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CAN A SOCIAL PLANNER MANIPULATE NETWORK DYNAMICS AND SOLVE COORDINATION PROBLEMS?

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Can a social planner manipulate network dynamics and solve coordination problems? ⋆

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Abstract

This paper aims to build an algorithm of network dynamics with decision-making under incomplete information. Accordingly, it tries to identify if a social planner reduces the influence of individual biases, such as confirmation bias or assimilation bias on agents' actions, and solve a coordination problem. The research questions are the following: " Can the social planner increase social welfare, by manipulating the set of possible invitations and annoyances, without directly changing a network structure?", " What are the main drivers of increasing social-planner utility functions?" "How do the results change if the social planner has incomplete information or wrong priors about the fundamental variable?" For this research, a "Liberal Social Planner" was created; a process through which network members get suggestions depending on its utility function. The results have potential applications for the management of social media platforms by the owners of these platforms. Platforms can develop robots that can help their users be more informed and more satisfied. As we live in a world of virtual connectedness, people seem to obtain more information from online network peers than from experts

Keywords: network dynamics, internet, higher-order beliefs, learning, expert opinions, biased assimilation, confirmation bias

JEL classification: D85, D83, D82, D72, C78

1. Introduction

Coordination problems in case of uncertainty about some fundamental parameter are everywhere. [1] Examples of such fundamental parameters include the outcome of a vote on a political issue, scientific findings of, e.g., a medical issue such as a vaccine for an epidemic, a price outcome, e.g. a stock-price in financial markets, etc. Decision-makers solving coordination problems with others try to take into account other agents' beliefs about some parameter.The main focus is on a network model where everyone has a fundamental parameter governing assimilation or confirmation bias in their preferences (for example, left or right-hand side political views, religion view vs scientific findings). To simplify the analysis, these fundamental biases are considered as constant, and that all network agents know these constant parameters of biased assimilation. However,the main focus is on developing a mechanism that endogenizes the weight people put on these biases while making decisions, and how this weight is affected by efforts of people to align their actions with the actions of others. More importantly, the question of whether a social planners can influence this endogenous coordination, leading to better social outcomes is studied.

Much of the literature on network theory develops coordination games where agents try to align their actions with these of their neighbors. Therefore, the network structure is crucial for their action-alignment efforts and for their ability to elicit information about unknown parameters.[2] In this paper, I use a utility function similar to this in Morris and Shin (2002), where agents care not only their friends' actions but also take into account other agents' actions. A key feature of the Morris and Shin (2002) model is that it can offer an analysis of the relationship between public/private information on unknown parameters and social

[1] Such a coordination motive is well known as *convention* in economics literature developed by Shin and Williamson (1996), and Young (1996).
[2] See, for example Golub and Morris (2017), Myatt and Wallace (2019), Ballester et al. (2006) and Denti (2017), among others.

welfare.[3]

In order to capture the alignment mechanism, in this model, the “beauty-contest” structure of Morris and Shin (2002) is kept. The model is developed not only for understanding the dynamics of endogenous biased assimilation regarding political issues, such as putting efforts into organizing protests and voting behavior, but also for understanding herd behavior in financial markets that appears during speculative attacks. For understanding the dynamics of such political and market phenomena, agents need not only second-guess the actions of their network friends, but also the actions of network non-friends too. Hakobyan and Koulovatianos (2020) develop a search-and-matching algorithm of network dynamics, focusing on an explanation of how expert opinions have been downgraded over time, and how network agents have been taking more polarized actions while forming more polarized subnetworks. In this paper, I address how a social planner can solve such polarized and populistic behavior in order to bring agents’ actions closer to the true values of unknown parameters.

A simulation model of network dynamics with incomplete information is build, where the social planner tries to manipulate the sample of possible invitations and annoyances in order bring agent actions closer to the model’s fundamentals. For example, if we consider herd behavior in financial markets, we can notice that agents trying to follow other agents’ actions, can play another equilibrium strategy, where they just move away from market fundamentals, leading the market to exhibiting price bubbles. In financial markets, bubbles occur during times of aggressive speculative attacks. Morris and Shin (2002) show that increasing the precision of public signal can harm social welfare if agents have private signals. In this paper, as it was demonstrated such results can vanish if agents are connected in a particular

[3] The idea of Morris and Shin (2002) has been applied to models studying how political issues influence social welfare, and to the study of financial markets and business cycles. See, for example Angeletos and Pavan (2007), Myatt and Wallace (2012, 2015, 2019).

network structure. If the social planner manipulates the network dynamics in a particular way, then social welfare can increase.

In the model of this paper two types of social planners is analyzed: (i) a social planner with perfect information, and (ii) a social planner with incomplete information. The two different scenarios is considered: (a) the case where, similarly to network agents, the social planner receives private and public signals, and calculates expectations, and (b) the case where the social planner has wrong expectations. In the latter scenario, the social planner's expectations about the state variable are equal to a constant, which is different from the optimal state variable.

It is demonstrated that social planners can improve welfare, not by directly influencing/changing the network structure, or by giving to agents fake news in order to manipulate agents' actions. In my model social planners give opportunities to non-connected agents, to be introduced to each other and to meet. The decision of the evolution of the network structure rests entirely upon the agents. Therefore, in this paper the focus is to understand how different "agent sampling" in a search-and-matching environment influences social welfare. One of the key and novel features in this paper is that agents are heterogeneous, making the setup more realistic.

The remainder of the paper is organized as follows. Section 2 presents the setup of the model, the utility function of agents, the network structure, the signals and the information structure. Section 3 focuses on presenting the linear equilibrium and the fixed-point strategies under evolutionary myopia. Section 4 demonstrates the network formation process , while Section 5 shows some simulation experiments. Section 6 concludes.

1.1 Related Literature

This paper links four strands of literature. (1)The first strand uses quadratic-utility functions trying to understand how agents play coordination games under information asymmetry. The closest paper in the literature using such a benchmark without network structure is Morris and Shin (2002). Compared to Morris and Shin (2002) this paper has three differences: I introduce (i) a network structure, (ii) assimilation and confirmation bias in the utility functions, and (iii) evolutionary dynamics of network structure with endogenous weights on signals. The literature combining global games and coordination games in the fashion of Morris and Shin (2002), with network structure are Golub and Morris (2017), Dewan and Myatt (2012), Myatt and Wallace (2012), Bonfiglioli and Gancia (2013), Llosa and Venkateswaran (2012), Pavan (2007, 2014). Almost all papers with quadratic utility use symmetric agents in their model. The closest paper using asymmetric agents is Myatt and Wallace (2019), which use two types of asymmetry: (a) asymmetry in conformity (coordination motive), and (b) different weights for friends (with whom agents coordinate). Myatt and Wallace (2018) use only coordination asymmetry. Compared to Myatt and Wallace (2018, 2019) in this paper agents have these two types of asymmetry. However, in this paper agents try to coordinate their actions with people they are not connected too, and agents have asymmetry in assimilation bias. The difference between this paper and Myatt and Wallace (2018, 2019) is also that this paper has an evolutionary dynamic of network structureIn addition, there is a difference in the information structure. In this paper, agents share information using their network connections, while Myatt and Wallace (2018, 2019) do not consider such types of information transmission; they describe “accuracy” of the information source, and agents decide how much attention to put in which signal, paying the cost for a signal. In Leister (2017) agents are asymmetric in general, but they get one private signal. In this paper I develop mostly the idea which was used in social media platforms in the sense that information is cheap, and agents share information more cheaply. The only requirement in my model for cheap information transmission, is being friends with those transmitting information in the network. This paper is also an extended version of Hakobyan and Koulovatianos (2020), trying to give an answer which arose in their model, which is how to deal with the reinforcement of populism due to evolving and gradually strengthened polarization and homophily.

(2) The literature focusing on understanding games on networks, coordination on networks, key players, homophily and degree centrality. Examples of this literature are Jackson (2008), Currarini et al. (2009), Kossinets and Watts (2009), Golub and Jackson(2012a,b), Bramoulle et al. (2012), Jackson and Lopez-Pintado (2013), Centola (2013), Lobel and Sadler (2015), Currarini and Mengel (2016), and Halberstam and Knight (2016). In this model, I show that the network structure, specifically indegree, and outdegree centrality, are tightly linked with social welfare. I demonstrate that a social planner can manipulate indegree and outdegree links indirectly, and thus increase social welfare.

(3) The literature on strategic disclosure or information manipulation, fake news. My model is different from standard sender/receiver games such as Crawford and Sobel (1982), Kartik(2009), and Edmond (2013). In this paper, I demonstrate the advantages that new social media platforms give to agents. Crawford and Sobel (1982) characterize two types of equilibrium in sender/receiver games with conflict of interest: (i) Separating, which is not a part of a Nash equilibrium and, (ii) Babbling equilibrium, which is a part of an equilibrium, but in this case, there is no information transmission. In my paper agents have a conflict of interest, so directly sending them perfect information wouldn't give any results. For this reason, I develop a mechanism that tries to solve the coordination problem without

directly sending information. The literature developed on information manipulation, such as Edmond (2013), Edmond and Lu (2017) use biased signals, trying to manipulate agents' behavior. In our model the social planner does not use any biased or unbiased signals. Therefore, differently from the literature on information manipulation, our manipulation of network dynamic is not direct. In many countries, there is a law against information manipulation.[4] In this case, even if a social planner has good intentions, whishing to bring agents' actions closer to fundamentals, this is infeasible, according to the law. Our model gives a solution for such problems by creating wrong priors, because it influences agents' decisions indirectly. While there is no direct link to the fake news literature, this paper can solve the consequences of polarization in networks, bringing agents' actions closer to the model's fundamentals. Therefore, researchers who are interested in hte fake news literature can find our framework useful for developing further work.

(4) The literature on social policy. Researchers who are interested in social-planner and social-welfare maximization can find this framework useful for developing further work. The most relevant reference in this literature is Dyckman (1966), Cavallo (2008), and Bernheim (1989).

