succ B$ denote the fact that a voter prefers Candidate A to Candidate B.

Definition 1 *A voter's preferences on the set of all pairs of candidates are **complete** preferences if there is a preference on each of the possible pairs.*

So, every voter must have either $A \succ B$ or $B \succ A$ for all possible pairs of candidates. When an individual voter's preferences are complete, she can compare all candidates and indifference between them is forbidden. We will consider that all voters have complete and transitive preferences over alternatives, but it doesn't imply that the phenomena of individual voter indifference between candidates cannot be formalized and considered. Transitivity requires that each voter is assumed to be consistent in his rankings in the following sense: if he considers alternative A no worse than alternative B, and alternative B no worse than alternative C, then she also considers alternative A no worse than alternative C. In other words, $(A \succ B) \wedge (B \succ C) \rightarrow A \succ C$ for each voter. Transitivity of individual preferences is most common used requirement, but it is possible to justify the existence of intransitive preferences (see for example Gehrlein 1990).

Individual voter preferences on candidates that are complete and transitive are defined as *linear preference rankings*. If n is a number of alternatives, there are $n!$ possible linear preference rankings that each voter might have. Let's consider the six possible linear preference rankings that each voter might have for three-candidate elections (Table 1):

n_1	n_2	n_3	n_4	n_5	n_6
A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

Table 1: The six possible linear preference rankings on three candidates

The standard way of giving this description in the case of relatively small alternative sets is by listing the voters as columns of a table and indicating their opinions as rows so that the first row indicates the alternative ranked first, second row the alternative ranked second etc. Here, n_i denotes the number of voters that have the associated linear preference ranking on the three candidates, so that n_1 voters all have individual preferences with $A \succ B \succ C$. Transitivity requires that $A \succ C$ also. If we let n define the total number of voters, then $\sum_{i=1}^{6} n_i = n$.

Now we can shift to another important concept — a preference profile.

Definition 2 ***A preference profile*** *is a description of a voting situation so that each voter's preference ranking over the alternatives is indicated.*

Voters with identical preference rankings can be grouped together. If there are n voters and k alternatives, we obviously get a $k \times n$ table. Table 1 is an example of preference profile. It's easy to imagine that $\{n_1, ..., n_6\}$ stand for some percentage of voters.

The preference profile is conventional way to describe voting situations. We can derive two other concepts from the preference profile: the pairwise comparison matrix and the tournament matrix. If the number of alternatives is k, both of these matrices are $k \times k$-matrices in which each row and column represents an alternative. Each cell of the pairwise comparison matrix indicates the number of voters who prefer the alternative represented by the row to the one represented by the column.

If the votes in Table 1 are as follows: $n_1 = 3, n_2 = 7, n_3 = 5, n_4 = 4, n_5 = 2, n_6 = 6$, the pairwise comparison matrix derived from it will have the following form:

	A	B	C
A	-	12	15
B	15	-	12
C	12	15	-

Table 2: A Pairwise Comparison Matrix

The tournament matrix, in turn, is derived either directly from the preference profile or from the pairwise comparison matrix. It is of the same $k \times k$ dimensionality as the pairwise comparison matrix if there are k alternatives. Each cell of the tournament matrix is either 1 or o. If the alternative represented by the row is ranked higher than the alternative represented by the column by a majority of voters, then the corresponding cell has value 1. otherwise it is 0. The tournament matrix of previous example is presented in Table 3.

	A	B	C
A	-	0	1
B	1	-	0
C	0	1	-

Table 3: A Tournament Matrix

Let's list main properties of the preference profiles, tournament matrices and pairwise comparison matrices. Firstly, if we deal only with strict preferences (ties in preference rankings are forbidden), then tournament matrix is always asymmetric. In other words, '1' in i row and j column in the tournament matrix implies that there is a '0' in j row and i column of the same matrix. Secondly, given the pairwise comparison matrix we can always determine the tournament matrix in a unique way, while the opposite is not possible. Third, given a preference profile, it is always possible to construct the pairwise comparison and tournament matrices so that the latter is asymmetric. Given an asymmetric tournament matrix and a fixed number of voters it is, however, not necessarily possible to construct a profile of complete and transitive preference relations for this number of voters that would correspond to the tournament matrix. But according to McGarvey's Theorem, if one is allowed to increase the number of voters, then any asymmetric tournament matrix can be translated into a preference profile consisting of complete and transitive preference relations[McGarvey, 1953].

Another fundamental notions are Condorcet winner and Condorcet loser. *Condorcet winner* is an alternative that would defeat all other alternatives in pairwise comparisons. *Condorcet loser*, is vice versa an alternative that would be defeated by all others in pairwise comparisons. The winning criterion is getting strictly more than 50% of the votes. These two notions are very important. Intuitively, the optimal result of any voting is electing of Condorcet winner, because he considered to be better than any other candidate for the majority of voters. But the problem is that Condorcet winner does not always exist. Voting rules which always elect the Condorcet winner, given that such a winner exists are called *Condorcet effective*. But such a procedures don't constitute the whole list of voting rules and there are reasons for it.

The basic concept model for describing and analyzing the voting rules and paradoxes has been outlined. But what exactly is a voting rule? Actually, we can find a lot of definitions, but the number of voting rules or voting procedures is quite large, so it's extremely difficult to define them in a strict uniform formal way. Instead, in this text we will use a broad, but correct definition:

Definition 3 ***A Voting Rule*** *is function from preference profiles to alternatives that specifies the winner of the election.*

Now, when we considered the main concepts and notation we can shift to describing basic and commonly used voting rules to analyse their vulnerability to voting paradoxes of any kind.

Plurality rule is one of the basic voting procedures. This rule gives every voter one vote. When the votes are casted, the winner is the option or candidate which has been given a larger number of votes than any other.

Let's consider the following example (Table 4):

Group A (3 voters)	Group B (5 voters)	Group C (7 voters)	Group D (6 voters)
x	x	y	z
y	z	w	y
z	y	z	w
w	w	x	x

Table 4: Alternative x wins by plurality rule

In this example, alternative x gets 8 votes, alternative y gets 7 votes and alternative z gets 6 votes. Alternative x is a winner of this voting by plurality rule, because it gets more votes than any other alternative. Plurality voting is one of the most ancient voting procedures.