2. Model

There is a directed network of $N < \infty$ agents. I denote this network by $\mathcal{G} := \{V, E\}$, where V is the set of agents/nodes, and E is the set of edges in this network. In period t the network is represented by an adjacency matrix $\mathbf{M}_t$ with entries in $\{0, 1\}$. The graph to be used is directed and unweighted, i.e., $M_t^{ij} \neq M_t^{ji}$. Edges between agent i and j represent the private information transmission,where the link between i to j means that agent i gets

[4] One country with such a law is France https://www.gouvernement.fr/en/against-information-manipulation

agent j private signal.[5] The diagonal elements of matrix $\mathbf{M}_t$ are 0, which means that there is no self-loops in this model.

In each period, agents make two decisions, (i) to guess a fundamental variable $\theta_t \in \mathbb{R}$ by using all available information (the state variable θ_t is unknown regarding to a political issues, or to a scientific finding, such as a vaccine or global warming), and (ii) to align their actions closer to other agents actions (the known "beauty contest" motive, defined by Keynes, 1936). The fundamental varible θ_t is i.i.d., so agents need to guess new fundamentals in each period $t \in \{0, 1, ...\}$, and there is no learning in the model.

I divide agents into two groups: (a) agent i's neighbors/friends ($j \in \{1, 2, ..., N_i\}$), and (b) non-neighbors ($k \in \{1, 2, ..., N_{-i}\}$). In addition there are two *types* of agents denoted by "+" and "−", depending on the direction of their structural biases b_i, i.e., whether the bias is above or below the value of θ.[6] Specifically, the payoff function for agent i with positive bias "+" is given by,

$$u_i^+ (a_t, \theta_t) = -(1-r_i)(a_{i,t} - (\theta_t + b_i))^2 - r_i \left[\frac{q_i}{\#N_i} \sum_{j \in N_i} (a_{j,t} - a_{i,t})^2 + \frac{(1-q_i)}{\#N_{-i}} \sum_{k \in N_{-i}} (a_{k,t} - a_{i,t})^2 \right], \tag{1}$$

while the payoff function of a agent i with negative bias "−" looks like

$$u_i^- (a_t, \theta_t) = -(1-r_i)(a_{i,t} - (\theta_t - b_i))^2 - r_i \left[\frac{q_i}{\#N_i} \sum_{j \in N_i} (a_{j,t} - a_{i,t})^2 + \frac{(1-q_i)}{\#N_{-i}} \sum_{k \in N_{-i}} (a_{k,t} - a_{i,t})^2 \right], \tag{2}$$

[5] For example, social network structure as a Twitter. Agent i can follow agent j, gets his private signal, but if agent j is not following back to agent i, he cann't get agent i's private signal.

[6] Intuitively, this structural bias in preferred actions reflects political, religious, and other similar biases, falling in the categories of biased assimilation and confirmation bias (see Lord et al., 1979, and Nickerson, 1998).

where $a_{i,t} = [a_{1,t}, ..., a_{N,t}]$. In equation (1) and (2) parameter $r_i \in (0,1)$captures the second-guessing motive. Each agent tries to have their action closer to $\theta_t \pm b_i$, where $b_i \in (0,1)$ is the individual bias.

The second part of the utility function is normalized for controlling network effects on calculations. Parameter $q_i \in (0,1)$ differentiates the weight that agent i put on friends versus non-friends. Parameters, r_i, q_i and b_i are iid across individuals, generalized in period 0, and are common knowledge to all agents.

2.1 Signals and Information Structure

Agents face uncertainty about state variable, θ_t, in each period $t \in \{0, 1, ...\}$. Every period generates a new task, which agents try to best-guess. Agents get public and private signals in each period, with the learning duration being confined to one period. The information set available to player $i \in \{1, ..., N\}$ in each period is $\mathcal{I}_{i,t} = \left(y_t, x_{i,t}, \sum_j x_{j,t} \right)$, where y_t is a public signal with,

$$y_t = \theta_t + \eta_t \ , \quad \text{with} \ \ \eta_t \sim N\left(0, \sigma_\eta^2\right) \quad , \ \ t = 0, 1.., \tag{3}$$

and $x_{i,t}$ is a private signal to agent i only, with,

$$x_{i,t} = \theta_t + \varepsilon_{i,t} \ , \quad \text{with} \ \ \varepsilon_{i,t} \sim N\left(0, \sigma_\varepsilon^2\right) \quad , \quad t = 0, 1, ., \tag{4}$$

and the precision of the public and the private signals are $\alpha = 1/\sigma_\eta^2$ and $\beta = 1/\sigma_\varepsilon^2$. Importantly, η_t, $\varepsilon_{i,t}$ are i.i.d. over time. η_t is independent from $\varepsilon_{i,t}$ for all $i \in \{1, ..., N\}$, and $\varepsilon_{i,t}$ is independent from $\varepsilon_{j,t}$ for all $i \neq j$. Contrary to the Morris and Shin (2002), I assume that if agents are connected in network then they can see the private information of their neighbors.

The main goal is to understand network evolution. For this reason, I develop an algorithm and run simulations for understanding network evolution. The algorithm needs that a modeler generate the information set $\mathcal{I}_{i,t} = \left(y_t, x_{i,t}, \sum_j x_{j,t} \right)$ for every period and the modeler needs a "true" parameter, θ_t^*, unknown to agents in the model. Using different values for $\{\theta_t^*\}_{t=0}^T$ does not change the optimal strategy chosen by agents, as learning is only one period, and the modeler can choose the same θ_t^* for every period.

3. Linear Equilibrium, fixed-point strategies and evolutionary myopia.

The model focuses on an incomplete-information benchmark, where the evolving state variable is the network structure, $\mathbf{M}_t$. Each agent i needs to second-guess the actions of all other agents, which means that each player needs to second-guess the beliefs of other players. The information asymmetry among agents are low compare with Golub and Morris(2017), Hakobyan and Koulovatianos (2020), . If agent i is connected with agent j, this means that the information set which is available to agent i intersects with the information set of agent j: $\mathcal{I}_{i,t} \cap \mathcal{I}_{j,t} = \left\{ x_{i,t}, x_{j,t}, \sum_{l \in \Omega_{ij}} s_l \right\}$, where $\sum_{l \in \Omega_{ij}} s_l$–represents agents i's and agent j's common-friend signals. At the same time, agent i tries to second-guess non-friend ($k \in N_{-i}$) beliefs about the state variable θ_t. Agent i understands that the intersection of his information set with non-friends can be non-empty, because of common friends $\mathcal{I}_{i,t} \cap \mathcal{I}_{k,t} = \left\{ \sum_{e \in \Omega_{ik}} s_e \right\}$, where $\sum_{e \in \Omega_{ik}} s_e$ represents agents i's and agent k's common-friend signals.

The structure of $\mathbf{M}_t$ is common knowledge for all agents. This common knowledge is one of the key assumption in this paper.

Nevertheless, there is limited foresight about the network structure's evolution. In period t, agents only perceive a myopic, narrow-sighted local evolution of their peer connections. This happens at the stage of evaluating the modeler's sampling of invitations for friendship or annoyances received in each period, that I explain below in the section network formation process. I call this nearsightedness of the local evolution of $\mathbf{M}_t$ for one period only, evolutionary myopia[7] .

At first, let's find the fixed-point strategies(the myopic best reply function) for every period, which depend on higher-order belief. The myopic best-reply function is similar as Myatt and Wallace (2019), Golub and Morris (2017)[8] , the only exception is that in our model agents also care about their assimilation bias, and non-friends action . I denote the optimal action taken by players by a_i^{+*} and a_i^{-*}. Each agent makes their decision based on the information set $\mathcal{I}_{i,t}$ available to her. I will skip the notation of the information set and denote agent i's mathematical expectation by $\mathbb{E}_i(\bullet)$ instead of $\mathbb{E}(\bullet|I_i)$.

Agent i maximizes the expected utility function (1) by her own action. First-order conditions imply the following solution for agent i's action:

$$a_{i,t}^{+*} = (1 - r_i)\, E\left(\theta_t + b_i\right) + r_i \left[\frac{q_i}{\#N_i} \sum_{j \in N_i} E\left(a_{j,t}\right) + \frac{(1 - q_i\ )}{\#N_{-i}} \sum_{k \in N_{-i}} E\left(a_{k,t}\right) \right], \tag{5}$$

$$a_{i,t}^{-*} = (1 - r_i)\, E\left(\theta_t - b_i\right) + r_i \left[\frac{q_i}{\#N_i} \sum_{j \in N_i} E\left(a_{j,t}\right) + \frac{(1 - q_i\ )}{\#N_{-i}} \sum_{k \in N_{-i}} E\left(a_{k,t}\right) \right]. \tag{6}$$

Therefore, an agent's optimal decision depends on the expectation of the state variable, $E_i\left(\theta_t\right)$, the expectation of the actions of friends, $E_i\left(\sum\limits_{j \in N_i} a_{j,t}\right)$, and of non-friends, $E_i\left(\sum\limits_{k \in N_{-i}} a_{k,t}\right)$. Moreover, based on both private and public information, the expectation of θ_t is given by (probability density function is defined for the case of the flat (absolutely non-informative:

[7] The evolutionary myopia is a reasonable assumption in directed networks.For example, Twitter is one of the directed network structure, and each agent made a decision which links to create not taking account of other agents action. At the end of the period, where all agents made their decision. The network structure $\mathbf{M}_t$ becomes common knowledge for everyone. If we consider undirected network structure, such as Facebook, creating a link needs to be accepted from both sides, so agent i need to understand if creating a link is valuable for agent k, or not.

[8] For more reference about myopic best-response functions and average based updating of information, see Calvó-Armengol et. al (2009), Bramoullé et al., DeGroot (1974) , Young (1996), Fudenberg et al. (1998) and others. For referense best-response function and bayesian learning, see Acemoglu et al. (2011), Mueller-Frank, M. (2013).

$p(\theta) \propto 1$) prior of θ, see Appendix for proof),

$$E_i\left(\theta_t\right) = \frac{\alpha y + \beta_i x_i + \sum_j \beta_j x_j}{\alpha + \beta_i + \sum_j \beta_j} \tag{7}$$

The linear equilibrium for each agent's action in period t is defined as a weighted sum of all signals in agent i's information set and of the weight on the structural bias, b_i. I call each weight associated with the j-th signal in this sum a 'j-th signal weight'. The presence of network signal transmission results in non-equal weights between private signals. Our educated guess is that the weight of signal j in agent i's action w_{ij} depends on agent i's and agent j's network degrees and on the precisions of signals β.

Given a network structure, I group agents into clusters depending on their closeness centrality measure and precision. Normalized closeness centrality CC_i for the node i is defined as the inverse of the average of the lengths of the shortest paths between the node i and all other nodes in the graph $\mathcal{G}$:

$$CC_{i,t} = \left(\frac{\sum_{j \in V \setminus \{i\}} dist(i,j)}{N-1}\right)^{-1}, \tag{8}$$

where $dist(i,j)$ — the number of the edges in the shortest path between nodes i and j in the network $\mathcal{G}$.

Definition 1 (Clusters) *We will say that there exists cluster $C_{q,t}$ of agents $i \in V$, $q \in \{1, 2, ..., Q_i\}$(q is the number of clusters in the network) if and only if these conditions are satisfied: (i) all members of the cluster $C_{q,t}$ are characterized by the same closeness centrality;(ii) all members of the cluster $C_{q,t}$ are characterized by the same private signal precision; (iii) any agent who does not belong to the cluster $C_{q,t}$ has closeness centrality different from the closeness centrality of this cluster's members, or different precision of private signal:*

$$C_{q,t} = \{i \in C_{q,t} : \forall j \in C_{q,t} : CC_{i,t} = CC_{j,t};\ \forall k \notin C_{q,t} : CC_{i,t} \neq CC_{k,t}\} \tag{9}$$

An agent who has different closeness centrality and different precision from others will belong to his own cluster. In this case, we simply separate this agent from the set of agents with different centrality measures.

The following proposition defines the linear equilibrium characteristics.

Proposition 2 (Linear equilibrium characteristics) *Given a network $\mathcal{G}$, there exists a linear equilibrium in period t, in which agent i's action can be represented in the following way:*

$$a_{i,t}^{+} = \sum_q \omega_{i,t,q}(\mathbf{M}_t)\, x_{i,t,q} + w_{i,t}(\mathbf{M}_t)\, b_i + \left[1 - \sum_q \omega_{i,t,q}(\mathbf{M}_t) - w_{i,t}(\mathbf{M}_t)\right] y_t \tag{10}$$

$$a_{i,t}^{-} = \sum_q \omega_{i,t,q}(\mathbf{M}_t)\, x_{i,t,q} + w_{i,t}(\mathbf{M}_t)(-b_i) + \left[1 - \sum_q \omega_{i,t,q}(\mathbf{M}_t) - w_{i,t}(\mathbf{M}_t)\right] y_t \tag{11}$$

where $0 \leq \sum_{q \in NC_i} \omega_{iq} \leq 1$ and $\omega_{iq} \geq 0$ $\forall i = 1..\mathcal{N}$, where $q = 1..Q_i$, all clusters which appear in agent i's networks.Formulation of the weight for private signals is depicted in the figure below.

$$\begin{aligned} \overline{x}_{i,q,t} &= \frac{\sum_{s \in INC_{i,q,t}} x_s}{|INC_{iq}|}, \\ NC_i &= \{INC_{i,q,t}\}_{q: \exists j \in \{C_q\} \cap \{N_i\}} \\ INC_{i,q,t} &= \{s : x_s \in \{N_i\} \cap \{C_q\}\} \end{aligned}$$

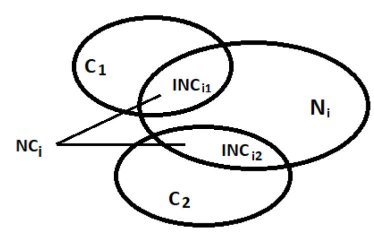

Figure 1. Illustration of cluster definition

Here $INC_{i,q,t}$ — individual neighbour cluster, or the set of agents that are simultaneously in the cluster q and in the agent i's neighbors set. The notation $s \in INC_{i,q,t}$ stands for the index number of the agent s from this set. $NC_{i,t}$— neighbour cluster, or the set of individual neighbour clusters that are not empty for agent i. The notation $q \in NC_{i,t}$ stands for cluster q from the set $INC_{i,t}$.

For demonstrating how the definition and proposition of fixed-point strategies works let's look at the following example.

Example 3 *Let's consider a simple example with 6 agents, that are connected in the network, depicted in Figure 2 .*

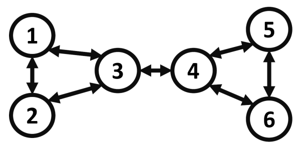

Figure 2. Example with 6 agents

In this network there are two different clusters. The first one consists of agents $C_1 = \{3, 4\}$ *who are characterized by closeness centrality* $CC_3 = CC_4 = 5/7$*. The second cluster consists of agents* $C_2 = \{1, 2, 5, 6\}$ *with closeness centrality* $CC_1 = CC_2 = CC_5 = CC_6 = 1/2$*. I demonstrate the difference between clusters using different weight simbols on private signals* υ *for cluster* C_1 *and* ω *for cluster* C_2*.*

$$a_{1,1} = \omega_1 \frac{(x_1 + x_2)}{2} + \omega_2 x_3 + w_1 b_1 + (1 - \omega_1 - \omega_2 - w_1) y$$

$$a_{2,1} = \omega_1 \frac{(x_1 + x_2)}{2} + \omega_2 x_3 + w_2 b_2 + (1 - \omega_1 - \omega_2 - w_2) y$$

$$a_{3,1} = \upsilon_1 \frac{(x_1 + x_2)}{2} + \upsilon_2 \frac{(x_3 + x_4)}{2} + w_3 b_3 + (1 - \upsilon_2 - \upsilon_1 - w_3) y$$

$$a_{4,1} = \upsilon_1 \frac{(x_5 + x_6)}{2} + \upsilon_2 \frac{(x_3 + x_4)}{2} + w_4 b_4 + (1 - \upsilon_2 - \upsilon_1 - w_4) y$$

$$a_{5,1} = \omega_1 \frac{(x_5 + x_6)}{2} + \omega_2 x_4 + w_5 b_5 + (1 - \omega_1 - \omega_2 - w_5) y$$

$$a_{6,1} = \omega_1 \frac{(x_5 + x_6)}{2} + \omega_2 x_4 + w_6 b_6 + (1 - \omega_1 - \omega_2 - w_6) y$$

Hence, in this example, the weights are the same within each cluster and differ between clusters. In this example we consider that $\beta_1 = \beta_3 = \beta_2 = \beta_4$ *and* $b_1 = b_2 = b_3 = b_4$*. This example is demonstrates what happens in the first period*[9] *. In Appendix I generalize the solution of finding linear equilibrium weights for* $Q_i = \mathcal{N}$ *different clusters.*

The linear equilibrium action should be optimal for all agents in the network. It means that the action, characterized by (10) and agent i's optimal actions (1) should be the same. This gives us a system of linear equations. This system provides us with a solution for equilibrium weights as a function of the parameters. Once the weights are found, optimal actions should be calculated using the following algorithm.

[9] *Notice that we normalize the weights. Normalizing is possible because the objective functions are ordinal utility functions.*

Algorithm 4 (Finding optimal actions) (i)*Substitute all other agents' actions (10) into the term* $\sum_{j\in N_i} E_i\left(a_{j,t}\right)$, $\sum_{k\in N_{-i}} E_i\left(a_{k,t}\right)$. *If the agent has no information about the signal* j, *then* $E_i x_j = E_i \theta$;

(ii)*Rearrange the terms to get a coefficient preceding the* $E_i\left(\theta_t\right)$ *term. Then substitute the mathematical expectation as a function of agent* i*'s signals (7).*

(iii)*Rearrange the terms to get a coefficient preceding each agent* i*'s signal. These coefficients should be equal to the corresponding weights in the linear equilibrium (10). The solution to the resulting system of linear equations is the vector of equilibrium weights.*

(iv) *Find the optimal action.*

Substituting the optimal action strategies in the objective function of each player gives the value functions (indirect utility functions),

$$V_i^{+}\left(\mathbf{M}_t\right)=E\left(u_i^{+}\left(a_{i,t}^{+*},\theta_t\right)\ \mid\mathcal{I}_{i,t}\right)\ ,\quad i=1,...,N_{+}\ ,\tag{12}$$

and

$$V_i^{-}\left(\mathbf{M}_t\right)=E\left(u_i^{-}\left(a_{i,t}^{-*},\theta_t\right)\ \mid\mathcal{I}_{i,t}\right)\ ,\quad i=1,...,N_{-}\ .\tag{13}$$

The value function will influence the evolution of the network structure. I demonstrate this influence in the next section.

4. Network formation process

The state variable in each period is the network structure. The network dynamics are governed by two main processes: *(1)* The Social Planner uses different sampling processes in order to create possible invitations(a link which can be created) to non-friends for each agent and annoyances(a link which can be broken) from friends of each agent. *(2)* Subject to the sampling process of possible invitations/annoyances chosen by the social planner, agent i uses his own value-function criterion in order to decide upon whom to add from the sample of non-friends ($k\in N_{-i}$) and whom to exclude among friends $j\in N_i$.[10]

[10] I describe this process below in subsection "second stage decision making process: creating/deleting a link".

In process *(1)* above, the Social Planner selects different processes in order to manipulate network dynamics. Given, however, that process *(2)* gives freedom to people to choose their network friends, it is a liberal social-planner manipulation[11] .