Plurality with run-off is similar to plurality rule, but more complicated. If any alternative is placed first by more than a half of voters, this alternative wins. It is not the case in previous table (Table 4). Alternative x gets highest number of first places (8 out of 21), but it is less than a half. In such case plurality with run-off requires to eliminate all candidates except two with highest number of first places. In our table, candidate x and y have highest number of first ranks (8 and 7 respectively), so candidates z and w are eliminated. Than these two candidates pass to the second round. Suppose that preferences of voters have not changed, so new preference profile has the following form(Table 5):

Group A (3 voters)	Group B (5 voters)	Group C (7 voters)	Group D (6 voters)
x	x	y	y
y	y	x	x

Table 5: Second round in plurality run-off

The winner according to this procedure is a candidate who surpasses the other in the number of first ranks (it is plurality rule for two alternatives). Here candidate y has 13 first ranks against 8 first ranks for candidate x, so y is the winner by plurality with run-off procedure.

Plurality with run-off is better in some sense that plurality rule, because it does not allow any group less than 50% of voters to impose its will to the majority. This procedure is commonly used on presidential elections in such countries as France, Croatia, Czech Republic, Poland, Portugal, Russia, Ukraine and others.

Next voting procedure calls **Borda rule** or Borda count [Borda, 1781]. This rule is named for the 18th-century French mathematician Jean-Charles de Borda, who devised this system in 1770 . As previous voting rules, Borda count is a ranked voting system. It means that the voters are required to report their preference rankings over the alternatives. Than we need to calculate Borda scores of all alternatives as follows. Every alternative is assigned 'a' scores for each voter placed it to the lowest position. In Borda's terminology number of these scores is the degree of merit that each voter attributes each candidate. The second to last alternative gets the degree of merit of a + b, the third to last a +2b and so on. Values of a and b - must be non-negative integers, usually the values a = 0, b= 1 are used.

More formally, we can describe Borda cout as such an algorithm: If there are n alternatives, give $n-1$ scores to alternative ranked first, $n-2$ scores to alternative ranked second ,. . . , 1 point to a alternative ranked 2nd to last and 0 points to alternative ranked last. So, the Borda score of A, denoted $Bs(A)$, is calculated as follows (where $\#U$ denotes the number elements in the set U):

$Bs(A) = (n-1) \times \#\{i | i \ \ ranks \ \ A \ \ first\} +$

$(n-2) \times \#\{i | i \ \ ranks \ \ A \ \ second\} + \ldots +$

$1 \times \#\{i|i \ \ ranks \ \ A \ \ second \ \ to \ \ last\}+$

$0 \times \#\{i|i \ \ ranks \ \ A \ \ last\}$

The alternative with the largest Borda score wins. Recall the example discussed in the previous paragraphs (Table 4). We can apply Borda rule and count scores for every alternatives as follows:

BS(x) $= 3 \times 8 + 2 \times 0 + 1 \times 0 + 0 \times 13 = 24$

BS(y) $= 3 \times 7 + 2 \times 9 + 1 \times 5 + 0 \times 0 = 44$

BS(z) $= 3 \times 6 + 2 \times 5 + 1 \times 10 + 0 \times 0 = 38$

BS(w) $= 3 \times 0 + 2 \times 7 + 1 \times 6 + 0 \times 8 = 20$

Here alternative y wins, because it has the highest Borda score. This result is the same as in plurality with run-off example, but it's not always the case. It's also interesting that Borda count places alternative z to the 2nd place.

In contrast to previous voting rules, **approval voting** is not ranking method. Approval voting forces voters to think about the decision problem differently: voters are asked not to rank all alternatives, but to choose only alternatives they can approve. That is, the voter is asked which candidates are above a certain "threshold of acceptance" (Brams and Sanver 2009). This rule can be formulated briefly: Each voter selects a *subset* of the cadidates (where the empty set means the voter abstains) and the candidate with the most votes wins.

Single Transferable Vote procedure was proposed by English mathematician Thomas Wright Hill in 1819. It is used if someone needs to elect a predetermined number of winners from the list of candidates. Then a quota is established, that is, the number of votes that a candidate must receive in order to be elected.

In this procedures voters need to order their preferences ranking as in the plurality or Borda rule. At the first stage, only the first candidates in the rankings of voters are taken into account. A candidate who receives the number of first places in the rankings more than the quota is elected. If the number of votes cast for this candidate exceeds the quota, then an excess of votes is cast by lot to candidates who are in second place in the ordering of those voters who were chosen by this lot. If after this a candidate appeared who received a number of votes equal to or exceeding the quota, he is elected. If at the same time the number of votes for it exceeded the quota, the process continues until a fixed number of the best candidates is chosen in advance.

In some texts, STV is considered as purely multimember constituencies electoral procedure, in other texts it's also considered as single winner procedure. Unfortunately, in this text we will not pay attention on multi-winner procedures and systems of proportional representation, because it is very wide branch of contemporary social choice theory. We will focus on the single-winner type of STV — Alternative vote or Instant Runoff Voting (STV, AV and IRV are synonyms here, but some theorists could argue with it).

IRV procedure was proposed by American architect W.R. Ware in 1871 [Reilly, 2001] and it is sometimes called Ware's procedure. According to this procedure, the winner is a candidate who receives more than 50% of the first places in the voters' rankings. If such a candidate exists, then the procedure ends. otherwise, the candidate who has the smallest number of first places in the preference profile is excluded from the list of candidates. The procedure is repeated for a reduced list of candidates until the winner appears (receives more than 50% of the first places). STV or IRV is sequential procedure. It attempts to obtain the benefits of a two-stage voting rule, without necessity to hold two separate elections. We can illustrate this algorithm on the following example (Table 6):

(8 voters)	(7 voters)	(5 voters)	(3 voters)	(2 voters)
V	Z	W	Y	X
Y	Y	X	Z	W
X	X	Y	X	V
W	W	Z	W	Y
Z	V	V	V	Z

⇒

(8 voters)	(7 voters)	(5 voters)	(3 voters)	(2 voters)
V	Z	W	Y	W
Y	Y	Y	Z	V
W	W	Z	W	Y
Z	V	V	V	Z

⇒

(8 voters)	(7 voters)	(5 voters)	(3 voters)	(2 voters)
V	Z	W	Z	W
W	W	Z	W	V
Z	V	V	V	Z

⇒

(8 voters)	(7 voters)	(5 voters)	(3 voters)	(2 voters)
V	Z	Z	Z	V
Z	V	V	V	Z

Table 6: Sequential STV procedure

No one gets more than 50% (more than 13 first ranks), so alternative with the least number of 1st ranks is eliminated on the 1st stage. on the 2nd stage Y is eliminated. Then W is eliminated, because on the 3rd stage no one gets more than 13 votes and W has the least number of first places. only two alternatives remain on the last stage and alternative Z wins with 15 first ranks against 10 for alternative V.

Another method was proposed by American psychologist Clyde Hamilton Coombs. **Coombs rule** is another sequential or two-stage procedure as plurality with run-off or STV, which iteratively removes the candidates with the most last-place votes. It can be formulated quite simply. Each voter submits a linear preference ordering over the set of alternatives. Alternatives placed last by the most voters are step-by-step removed. The last candidate to be removed is the winner. In first example (Table 4) the winner is alternative z. In last example (Table 6) alternative Y is the winner.

The next preferences aggregation method was proposed by great British mathematician and philosopher who made a big contribution to the theory of voting — Charles Dodgson. He suggested new method of voting which is Condorcet extension (always chooses Condorcet winner if one exist) and based on pairwise comparison of alternatives. It can be formulated in a such way: Each voter submits a linear ordering over all the candidates. For each candidate, determine the fewest number of pairwise swaps needed to make that candidate the Condorcet winner. The candidate with the fewest swaps is declared the winner.

This is only a small part of the huge list of currently known aggregation procedures for individual preferences, but these procedures are the most common. At this point, the reasonable question can be asked: "Are all these procedures actually so different and does this really affect the outcome of the elections?". Perhaps inventing new and new procedures does not bring us closer to fairer elections, but only confuses? The problem is that applying of different voting rules can dramatically change the result of election. We can illustrate it by one very interesting example. It is not difficult to imagine elections with 5 candidates. Suppose that we have 5 candidates on presidential elections and 100 voters (which can be considered as 100% of voters). All voters can express their preferences over all 5 alternatives (all preferences are complete and transitive). Here is the distribution of votes(Table 7):

33 voters	16 voters	3 voters	8 voters	18 voters	22 voters
a	b	c	c	d	e
b	d	d	e	e	c
c	c	b	b	c	b
d	e	a	d	b	d
e	a	e	a	a	a

Table 7: Contradictions between different voting rules

Who is the winner of these elections? or it's better to say what result is the most fair? We cannot find an answer before deciding which voting rule to choose. Let's compare 5 different most common used rules. We will not read the results for each rule in detail, but further computations are correct. If we apply plurality rule to this preference profile the winner is a. The Borda rule define alternative b as winner. At the same time, Condorcet winner here is c. Application of Single Transferable Vote will result in the winning of d. And plurality with run-off will find alternative e as the winner. It seems weird, but the voting with 5 alternatives can result in 5 different winners depending on which voting procedure is chosen. That's why research of different properties of voting rules and search for new ones is not only interesting puzzle for mathematicians and economists, but has a real application to voting systems design. Such results cannot be ignored by anyone who interested in improving existing electoral systems.

As it was said earlier, all voting rules are manipulated or dictatorial. But some procedures are more vulnerable to manipulations than others [Aleskerov, 2017, 2015]. Some of the voting procedures requires more computational power than others, moreover calculating of winner in some voting procedures is NP-complete task as Dodgson's procedure [Bartdholdi 1989]. There are more interesting properties and differences between voting rules, but our goal is to overview one of the most underestimated in scientific literature characteristics of voting rules - their vulnerability to different voting paradoxes. Next chapter is dedicated to classification and description of paradoxes that occur in different voting rules.

3. Variety of Voting Paradoxes

This chapter is devoted to description of different voting paradoxes known for today. But what can we call 'paradox'? Discussing central paradox in the theory of voting, Condorcet paradox, American philosopher Michael Dummett said: "The position of so-called paradox [of voting], which is not a genuine paradox, but merely a *surprising fact*, as one of the fundamental premisses of the entire theory of voting is perhaps indirectly responsible for the neglect of that theory by those practically concerned with voting" [Dummett, 1984, p.11]. It is true that we can hardy imagine strict universal definition of paradox as such. So, we will use the term "voting paradox" in the sense of M. Dummett. From the one point of view, occurrence of paradox in voting system is a negative result for theory of voting. Of course, the paradoxes has consequence that there are many more negative theorems to be proved in the theory of voting that the positive ones.[4]. But at the same time it is a way of gaining the new knowledge about our theory and particular properties of voting procedures. Eventually, discovered paradoxes supported a lot of new theorems to be proved.

We can classify all paradoxes in our theory into four groups[5]:

1. incompatibility paradoxes
2. monotonicity paradoxes
3. choice set variance paradoxes
4. representation paradoxes

The first group of paradoxes deals with several intuitively correct properties that one could impose on "best" alternatives and these requirements cannot be satisfied at once. The second type of paradoxes is similar to the first, but it deals with incompatibility of the requirement that the voting procedure must be monotonic. The third group of paradoxes consist of different cases variability of choice sets in counterintuitive ways. The choice set variance paradoxes show what would happen if the alternatives were presented in various ways to the voters.

The last group, paradoxes of representation, deals with distribution of seats for voting bodies that would in some natural way mirror the distribution of crucial characteristics of a population. In this text we will not consider representation paradoxes, because they usually deal not with the voting procedures, but with the systems of proportional representation as Hare quota, Droop quota, D'Hondt method, etc. [20, 30].

The first known voting paradox is the Condorcet paradox or paradox of voting. It was proposed by great French mathematician Marquis de Condorcet in 1785 [Condorcet, 1875]. It is one of the central paradoxes in the theory of voting, Condorcet works marked the beginning of social choice theory as scientific discipline. However, Paradox was rediscovered by several other people, starting a century later with Ch. Dodgson (Lewis Carrol) (1885). Condorcet's Paradox was also rediscovered in a study by Huntingdon [Huntingdon, 1938]

[4]The main result is Arrow's theorem (1963). See also Kelly (1978) and Aleskerov (1999)

[5]Classification was proposed by Hannu Nurmi [Nurmi, 1999] and commonly used by other authors (Gehrlein, Lepelley 2017; Felsenthal, Nurmi 2018, etc.)

and other people [Granger, 1956; Black, 1958; Riker, 1961].

Before describing the paradox, we must note again that in social choice theory a very common assumption is that a rational preference ordering must be transitive: if A is preferred to B, and B is preferred to C, then A must be preferred to C [Hansson and Grüne-Yanoff 2009]. Now let's consider the following example (Table 9):

1'st Group	2'nd Group	3'rd Group
A	B	C
B	C	A
C	A	B

Table 8: Condorcet paradox

In this preference profile all individual preferences are transitive. But who is the winner of this elections if all groups are of equal size? We can see then that for the majority of voters A is better than B, B is better than C, so we could expect that A is better than C for the majority. But here 2'nd and 3'rd groups prefer C to A. Hence, pairwise majority comparisons doesn't lead to a transitive collective preference despite the fact that all individual preferences are transitive.