4.1 First stage of decision making: Sampling process

My main goal is to examine how a social planner who cares only about bringing optimal actions of agents closer to fundamentals, can solve coordination problems among agents by varying the sampling process of invitations and annoyances sent to non-friends and friends of each agent. I will call this selection of sampling processes by the Social Planner "Social planner manipulation of network dynamics". I will compare this sampling process to two other sampling algorithms, the uniformly random sampling and the biased sampling. The next subsections will provide a more detailed explanation of these algorithms.

4.1.1 *Social planner manipulation of network dynamics*

In the role of a modeler, I introduce a social planner, who doesn't care about individuals' biases. The social planner cares how to bring agents' actions closer to the state variable θ_t. The utility function of a social planner is given by,

$$W = -\frac{\sum\limits_{i \in N} (a_{i,t} - \theta_t)^2}{N} \quad \text{for all } i = 1, 2, ..., N \tag{14}$$

The utility function of social planner looks like as Morris and Shin (2002) utility, but contrary to their case, in my model agents optimal actions contain bias. The social planner understands that optimal actions of agents are influenced by their biases, and he tries to minimize the effect of these biases by manipulating the set of possible invitations and annoyances. In Hakobyan and Koulovatianos (2020) authors show that biases, such as assimilation

[11]In the role of a social-planner can be social-media platforms owners/government.

bias or confirmation bias, can increase polarization and populistic behavior, as time passes. In this paper, I show that the social planner can solve the problem of polarization in network dynamics, using his power to manipulate the sampling process of possible invitations and annoyances. I consider two social-planner types: *(a) a social planner with perfect information* about the state variable θ_t in each period, and *(b) a social planner with incomplete information* about θ_t. For the second case, I will assume that, like common agents, the social planner gets public and private information or has some prior about θ_t, which is constant, but generally $\theta_t^{sp} \neq \theta_t^*$.

The social planner directly manipulates network dynamics. In the first period, he takes the adjacency matrix $\mathbf{M}_t$ and calculates the social welfare W. After fixing the welfare level, he calculates all possible changes in the network structure that are driven by agents' decisions, and suggests a vector of possible invitations together with a vector of possible annoyances, which increase his utility function.

Algorithm 5 (Social-planner manipulation of network dynamics) *(1) The social planner takes the adjancency matrix $\mathbf{M}_t$, calculates social welfare W and fixes it. (2) The social planner takes each row of the adjancency matrix, $\mathbf{M}_t$, and calculates social welfare taking into account all possible changes of the adjancency matrix driven by potential agent link connections. (3) The social planner chooses one agent from a sample of k with whom agent i could create a link, making social welfare to increase as a result of establishing this link. (4) The social planner chooses some agent j with whom agent i can delete a link, making social welfare to increase as a result of deleting this link.*

A matrix of possible invitations and annoyances (PIA matrix) is created. The size of the PIA matrix is $N \times 2$, where the first column shows possible invitations and the second column shows the possible annoyances. After the creation the PIA matrix, the game proceeds to the second stage of decision making, where agents decide which link to create and which link to delete, depending on their value functions.

4.1.2 *Uniformly Random Sampling*

In each period the social planner randomly creates a vector of possible invitations and annoyances, using the following algorithm.

Algorithm 6 *(1) The social planner takes the adjancency matrix* $\mathbf{M}_t$ *and randomly chooses one agent from a set of k (individuals in the i-th row of* $\mathbf{M}_t = 0$*) with whom agent i can create a link, and saves the index of agent k in the sample of possible invitations. (2) The social planner randomly takes, from a set of j (individuals in the i-th row of* $\mathbf{M}_t = 1$*), one agent with whom agent i can delete a link.*

4.1.3 *Biased Sampling*

There is a vast literature examining whether the network structure is random or not.[12] Here the social planner uses a biased sampling algorithm. In this model agents get invitations from friends of friends. Such algorithms are used in real-world social-network platforms.[13]

Algorithm 7 *(1) The social planner calculates the number of common links agent i has with each of his non-friends, suggesting the agent k, with whom agent i has the most common friends (e.g., meeting friends of friends). (2) For determining the set of annoyances, the social planner uses the opposite. He calculates, among friends, the agents with whom agent i has the fewest common friends, creating a set of agents who cause an annoyance to agent i.*[14]

4.2 Second stage of decision making: creating/deleting the links

Agent i makes the decision of creating a new link or of deleting an old friend, conditional on the set of invitations and annoyances that has been created by the social planner. Player i

[12] See, for example, Jackson et al. (2007), Snijders, et al. (2010), Bhattacharya et al. (2017), Golub and Livne (2011), among others.

[13] This biased sample is very common in such networks like Facebook or Vkontakte. For example Facebook suggests a potential friends list (from a group of non-friends who have common characteristics to these of agent i). These characteristics can include friends of friends, or sometimes people who belong to the same groups of interests (in my model this corresponds to agents with the same fundamental bias, $\pm b$).

[14] *Suggestions of breaking links is not crucial in this model. In social media platforms as Facebook, Vkontakte agents can ignore friends messages, or unfollow friends news, but platform will continue show them as friends. In our model I consider such behavior as breaking a link, because the link between agents show the information transmission process.*

receives one invitation and experiences one annoyance, i.e., he examines 2^2 cases. These cases consist of $\{0, 1\}$ choices. Choice "0" stands for either not creating a new link or excluding an old friend based on a caused annoyance. On the contrary, choice "1" stands for either creating a new link or keeping an old friend, despite a caused annoyance. The algorithm creates a 2^2 versions of the original matrix $\mathbf{M}_t$, with each agent calculating his value function for all possible cases, choosing the $\mathbf{M}_t$ version that gives him the maximum value-function level. The generalized version of calculating the value function for N different clusters is introduced in the Appendix (7.4). This paper is focused on directed graphs, so the game evolves as $\mathbf{M}_{t+1} = \widetilde{\mathbf{M}_t}$, where $\widetilde{\mathbf{M}_t}$ is the updated version of the adjancency matrix. Notice that $\widetilde{\mathbf{M}_t}$ is not symmetric.[15]

5. Simulation experiments

Due to the complexity of equilibrium conditions, and the network formation process, I perform 100 Monte-Carlo simulation experiments in order to analyze the network dynamics, provide answers to the following questions:

- Is the manipulating power of the social planner, capable of solving the coordination problem among agents, bringing their actions closer to fundamental variables, and thus increasing social welfare?

- Can the social planner increase social welfare, just by manipulating the set of possible invitations and annoyances, without directly changing a network structure?

- What are the main drivers of increasing social-planner utility functions?

[15] An interesting extension is to add one more step for capturing a feedback effect, where agents i and j can update information only if both of them decide to add each other to their subnetwork of friends.

- How do the results change if the social planner has incomplete information or wrong priors about the fundamental variable?

For simulation experiments I use the following parameters: the "beauty-contest" parameter $r = 0.7$ (the results are similiar with $r \sim U[0,1]$) for all agents, the weight on the action of friends is set to $q \sim U[0,1]$.[16] The precision of the public signal is $\alpha = 1/\sigma_\eta = 30$, while the precision of the private signals is randomly distributed following a uniform distribution, parametrized as, $\beta_i \sim U[10,45]$. We differentiate agents into two groups: agents with positive biases and agents with negative biases, $b_i \sim U[0,1]$. There are $N = 20$ agents, and we split them into two groups $N_1^+ = 10$ and $N_2^- = 10$.

For performing comparative dynamics among different algorithms and for studyung their influence on social welfare, I fix the adjancency matrix in period 0. I randomly generate a non-symmetric original matrix $\mathbf{M}_t$ in period $t = 0$ and fix it in order to understand how different sampling processes implemented by the social planner influence network dynamics.

I perform comparative dynamics in two main cases: *(1) A social planner with perfect knowledge* of the true state variable θ_t, and *(2) a social planner with incomplete information/knowledge* about state variable θ_t.

5.1 *Social planner with perfect knowledge* about state variable θ_t

In this subsection I assume that the social planner has perfect knowledge about the fundamental variable $\{\theta_t^*\}_{t=0}^T$ in every period. As mentioned in Section 3, there is no learning between periods. In my simulations, I choose the same fundamental variable, $\theta_t^* = 0$, in every period, and, in this Section, I assume that the social planner knows that $\theta_t^* = 0$, in every period. The welfare function of the social planner, given by (14) can be transformed

[16] Svensson, (2006) argues that the main Morris and Shin (2002) result is present only if the second-guessing motive is relatively high $r \in (0.5, 1)$.

into a matrix form, after doing some algebra. The social planner's welfare function in a matrix form is,

$$W = -\ \mathbf{1}^T \left(\bar{\omega}_{b\circ}^2 \circ \bar{b}_{\circ}^2 + \bar{\omega}_{b\circ}^2 \circ E^*(\theta^2) - 2 * \bar{\omega}_{b\circ}^2 \circ \bar{b} \circ E^*(\theta) + W_{\text{private}\circ}^2 \left(\frac{\mathbf{1}_\circ}{\overline{\beta}} \right) + \frac{W_y^2}{\alpha} \right) \quad (15)$$

where $\mathbf{1}$ is a vector of ones, the size of this vector is $N \times 1$.[17] $\bar{\omega}_b$ is the weight for bias which was describe by equation (18)[18] .$\overline{\beta}$ is a vector of private signal precision (size of this vector is $N \times 1$). $E^*(\theta^2)$ is described in Appendix. Symbol $\circ$ represents element-by-element multiplication.

In the case of perfect knowledge of the state variable ($\theta_t^* = 0$), following equation (15), one can see, that the social planner's welfare depends on the weight that every agent puts to the bias, and on weights in the Kronecker products of private and public signals with the precision of signals. Therefore, the social planner manipulates network dynamics in order to decrease these components in his utility function.

For understanding the network evolution of well-known social platforms, and compare them with social planner manipulation, I run simulation experiments. I begin with our comparative dynamics, with a benchmark parameters, and examine how network evolution depends on social-planner's manipulation. Figure 2a depicts these network dynamics. As it can be seen in Figure 2a, in the last periods the key agents share their information with others more actively, and the numbers of indegree links increase. I prove analyticaly and by using simulation example the importance of indegree and outdegree links below.