The essence of this paradox is the fact that we could choose any alternative as the winner, but in any case majority of voters would prefer to elect another candidate. And this contradiction arises despite the fact that individual preferences are correct.

Another paradox of enlightment era was proposed by Jean-Charles de Borda. He described this paradox during his talk at the French Royal Academy on 16 June 1770 [De Grazia 1953; McLean and Urken 1995]. His main goal was to show that the plurality rule is in some sense unsatisfactory as a method of aggregation individual preferences. Borda uses the following example in his argument (Table 8).

(1 voters)	(7 voters)	(7 voters)	(6 voters)
A	A	B	C
B	C	C	B
C	B	A	A

Table 9: Borda's Paradox

Borda points out that in this example, A is a winner by plurality rule, but he is the worst alternative for absolute majority of voters. Borda finds out that pairwise comparison of these candidates will elect candidate C.

The essence of Borda's paradox is fact that plurality rule can elect a Condorcet loser (alternative that would lose any pairwise comparison). Borda's criticism against the plurality voting rises up the argument that the Condorcet loser should not be elected. Such a reasoning leads Borda to proposing his new method of aggregation, Borda count. Bourda count never elects an alternative that would have the majority of voters against him (Condorcet loser). It is not clear whether Borda's intention was not only to avoid the victory of Condorcet loser, but to guarantee the victory of Condorcet winner. Because Borda rule is not actually Condorcet efficient. Later, when this property was discovered, he said that "the elimination of a Condorcet loser was his primary interest" [Nurmi, 1999, p. 13].

Borda's paradox highlights incompatibility of two intuitions about winning in elections: pairwise comparison and positional ranking. According to the former intuition, the winners should be elected by pairwise confrontation of one alternative with with each other. In this case, the more competitors an alternative defeats, the better. According to the latter intuition, the winning alternatives must be positioned better than the others in individuals' rankings. That's why such a paradoxes as Borda and Condorcet paradox (which is explained in the next paragraph) are called incompatibility paradoxes.

The next group of paradoxes is based on violation of one of the most important properties of voting procedures - monotonicity.

Definition 4 *If an alternative x wins in a given profile P when a certain procedure is being applied, it should also win in the profile P' obtained from P by placing x higher in some individuals' preference rankings. Procedure satisfying this requirement is called* ***monotonic****.*

Monotonicity highlights our intuition that the more support candidate receives, the better chances she has. For example, plurality rule is monotonic: increasing the number of votes for particular candidate leads to increasing her chances.

The No-Show paradox [Fishburn, Brams 1983; Moulin 1988;] is on of the paradoxes abusing non-monotonic procedures or so-called "lack of monotonicity" of some formally monotonic procedures.

The essence of the no-show paradox is that for some voters it may be better not to vote at all than to vote sincerely according to their preferences. Consider example (Table 10) both for plurality with run-off and STV (alternative vote) because of the same result.

26%	47%	2%	25%
A	B	B	C
B	C	C	A
C	A	A	B

Table 10: No-show paradox

No alternative has majority of votes. So, alternative C with the smallest number of first ranks is eliminated. On the second stage candidate A wins with 51% of votes against 49% for B. Suppose now that the 47% of voters with the preferences $B \succ C \succ A$ decides not to vote at all. In this situation the first round cannot determine winner by a majority of votes. So, alternative B gets 2% of the votes and being eliminated. After this, alternative C with 27% of votes defeats A with 26%. Unexpectedly, the 47% of voters who decided to hide their preferences, benefit from their action — now their second-ranked alternative C wins, but if they go to election and show their preferences, their last-ranked alternative A wins. This example shows that 47% of voters can actually be better off by hiding their preferences than by voting.

We can reformulate No-Show paradox to catch its essence: "The addition of identical ballots with candidate x ranked last may change the winner from another candidate to x." [Fishburn and Brams 1983, p. 207].

Additional Support Paradox (sometimes it's called Lack of Monotonicity or Negative Responsiveness paradox) is another monotonicity paradox. It can be formulated in a such way: If candidate x is elected under a given distribution of voters' preferences among the competing candidates, it is possible that x may not be elected if some voters increase their support for x by moving it to a higher position in their preference ordering [Smith 1973, Fishburn 1974; Fishburn, Brams 1983]. Consider the following preference profile and plurality with run-off procedure (Table 11):

34%	35%	31%		30%	35%	4%	31%
A	B	C		A	B	B	C
C	C	B	⇒	C	C	A	B
B	A	A		B	A	C	A

Table 11: Additional Support Paradox

In the left part alternative B wins by defeating A in the second stage of runoff. Suppose now that 4% of voters decide to change their preference to $B \succ A \succ C$ rather than $A \succ C \succ B$ so that B's support increases. Now, only 30% of voters have the ranking $A \succ C \succ B$, while B's first-round support increases from 35% to 39%. Now alternative C goes to the second round and contest with B. On the second stage C wins with 61% of votes against 39%. We see that increasing of support can harm a winner in STV procedure and in plurality with runoff.

The interesting fact is that almost all multi-stage voting procedures are non-monotonic and fail Additional Support Paradox. But some one-stage procedures are vulnerable for it too. In 1982 Fishburn proves a theorem about general characteristics of procedures vulnerable to the additional support paradox (see Fishburn 1982 for details). Failure of monotonicity is one of the most frequent paradoxes in theory of voting.

The Preference Truncation Paradox [Brams 1982, Nurmi 1999] is less sophisticated but not less important paradox of monotonicity. It states that voter may obtain a more preferable outcome if she lists only part of her preference ranking in her ballot rather than listing preference ordering among all possible alternatives.

33%	29%	24%	17%
A	B	C	D
B	A	B	C
C	C	A	B
D	D	D	A

Table 12: Preference Truncation Paradox

One of the arguments in favor of some voting procedures is the fact that they encourage the voters to sincerely reveal their full preference ordering since it cannot do harm a voter to give a full ranking. And the commonly mentioned in this case procedure is alternative vote (or STV) [Nurmi 1999, p. 63]. The search of such procedures is important, because procedure that guarantee sincere voting as the best strategy for each voter would avoid any kind of manipulations. But Fishburn and Brams provide an example in where truncating one's preferences leads to more preferable outcome using STV [Fishburn and Brams 1984, p. 401].

To illustrate this paradox we can apply STV to preference profile in Table 12. No alternative is ranked first by more than 50% of the voters, so D with the smallest number of first ranks is eliminated. It does not still leads to a winner, so B is eliminated and A wins. Suppose that group of 17% with preferences $D \succ C \succ B \succ A$ decides not to express their full preference ranking but show only first-ranked alternative D. In new profile, D is still being eliminated first, but since the voters with D as their first-ranked alternative do not show their preferences on the rest of alternatives, their votes cannot be transferred. So, we have a situation in which C is eliminated and B becomes the winner. It's easy to see that this outcome is more preferable than A's victory for those 17% who hided part of their preferences.

The next group of voting paradoxes is so-called "Choice Set Variance Paradoxes". They represent situations in which a few different issues are being presented to a group of voters. Each of these issues must be independently approved or disapproved by the majority of voters. A paradoxical outcome can emerge if the overall finally approved outcomes on the issues represents a general result that is inconsistent with the underlying preferences voters have on the issues. We will consider Choice Set Variance Paradoxes in the next paragraphs.

The first of them is Ostrogorski's paradox. It was proposed by Russian political scientist and historian Moisey Ostrogorski [Ostrogorski 1902]. Ostrogorski was an opponent of political parties. He argued that voters must be allowed to vote straightforward for candidates without the intervention of representatives. It sounds quite strange from political scientist, but this position has strong background.

Consider an example of the contest of two parties X and Y which have completely opposite positions (Table 13). Let's group all voters into four groups according to their positions on four issues. Groups A - C consist of 20% of the voters each, and group D consists of 40% of voters. All voters have two possible ways of voting: to vote for party which stands closer to his views or to vote on each issue according to her views and the winner is the party that wins on a majority of issues.

Group	Issue 1	Issue 2	Issue 3	party supported
A (20%)	X	X	X	X
B (20%)	X	Y	X	X
C (20%)	Y	X	X	X
D (40%)	Y	Y	Y	Y

Table 13: Ostrogorski's paradox

The essence of Ostrogorski's paradox is the observation that the following two ways of the winner determination may lead to different outcomes:

1. Each voter votes for the party whose stand is closer to his in a majority of voting issues . The winner is the party commanding the support of the majority of voters.

2. For each issue the winner is the party whose stand is supported by a majority of voters. The winner of the election is the party that wins on a majority of issues. [Nurmi 1999, p. 71]

These different approaches can lead lead to opposite outcomes. The first procedure results in the winning of X by 60% of voters, so X defeats Y. The second procedure results in the winning of Y by a majority of voters in two issues out of three. So, Y defeats X.

Ostrogorski's paradox shows that some party may win the fair election, when the loser represents views of the majority of voters on a majority of issues. A particularly dramatic case of this paradox is possible situation when losing party represents the views of a majority of voters on all controversial issues. This situation is usually called Strict Ostrogorski's Paradox.

Another interesting paradox was proposed by famous American philosopher professor Elizabeth Anscombe [Anscombe 1975]. It states that it is possible for a majority of voters to be on the losing side of a majority of issues. It sounds surprising but let's consider the committee of five members who vote on three occasions, each time on a simple motion to be accepted or rejected. Each vote is decided by a majority of voters. After sincere voting we can face with the following situation (Table 14):

	voter 1	voter 2	voter 3	voter 4	voter 5	outcome
motion a	Pro	Con	Con	Pro	Pro	carried
motion b	Con	Pro	Con	Pro	Pro	carried
motion c	Con	Con	Pro	Pro	pro	carried

Table 14: Anscombe's Paradox

Voters 1, 2 and 3, obviously, constitute a majority of the committee members, but each of them has voted with the majority only one time out of three and has been in the minority a majority of times. But minority- Voters 4 and 5, have been in the minority every time. Professor Anscombe wrote: "the appearance that one is not subjected to any authority that exercises power over the individual may be compelling in a case in which one has cast one's vote with the majority, but it is nevertheless an illusion" [Anscombe 1975, p.51].

The most crucial fact about Anscombe's paradox is the fact that it seems that if the procedure under which majority is decisive on any given outcome is employed than majority can always enforce its will. But this paradox is a counterexample for this point of view. We can imagine that in our example voters 1, 2 and 3 can together secure a series of outcomes preferable for them by forming a coalition. If they form a coalition and agree always to vote for the outcome preferred by the majority of them, they will be able to hold decision 'contra' on all 3 occasions. But is such situation possible for rational voters? If we apply backward induction and analyse this voting from the 3'rd voter's point of view, we can see that she has no personal motive for abiding by the agreement in the vote on motion c, because voters 1 and 2 have already voted and she can reach the most favorable outcome by breaking their arrangement and leaving the coalition. At the same time, voter 2 will understand this way of reasoning of voter 3 and will have no motivation to carry their agreement. But if it is so, voter 1 should understand it on the 'motion a' voting. Thus, the essence of this paradox is not only in fact that majority can be on the losing side on the majority of issues, but in the fact that majority cannot even form a coalition to enforce its will and change results of voting.

The Pareto Violation Paradox or Dominated alternative paradox [Fishburn 1974] is based on violation of the Pareto criterion.

Definition 5 ***Pareto criterion*** *states that if all voters strictly prefer X to Y then Y is not elected*

Note that criterion does not require that X must be chosen. But it states that Y must have no opportunity to be elected. It seems impossible for any voting procedure that a candidate Y may win the election while candidate X will lose despite the fact that all voters prefer candidate X to Y, but some voting procedures violate Pareto criterion.

The following example (Table 15) demonstrates vulnerability of the Approval Voting procedure to the Pareto Paradox. Suppose that there are 3 voters with preference rankings among three candidates as follows.

2 voters	1 voter
A	C
B	A
C	B

Table 15: Pareto Violation

Alternative A is the Condorcet Winner. However, if left two voters will approve two of their best preferences (A and B) , while the third voter approves only her first-ranked altyrnative C, then a tie would occur between the number of votes obtained by alternatives A and B. It is reasonable to say that there are different approaches what to do in the case of a tie. But from the mathematical point of view tie means the equal chances to be elected, so the probability of winning is equal to 0.5 for alternative A and B. So if B becomes the winner it leads only to lose of Condorcet Winner A, but also to the winning of Pareto-dominated candidate with the probability of 0.5 using Approval Voting procedure.

The Multiple Elections paradox wast introduced by Brams, Kilgour and Zwicker [Brams, Kilgour and Zwicker 1998]. If we consider Table 16 we can find that alternative Y winning on every issue and coincide with no voter's opinion. This is the essence of this paradox.