[17]Notice that the expression in the brackets is an $N \times 1$ matrix, and multiplying it with the transponse of vector ones ($\mathbf{1}^T$) will give us a scalar.
[18]Please notice, that the optimal-action weight on the bias depends on network structure.

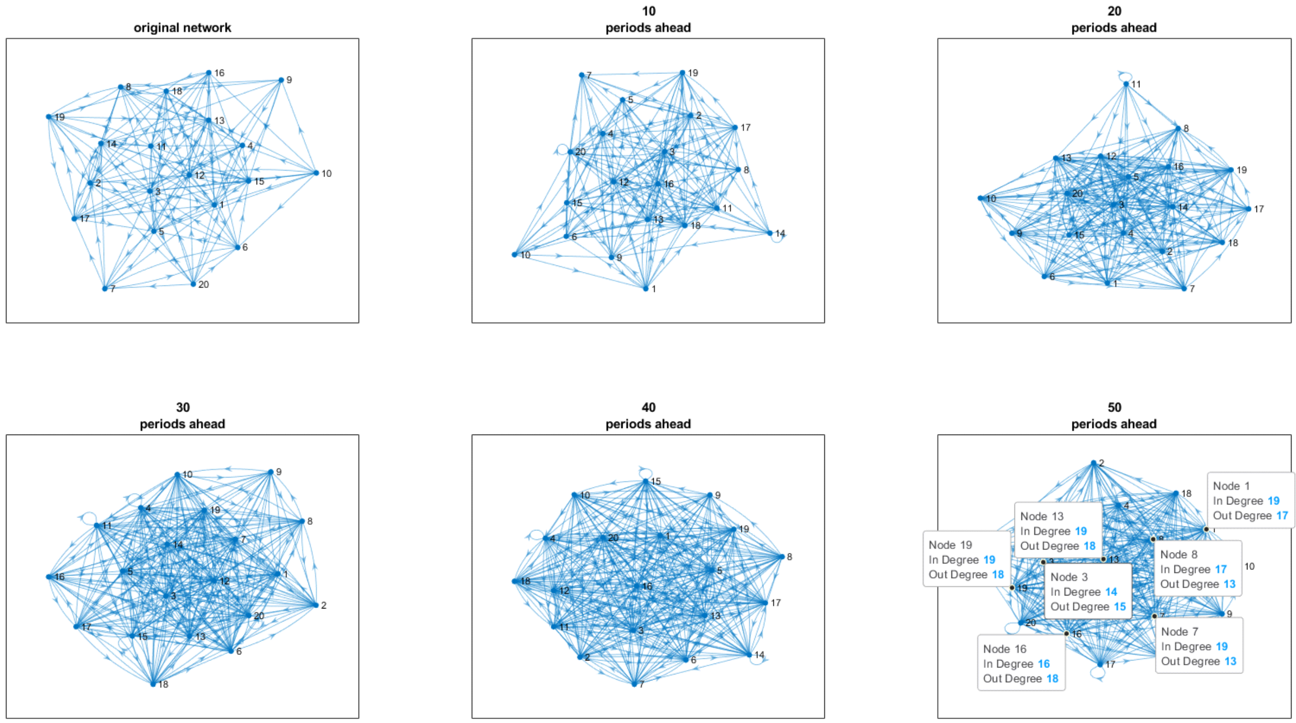

Figure 3a. Social planner manipulation of network dynamics

The number of nodes directed towards agent i (indegree nodes) shows the number of agents who receive agent i 's signal. As we can see in the last periods, the graph looks like a combination of a star network and a ring network. Comparing the three cases of network evolution that we can see, there are similarities in the node degrees between the network dynamics also in the cases of social planner manipulation (Figure 2a) and the biased sampling by the social planner (Figure 2b). On the contrary, social planner manipulation (Figure 2a) and random sampling (Figure 2c) there are no similarities: as time passes the outdegree links of key players increase, instead of the indegree links increasing. Therefore, a combination of indegree and outdegree links seem influence social welfare the most.[19] Below we take a closer look on how social welfare depends on the evolution of networks in these

[19]For these particular network dynamics look at the social welfare dynamics in the Appendix. In main body I present results of 100 Monte-Carlo simulations.

cases.

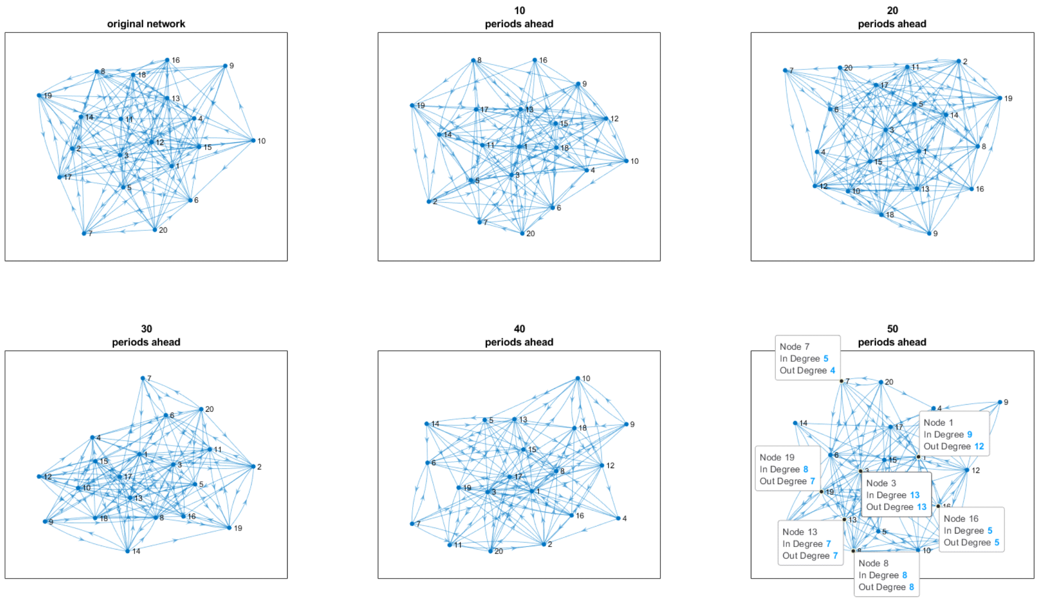

Figure 3b. Evolution of network dynamics—bias sampling

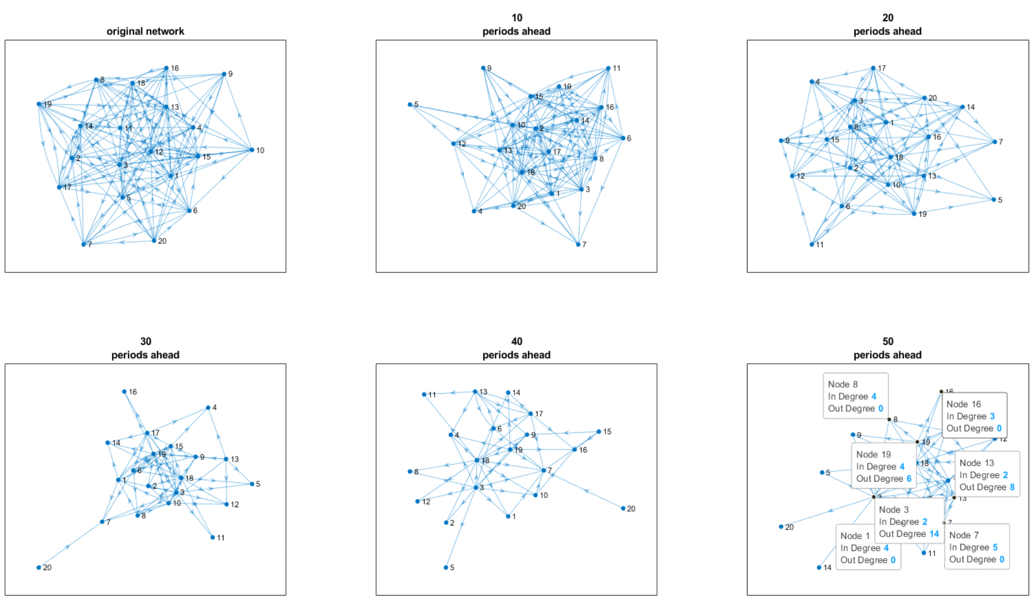

Figure 3c. Network evolution with randomly invitations and annoyances

Comparative dynamics and social welfare As mentioned above, the original adjancency matrix $\mathbf{M}_0$ is the same for all sampling algorithms. But from $t = 1$ the set of possible invitations and annoyances differ, depending on the different algorithms. Figure 3 illustrates how welfare dynamics differ across cases. The time horizon is set to $T = 50$. As we can see in the benchmark case, where $r = 0.7$, $q = U[0, 1]$ for all agents the difference in friends of friends between the social planner manipulation case and that case of biased sampling is not big. Yet, the difference between random sampling and the two other sampling process is substantially big.