To describe this paradox formally, let the binary vector $(z_1, ..., z_k)$ denote a voting strategy in a k-issue election. The value of z_i denotes the voter's decision on issue i (Approve of disapprove). As illustrated in Table 16, strategy of voter 1 is (X,Y,Y). The outcome

Group	Issue 1	Issue 2	Issue 3	party supported
voter 1	X	Y	Y	Y
voter 2	Y	X	Y	Y
voter 3	Y	Y	X	Y

Table 16: Paradox of Multiple Elections

of a k-issue election is binary vector of k items where $k_i'th$ indicates the winning result on $i'th$ issue. We say that voting outcome coincides with voter's strategy if every item i in voter's strategy is equal ti $i'th$ element of the voting outcome. This coincidence shows that voter votes similar with the majority on every issue. The number of coinciding strategies can range between 0 and n, where n is the number of voters. Situation when number of strategies coinciding the outcome is minimal in a way that no possible outcome could have less number of coinciding strategies is called The paradox of multiple elections. In Table 16 we can see that the number of coinciding strategies is equal to zero.

Before discussing the Inconsistency (or Multiple Districts, or Reinforcement) Paradox [Young 1974], we need to define the property of consistency of voting procedure. Let the set of voters N is a union of two non-overlapping subsets N_1 and N_2. R is the preference profile of N. $F(A, R)$ is the result of applying voting rule F to the set of alternatives A and preference profile R. Assume that two distinct parts of the voters, using F, make at least partially same choices from A: $F(A, R_1) \bigcap F(A, R_2) \neq \varnothing$.

Definition 6 ***Consistency*** *of F requires now that $F(A, R_1) \bigcap F(A, R_2) \subseteq F(A, R)$*

Weak interpretation of Consistence requirement means that if x is elected in each of several disjoint districts, than x must be elected if all districts are combined. Strict version replaces the symbol of inclusion with the symbol of equality: $(\subset)$. In other words, strict consistency requires that intersection of the winners sets in districts N_1 and N_2 must be equal to the set of winners in united district N.

Let's illustrate inconsistency paradox with applying of plurality with runoff to the following example(Table 17):

East			West		
35 voters	40 voters	25 voters	40 voters	55 voters	5 voters
A	B	C	C	B	A
C	C	B	B	C	C
B	A	A	A	A	B

Table 17: Inconsistency Paradox

Assume that groups "East" and "West" are equal size and non-overlapping. If the voting is conducted in these districts separately, alternative B wins in both of them by a plurality with run-off procedure. So, in this case B is the winner. But if we now combine this groups and form united electorate, A will be eliminated with 40 out of 200 votes. In the second stage alternative B has 95 out of 200 and alternative C has 105 out of 200, so C is the winner now.

This paradox is extremely important, because it can be a foundation of Gerrymandering which is a kind of electoral manipulations by demarcation of electoral districts. Multiple-Districts or Inconsistency Paradox is one of the particular cases of more fundamental phenomenon known as Simpson's Paradox [Malinas, Bigelow 2009].

The Simpson's Paradox was firstly introduced by Cohen and Nagel [Cohen, Nagel, 1934, p. 449]. But it is known as Simpson's paradox, despite the fact that Simpson [Simpson 1951] wrote about it almost twenty years later. The essence of this paradox is the effect when there are two groups of voters and each of them has the same directional distribution of votes between 2 alternatives, but when these groups are combined, the direction of the preferences over alternatives is reversed. This paradox is obtained from the field of probability theory and statistics, but it is important for Social Choice Theory.

voter	voters		party A votes		percentage	
group	East	West	East	West	East	West
employed	400 000	90 000	80 000	15 000	20	17
unemployed	100 000	80 000	50 000	35 000	50	44
total	500 000	170 000	130 000	50 000	26	29

Table 18: Simpson's Paradox

Consider some political example of Simpson's Paradox. Assume that we have elections in two-district political system, where districts are of unequal size. Population in East district is 500 000 voters and in the West district, 170 000 of voters. In Table 18 we have a voters distribution for Party A and we need to calculate distribution of support for this party accordingly to voters' employment status. It is easy to see that party A's popularity is higher in East district independently of the voters' category. But the percentage of votes for A is lower in the East district than in the West when both categories are combined. Does A really have better support in the East district than in the West? If we conclude that it's true, then the overall support rate is being ignored. On the other hand, if we decide that our conclusion was false, then we are ignoring support rates among sub-populations. So, both approaches are leading to paradoxical outcome. This is one of the crucial problems any political systems seeking to correctly represent opinion of the different groups of population. Moreover, it does not matter which voting procedure to use, because this paradox occurs on the basis of statistics, not voting rule.

We highlighted the main paradoxes of the theory of voting. Some of them occur very frequently, others less, but the point is that every voting procedure known today is vulnerable to paradoxes of different kinds. But can we find any procedure which would not be vulnerable to paradoxes or, at least, would cope with the most dramatic of them? The next chapter deals with comparative analysis of the vulnerability of described voting rules to the paradoxes.

4. Vulnerability of Voting Rules to Paradoxes

This chapter provides the analysis of how the properties of voting procedures lead to the emergence of paradoxes in this procedures. First of all, it's necessary to understand that some paradoxes are procedure dependant and others are not. For example, Condorcet paradox can occur in any voting procedure, all procedures can lead to intransitive collective ranking. That's why it is usually called paradox of voting. At the same time, the Simpson's paradox and the Paradox of Multiple elections deal with statistical phenomenon of incompatibility over-all and sub-populations opinions. So it does not matter which voting rule is used. At the same time, since the representation paradoxes occur before elections, the procedures' properties have no relevance to these paradoxes [Nurmi, 1999, p. 122] and is therefore omitted.

It also necessary to note that the Ostrogorski's and Andscombe's paradoxes are in some sense different from other observed paradoxes. All other observed paradoxes are connected with procedures electing a single candidate from the individual preference rankings over the candidates. But the Ostrogorski's and Andscombe's paradoxes in their essence are connected with yes-or-no voting. This fact make our reasoning of the possibility of occurrence of these paradoxes in Borda count or Coomb's procedure or any other meaningless. Yes-or-no voting are being casted and counted only as the Absolute Majority Procedure, since it has only 2 possible alternatives.[6]

But other paradoxes can be analysed from their possibility to occur in particular voting procedures point of view. Obviously, to prove that voting procedure is vulnerable to the paradox, it is enough to show example of this paradox emergence using particular procedure. But the proof of invulnerability of procedures to the voting paradoxes can be more difficult. Since we cannot sort through all possible examples of votes' distribution, we need to find another way of reasoning. Partly, vulnerability of voting rules was shown in previous chapter, so we need only extend our reasoning and intuition to all described procedures.

Borda's Paradox occurs in Plurality voting, which was shown in Table 8 and in the original text of Borda. It's easy to see that Plurality with runoff can lead to the victory of Condorcet loser with a plurality, but not absolute majority of votes. So, CL can obviously be ranked first or second on the first stage of elections (but cannot obtain more than 49%) and continue a contest in the second stage. But since CL cannot get majority on the second stage (by definition), it cannot win the election under plurality with runoff procedure.