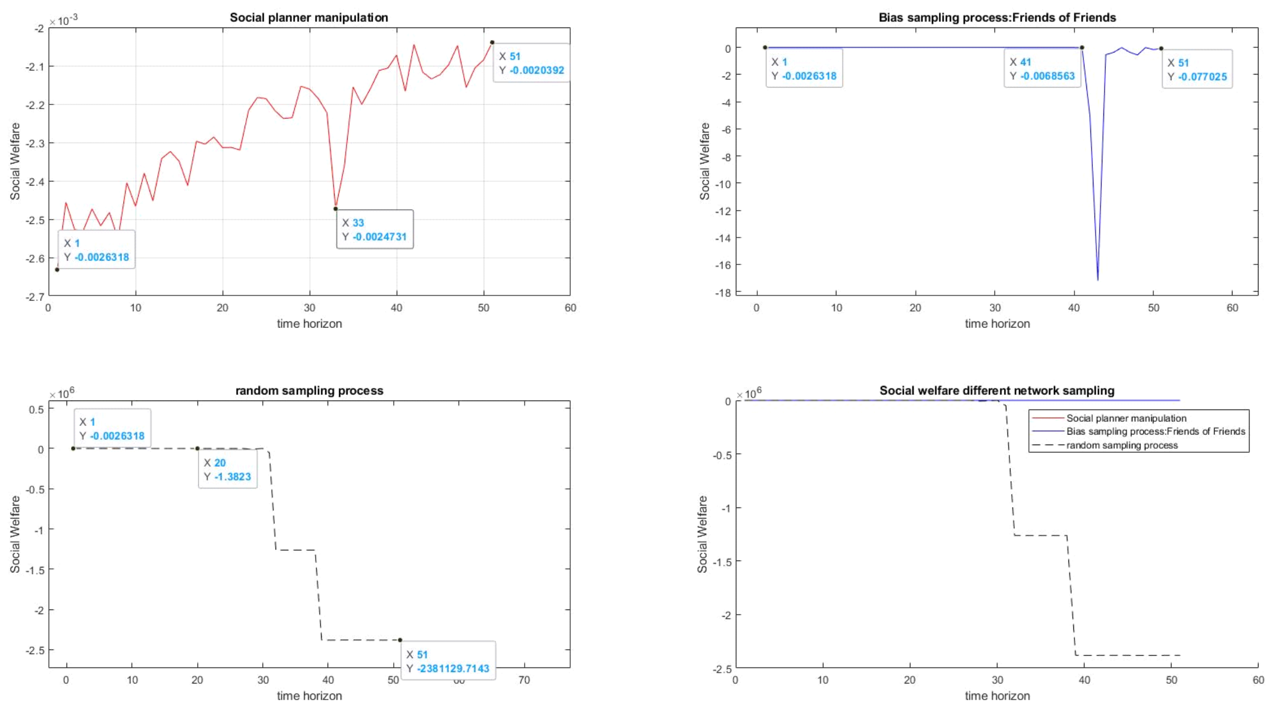

Figure 4. Comperative statics: Social Welfare: 100 Monte-Carlo Simulations.

The key explanation of this distance lies on how many indegree and outdegree links agents have. As we can see from Figure 4 in some periods there is a sudden drop in social welfare. There can be several reason of such drop, *(1) "tragedy of the commons" problem.* The algorithm of social planner maximize changes in adjancency matrix by row, which influence on the second stage decision making process. One of the solutions of such drops can be changing the social planner manipulation strategy, by adding a step in the algorithm, where the social planner tries to compare pairwise stability of changing rows, and decide which row to change and which row to keep as it is in period t.[20] *(2) "Not enough Monte-Carlo simulations."* One of the solution of such drops increase the numbers of Monte-Carlo simulations. In 100 Monte-Carlo simulations only one simulation demonstrate such drop in social welfare, and effect of this simulations influence on other 99 Monte-Carlo simulations. Increasing the number of Monte-Carlo simulations can smooth such drops.[21]

The key result of Figure 4 is increasing curve for social planner manipulation strategy, and one of explanation is connection between node degree and social welfare. I analyze the connection between node degree and social welfare looking at a one-period (static) game.

5.1.1 Analyzing the connection between node degree and social welfare

In order to keep the analysis simple, I examine the static game, with the following network topology: I consider a central agent with a ring network, demonstrated in Figure 4.

[20] I will add this step in a future version of the draft.

[21] In future work I will further extend the results for 500-1000 Monte-Carlo simulations and robustness checks of results.

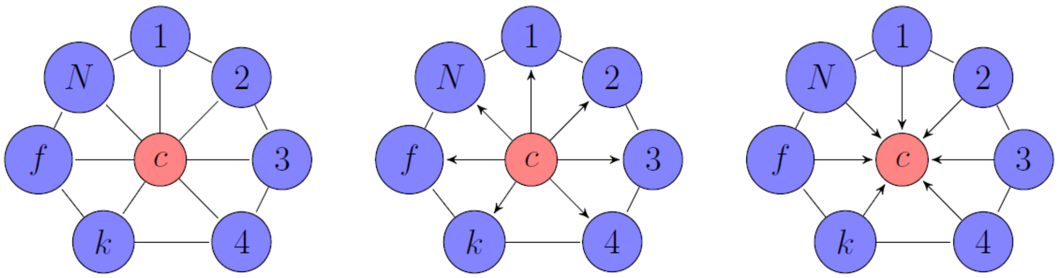

Figure 5. Star network with combination of ring network.

The first graph shows the undirected graphs, so everyone can see the signal of each other. The second graph, shows that the central agent receives signals frm other agents, not sharing his information ($C_{\text{Outdegree}} > C_{\text{Indegree}}$). The third graphs shows that the central agent shares his private signal, but doesn't get signals from others ($C_{\text{Outdegree}} < C_{\text{Indegree}}$).[22] For the sake of simplification, I consider a graph with two clusters only (see the definition of clusters in Section 3). Let's assume that the precision of the private signal is the same for all agents $\beta_i = \beta$, and in this static game there is no bias$b_i = 0$. This simplification will give us the opportunity to demonstrate that the Morris and Shin (2002) results do not go through in particular network structures.

Lemma 8 *Let $\mathcal{G}$ is described as in Figure 4. Suppose $\beta_i = \beta$, $r \in (0.65, 1)$,$b_i = 0$. The optimal weights w_s^*,w_r^*,w_c^* is described by formula (20) ,(23) and (21) . Using this optimal weights, the social welfare is described by the equation (24) ,(25) and (26). And the following holds:*

1. For every $\beta = \overline{\beta}$ and $\alpha < \overline{\beta}$, the $\frac{\partial W_c}{\partial \alpha} > 0$ if agent i put weight to central agents signal and $\beta = \overline{\beta}$ and $\alpha > \overline{\beta}$, the $\frac{\partial W_c}{\partial \alpha} > 0$ if agent i put weight to public signal.

2. For every $\beta = \overline{\beta}$ and $\alpha < \overline{\beta}$, the $\frac{\partial W_s}{\partial \alpha} > 0$ if agent i put weight to central agents signal and $\beta = \overline{\beta}$ and $\alpha > \overline{\beta}$, the $\frac{\partial W_s}{\partial \alpha} > 0$ if agent i put weight to public signal.

3. For every $\beta = \overline{\beta}$ and $\alpha \lesseqgtr \overline{\beta}$, the $\frac{\partial W_r}{\partial \alpha} < 0$.

In Figure 6a. I will illustrate the effect of the public and private signals precision on the welfare. More formal proof is available on online Appendix.

[22] An analytical solution can be found in Appendix .

I call these networks "central agent", "central receiver" and "central sender" respectively. The effect of the public and private signals precision on the welfare in these network structures is illustrated in Figure 6a.

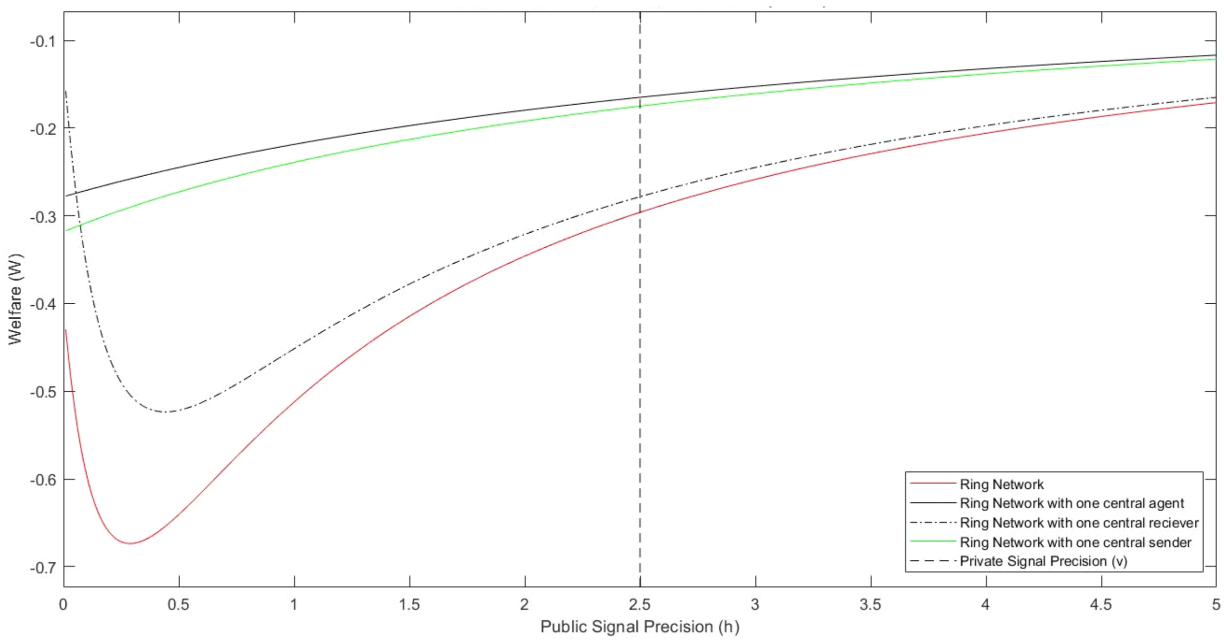

Figure 6a. Social Welfare, comparing information sharing in central agent contest.

It can be seen that in the case of one central receiver the behavior of welfare is similar to this in the ring network. However, for each value of α, the welfare for the former network is higher than the welfare for the latter one. It can be caused by the presence of the agent, who is better informed compared to other agents. Since this central agent does not send any signal, her effect on the other agents' actions is minor (only through the presence of the unobserved signal in the transparency term $(E(\sum a_k))$.

However, in the case of the central-sender type of network, the Morris and Shin (2002) result vanishes. Even for high values of r, the welfare function is monotonically increasing with respect to α. Therefore, crucial result in this paper is that each agent's private signal sent in network can be viewed as a substitute to the public signal sent by the authority.