Table 19 shows that Approval Voting procedure is vulnerable to the Borda's Paradox.

6	4	1	4
(a)	(b)	(c)	(c)
b	c	(a)	b
c	a	b	a

Table 19: Approval voting and Borda's paradox

[6] Assuming this fact, these paradoxes are marked as 'paradoxes of other kind' in final table (Table 27)

Here a is ranked last by 8 voters out of 15, so it is Condorcet Loser. However, the goal of voting is to elect a single candidate and all voters approve the candidates shown in parentheses then 'a' will be elected under Approval Voting.

The Borda count inevitably eliminates the Condorcet loser [Nurmi, 1999, p. 14], therefore Borda count never elects Condorcet Loser. The STV procedure and Coombs' method iteratively check if any alternative gets the majority of the votes. So, it follows that they elect neither a Condorcet Loser nor a Pareto-dominated alternative in all possible profiles.

We can prove vulnerability of Dodgson's procedure to the Borda's Paradox with the example in Table 20 [Fishburn 1977, p. 477].

1	1	1	1	1	1	1
a	g	f	e	d	c	b
b	a	g	f	e	d	c
c	b	a	g	f	e	d
d	c	b	a	g	f	e
x	x	x	x	x	x	x
e	d	c	b	a	g	f
f	e	d	c	b	a	g
g	f	e	d	c	b	a

Table 20: Dodgson's method and Borda's paradox

Alternative x is Condorcet Loser, because it loses the pairwise comparison against any other alternative. However, if only one of the voters swipes it up four times, x will be elected by Dodgson's procedure. All other alternatives requires not less than six swaps to become a Condorcet winner.

The Plurality Voting is invulnerable to the **No-Show Paradox** since no one could obtain better result by abstaining, because only abstaining of voters with first-ranked winning alternative can change the winner and it makes no sense for any of them.

Example in Table 10 is demonstration of the vulnerability of Plurality with Runoff and STV (alternative vote) to the No-Show paradox. The Approval Voting is invulnerable to the No-Show Paradox for the same reason as Plurality rule. If improvement of a ranking position holds approvability status, approved alternatives will be approved after any changes of ranking.

Since Borda's procedure assigns every alternative some scores based on its ranking and elects an alternative with the highest score, any abstaining decreases Borda score and chances to win. So Borda's rule is invulnerable to this paradox.

Example in Table 21 demonstrates vulnerability of Coombs' procedure to both of the No–Show and Truncation paradoxes.

Plurality rule is not suspected to be vulnerable to **Additional Support Paradox** since increasing of electoral support of a particular candidate will increase her chances and no other candidate can get more votes. In other words, plurality rule is monotonic. If we consider the possibility of increasing the set of voters, new voters will increase support of winning

4	4	5	2
a	b	c	c
b	c	a	b
c	a	b	a

Table 21: No-show and Trancation paradoxes in Coombs' method

candidate, so the winning alternative remains the same. So plurality rule is invulnerable to additional support paradoxes in both cases.

Same as plurality rule, approval voting is invulnerable to additional supprot paradox for the same reasons: changing alternative's position does not change its approvability.

Going further we can see that increasing support of any alternative (in fixed and open set of voters) implies increasing of its Borda score, so no additional help can decrease Borda score of the winning alternative. That's why Borda rule avoids any kind of monotonicity failure, in particular additional support paradox.

1	10	11	11	10	2
a	a	b	b	c	c
b	c	a	c	a	b
c	b	c	a	b	a

Table 22: Failure of monotonicity under Coombs' method

In Table 22 alternative b is the Condorcet Winner, but Coombs' procedure will elect alternative c. Let 11 voters with preference ranking $b \succ a \succ c$ show different preference ordering: $b \succ c \succ a$. We see that b remains the Condorcet Winner but now candidate a is eliminated first by Coombs' procedure and candidate b beccomes the winner. In this example Coomb's procedure violates monotonicity and this is sufficient proof of its vulnerability to additional support paradox.

The Dodgson's procedure is vulnerable to both No–Show Paradox and Additional support. Example with the proof of this statement can be found in [Fishburn 1982, p. 132].

Preference Truncation Paradox is incompatible with Plurality Voting because under this procedure only first ranks in the preference profile are being taken under account. This paradox is irrelevant to the Plurality with Runoff procedure too. Because if the first ranked alternative passes to the second round, any truncation is useless since only fist ranked candidates are taken under account. If, the first-ranked alternative cannot pass to the second round, truncation will only decrease chances of second-ranked candidate to win. [Felsenthal, Nurmi, 2018, p. 31]. So, the occurrence of the Truncation Paradox is not possible.

99	1
(a)	(b)
(b)	a
c	c

Table 23: The Truncation Paradox and Approval Voting

Suppose there are 100 voters whose preference orderings are as follows Table 23. If 100 voters with the following preferences approve candidate b, b will win the elections. But for 99 voters it's profitable not to show the fact that they are ready to approve candidate b and include in their ballots only approval of a. After such a manipulation candidate a wins the election with 99% of votes against 1%. This is a demonstration of the vulnerability of the Approval Voting procedure to the Truncation Paradox.

3	1	1	2
a	b	b	c
b	c	c	d
c	a	d	a
d	d	a	b

Table 24: Borda count and Truncation Paradox

In this example, c is the winner since the Borda scores of a, b, c, and d are 12, 12, 13, and 5, respectively. Let three voters with preference ranking $a > b > c > d$ decide to truncate alternative c from their ballots. Formally, now we cannot apply Borda rule to recount all scores, but the decision was proposed Fishburn [Fishburn, 1974, p. 543] — Borda Truncated scoring system for truncated ballots. Without going into detail, result of new scores for a, b, c, d 12, 12, 10, and 8. Now we have a tie between alternative a and b, which is more preferable result for all truncating voters.

Truncation Paradox also apperas in the STV procedure:

33	29	24	17
a	b	c	d
b	a	b	c
c	c	a	b
d	d	d	a

Table 25: STV and Truncation Paradox

In Table 25 none of the four candidates is ranked first by a majority of voters, candidate d is eliminated. Then STV eliminates candidate b and a becomes the winner. Now imagine that 17 voters with last-ranked a truncates all their preferences except first - d. In the new profile candidate d is still being eliminated first, but since 17 voters truncated their preferences over remaining alternatives, STV eliminates c, so candidate b wins. Outcome of updated situation is better for 17 voters than original one, demonstrating vulnerability of STV rule to the Truncation Paradox

Proof of the Dodgson's rule's vulnerability to the Truncation Paradox is quite complicated, but Felsenthal and Nurmi demonstrate it in their book. [Felsenthal, Nurmi, 2018, p. 93].

Pareto Violations paradox is very special and very dramatic failure of aggregating procedure. Fortunately, it occurs only in a very distinct procedures. Pareto dominated candidates cannot be elected neither under plurality rule nor under plurality with runoff:

Pareto dominated candidate by definition cannot be the first-ranked preference of any voter and cannot obtain any votes in any stage of plurality voting. In the same manner we can see that in any profile at least one candidate receives strictly more Borda scores, than Pareto dominated candidate, so Borda rule is sustainable to the paradoxes of this kind. Even complicated Dodgson's procedure is obviously invulnerable to this paradox: Pareto dominated candidate is lower that dominating one in all preference orderings, so in needs strictly more binary swaps to become the Condorcet Winner.

The only vulnerable to the Pareto-Dominated Candidate Paradox is Approval Voting. Its failure is demonstrated in Table 15.

The last considered paradox is called the **Inconsistency Paradox**. It occurs under any king of runoff procedures and in procedures which iteratively eliminate sub-optimal alternatives. We already considered vulnerability of Plurality with Run-off procedure to the Inconsistency Paradox in Table 17. STV as runoff procedures behaves in the same manner. Detailed proof of Dodgson's procedure's failure with Inconsistency (Reinforcement) Paradox was proposed by Fishburn [Fishburn, 1977, p. 484)]. Demonstration of Coombs' procedure the vulnerability to the Inconsistency Paradox can be found in Table 26.

Group I				Group II	
9	9	11	5	1	6
A C B	B C A	C A B	C B A	A B C	B A C

Table 26: Inconsistency Paradox and Coombs' procedure

Since no alternative is ranked first by more than 50% of Group I, Coombs' procedure eliminates alternative a in the first stage, then alternative b is elected. In Group II alternative b is the best alternative for the majority of the voters and is elected in the first stage. But in merged electorate, none of the alternatives is ranked first by the majority, so alternative c is eliminated by Coombs' procedure in the first stage. Now alternative a (not b) is elected and this fact demonstrates vulnerability of Coombs' procedure to the Inconsistency Paradox.

Intuitively, we can see that plurality voting avoids this paradox, because if any alternative gets the plurality of votes in two different subsets of voters, it will obviously get the plurality of votes in a merged set of voters (note: our assumption states that preferences of all voters do not change after the merging). Approval Voting procedure is invulnerable to Inconsistency Paradox for the same reasons: merging of disjoint districts does not affect approvability status of any candidate. To prove Borda's rule invulnerability to this paradox, it's enough to understand that if alternative gets the largest sum of scores in all pairwise comparison matrices which represent different districts, this alternative will have the largest score in the pairwise comparison matrix of the merged set of voters.

We observed the main voting procedures and their vulnerability to the main paradoxes of voting. We can see (Table 27), that different procedures have different patterns of the vulnerability to paradoxes. Some of them (Plurality with Run-off, STV, Coombs' and Dodgson's procedures) distinctly fail monotonicity which leads to the emergence of the No-Show, Additional Support and Truncation paradoxes, but others can easily deal with this paradoxes. Surprisingly, Plurality Voting and Borda Procedures are the most sustainable in terms of paradoxes occurrence. Of course, the list of analysed procedures and paradoxes is not complete and the further research can make significant contribution to understanding main properties of voting procedures and paradoxes. We also observed only procedures, designed to elect a single candidate. The situation can be different if we face with multi-winner procedures and other patterns of paradoxical outcomes occurrence can be found. Nevertheless, obtained result shows that the Paradoxes are very important topic in contemporary Social Choice Theory and challenges our methodology and commonly used approaches quite intensively.

Paradox	Procedure						
	Plurality	Plurality with Run-of	Borda Rule	Approval voting	STV	Coombs procedure	Dodgson's Procedure
Borda's Paradox	⊕	—	—	⊕	—	—	⊕
Condorcet's Paradox	No procedure dependence						
No-Show Paradox	—	⊕	—	—	⊕	⊕	⊕
Additional Support Paradox	—	⊕	—	—	⊕	⊕	⊕
Preference Truncation Paradox	—	—	⊕	⊕	⊕	⊕	⊕
Ostrogorski's Paradox	Paradoxes of other kind						
Anscombe's Paradox							
Pareto Violations	—	—	—	⊕	—	—	—
Inconsistency Paradox	—	⊕	—	—	⊕	⊕	⊕
Paradox of Multiple elections	No procedure dependence						
Simpson's paradox	No procedure dependence						

Table 27: (In)Vulnerability of Voting Rules to Various Voting Paradoxes

5. Conclusion

In conclusion, we can say that voting paradoxes constitute a large area for actual research in the field of the theory of voting. Studying the paradoxes and characteristics of voting procedures that lead to the occurance of paradoxes can make a significant contribution to our understanding of the theory of collective choice. At the same time, a comparative analysis of the vulnerability of voting rules to paradoxes provides us with an additional tool for assessing the pros and cons of different approaches to the aggregation of individual preferences.

Classification of paradoxes does not directly make a contribution to solving them. Of course, some paradoxes are being dealt with in practice, but most of them are not. The results provided in Chapter 3 show that when we face with procedure-related paradoxes the only way to avoid them is to avoid procedures vulnerable to paradoxes. But the number of incompatibility results in social choice theory indicate that we must be prepared for trade-offs. As Hannu Nurmi said: “one advantage is often offset by disadvantage of a different sort” [Nurmi, 1999, p. 124]. And the main goal of this research is to indicate the pay-offs in these trade-offs between different voting procedures. Further research in this area requires taking into account of a larger number of voting procedures and their vulnerability to known paradoxes. Such a research will help us to form a holistic view on the nature and place of the paradoxes in our theory. On the one hand it will help us in practice for design of electoral institutions. On the other hand, it will allow to build up an important theoretical apparatus for proving vulnerability or invulnerability of voting procedures to various paradoxes.

Another major topic, not covered in this paper, is the paradoxes of proportional representation. Which are less studied in academic literature than procedure-related paradoxes, but they may occur much more often, because they depends on representational issues, but not certain voting procedures. If paradoxes discussed in this work may question certain voting procedures, paradoxes of representation may question the possibility of representatives as such. All this illustrates that we do not know much about paradoxes and are practically powerless trying to overcome them. But at the same time, it provides tremendous potential for further research that can lead to new important results, as it happened several times in the history of the theory of voting.

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