Here I considered the extreme case: each agent in the network receives the same signal from the central agent. It means that agents are able to choose between two signals, that has the same characteristics as long as their precision parameters are the same. The effect of switching between signals is illustrated by Figure 6b.

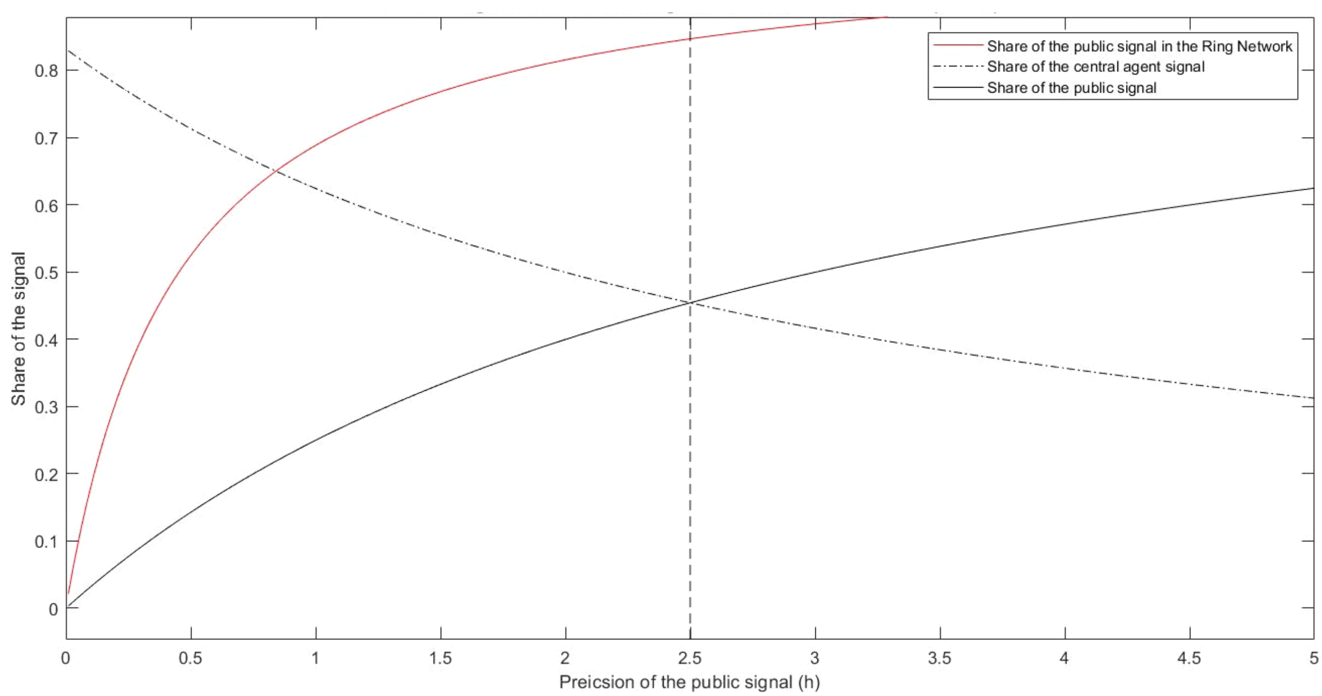

Figure 6b. Share of signals in non-central agents actions.

Figure 6b shows that, the higher the public signal precision, the higher its share in the non-central agent's action. Moreover, share of all other signals is approximately equal to zero. This means that non-central agents simple switch between two signals: the public one and the central agent's one.

Since they have freedom of choice, agents can choose the most precise signal available. This fact leads to the absence of the Morris and Shin (2002) result (the noisy signal is simply ignored). Therefore, the central agent's and central sender's strategy can solve the problem raised in the paper by Morris and Shin (2002).

Therefore, coming back to our social-planner manipulation of networks dynamics agents with high precision of private signals end up with high indegree links, and agents which have low precision of private signal end up with high outdegree centrality and this effect increases social welfare.

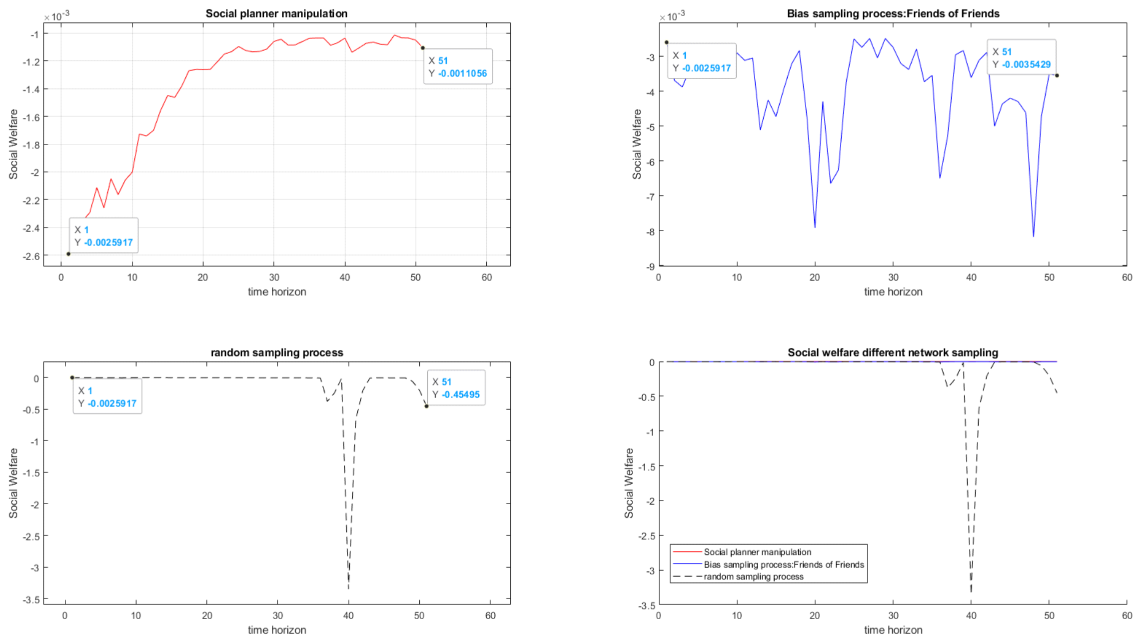

Figure 7. Social welfare dynamics for the case described in Figure 3a,3b,3c

The connections between node-degree, private and public signal precision can be demonstrated in the Table 1. Please note that the weight of signal j in agent i's action w_{ij} depends on agent i's and agent j's the precisions of signals and degrees in the network. So the Table 1 answer partially to the question.

Agent_N	Beta	Gam_indegree	Gam_outdegree	SPindegree	SPoutdegree	bindegree	boutdegree	r_indegree	r_outdegree
1	28	11	8	19	18	12	12	4	1
2	21	10	9	17	16	10	7	5	1
3	39	12	10	15	16	14	15	3	16
4	39	12	3	19	10	6	6	4	1
5	30	13	5	20	17	7	9	3	1
6	19	6	8	20	18	11	9	4	6
7	34	6	7	20	15	6	5	6	1
8	18	11	6	17	13	8	8	6	1
9	26	6	5	14	16	4	5	5	1
10	23	4	9	17	16	7	6	5	5
11	29	10	6	17	18	8	7	3	1
12	45	11	12	17	16	9	9	3	1
13	37	9	10	20	18	9	9	3	8
14	45	7	9	13	17	4	5	2	1
15	18	8	10	10	17	7	9	2	7
16	29	6	8	17	18	5	4	4	1
17	11	6	11	4	18	8	8	3	6
18	37	8	10	14	18	6	7	4	9
19	31	6	9	20	18	9	9	5	7
20	40	3	10	19	16	6	7	2	1

Table 1. Node degree data, precision of private signal. The precision of public signal is $\alpha = 30$.

In Table 1. Gam_indegree and Gam_outdegree describe the original matrix $\mathbf{M}_0$. SPindegree and SPoutdegree colums describe the node degree after 50 periods in the case of social planner manipulation of network dynamics (Figure 3a). bindegree and bourdegree columns show the results in Fugure 3b, and r_indegree and r_outdegree columns are the illustration of Figure 3c.

5.2 *Social planner with incomplete information* about state variable θ_t

In the real world it is diffucult to find a social planner who exactly knows all fundamental variables, in every period. Therefore, I examine the case of an imperfectly informed social planner. In this section I examine two social-planner types: *(1) A social planner with incomplete information,* and *(2) a social planner with wrong expectations* $E(\theta_t) = \widetilde{\theta}_t \neq \theta_t^*$. The second case can be useful if we consider public policy issues in the autocratic regimes.

Where agents/citizens have precise signals about the fundamental variables, but the regime has wrong expectations and tries to manipulate agents actions closer to their expectation.

Social planner with incomplete information

In this case the social planner has incomplete information about the state variable θ_t, and receives private and public signals like all other agents in the model. Therefore the social planner calculates $E\left(\theta_t|\mathcal{I}_{sp}\right)$, with $\mathcal{I}_{sp}$ denoting the information set of the social planner, consisting of a private and a public signal. The social planner's public signal is the same public signal as that of all other agents. For the benchmark case I consider that the social planner's private signal has higher precision, than the precision of the public signal and precision of the privete signals of all other agents. Under imperfect information, $E\left(\theta_t|\mathcal{I}_{sp}\right) \neq 0$, and social welfare also depends on the expectations of the social planner.

$$W = -\ \mathbf{1}^T\ \left(\bar{\omega}_{b\circ}^2 \circ \bar{b}_{\circ}^2 + \bar{\omega}_{b\circ}^2 \circ E\left(\theta_t^2|\mathcal{I}_{sp}\right) - \bar{\omega}_{b\circ}^2 \circ \bar{b} \circ E\left(\theta_t|\mathcal{I}_{sp}\right) + W_{\text{private}\circ}^2 \left(\frac{\mathbf{1}_{\circ}}{\bar{\beta}}\right) + \frac{W_y^2}{\alpha}\right)$$

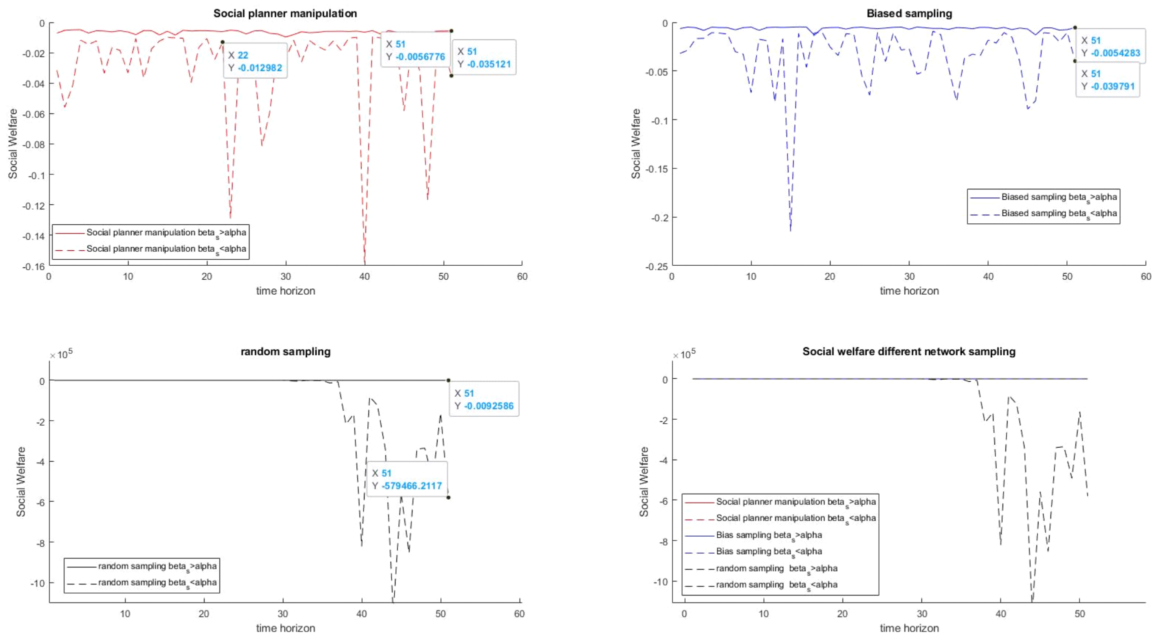

Figure 8a. Social welfare: Incomplete information benchmark

As we can see from Figure 8a, this setting of imperfect information leads to worse results than in the case of biased sampling. This happens because the social planner cannot make the indegree links of key players to increase. A demonstration through simulated evolution dynamics that the indegree links are higher in the biased sampling setting than in the examined case here, can be found in Appendix . The key conclusion that can be reached in the incomplete information benchmark is that following any social planner manipulation of network dynamics can be useful only in short horizons. Therefore, I decrease the time periods and I examine the short-horizon case.

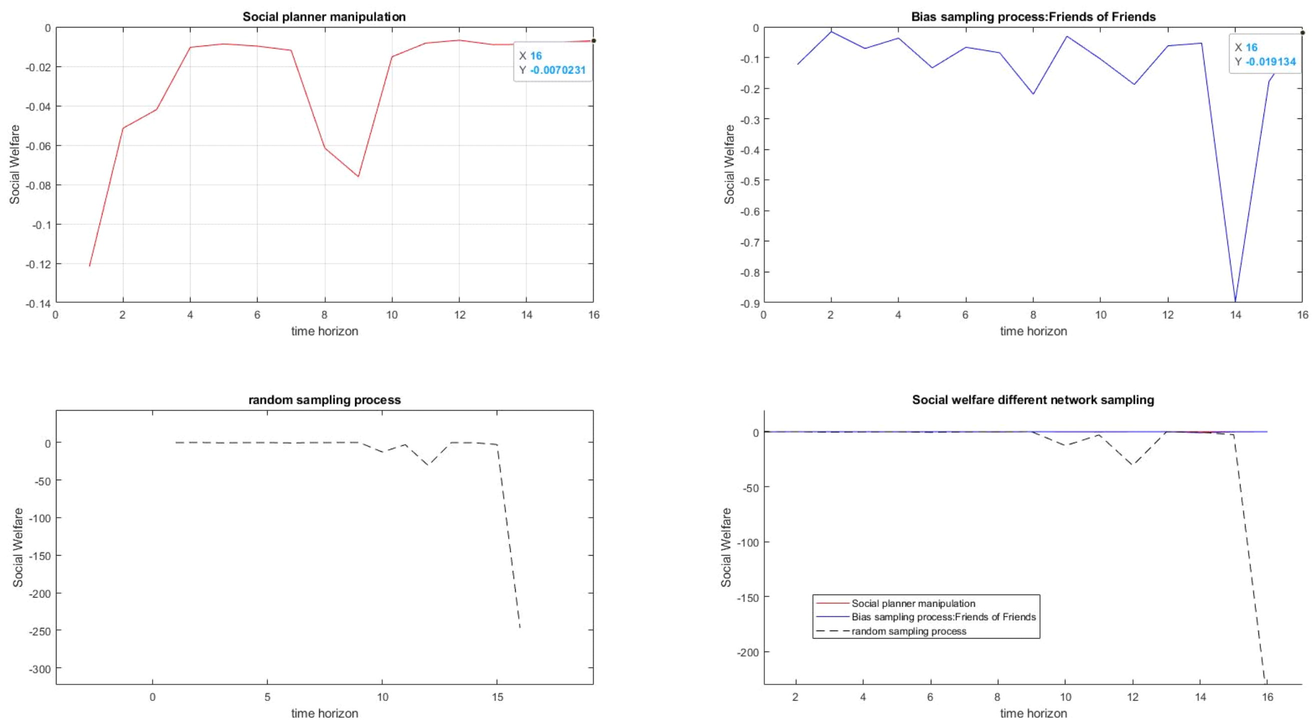

Figure 8b. Social welfare: short-time periods

As it can be seen, in the incomplete information case, social-planner manipulation works better for short horizons. This means that the social planner does not need to manipulate the sampling process in all periods. This result can be further examined through Monte-Carlo simulations.

Social planner with wrong expectations

In this case of imperfect knowledge on the side of the social planner, I examine that the planner has wrong expectations about the state variable, i.e., $E(\theta_t) = \widetilde{\theta}_t \neq \theta_t^*$. Under wrong expectations, the social planner has some prior beliefs about the state variable of the type $\theta_t = \widetilde{\theta}_t = const$. For example, if the true value is θ_t^*, the social planner believes that $\theta_t^{*,sp} = \widetilde{\theta}_t \neq \theta_t^*$. Can we see increasing social welfare in this setup in every period?

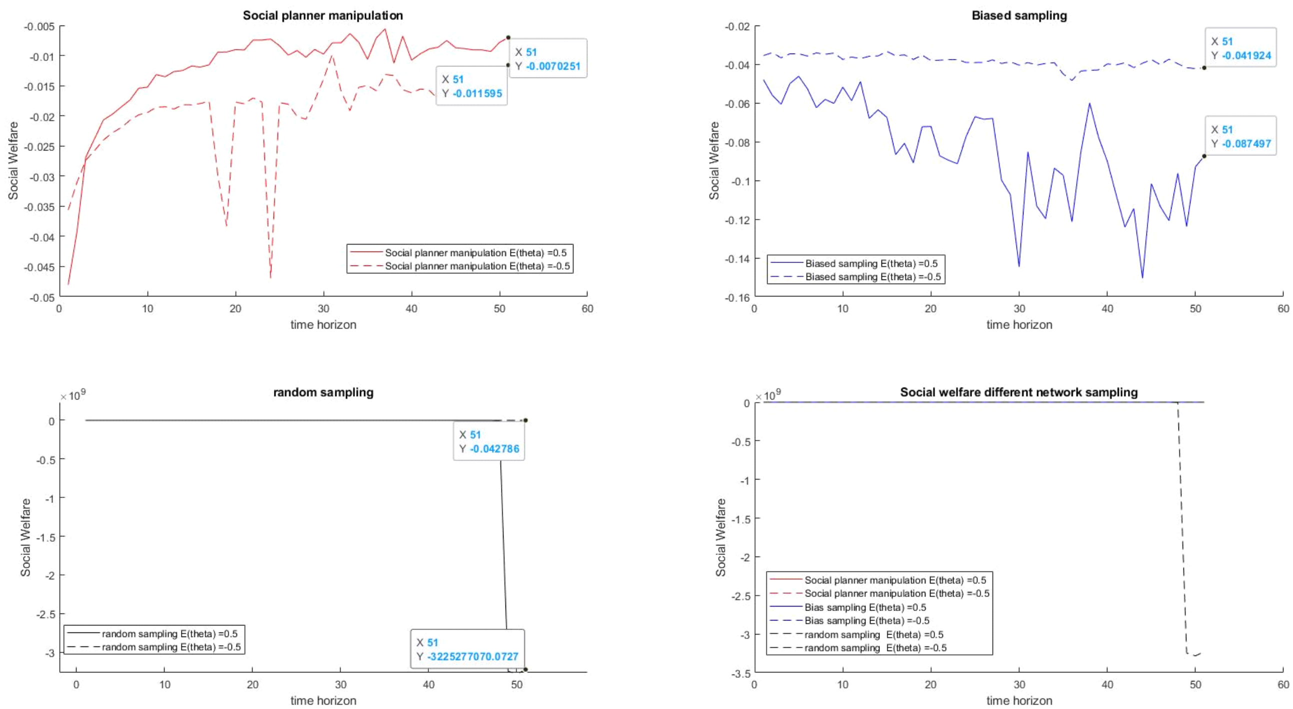

Figure 9. Social welfare. Wrong expectations of social planner.

As we can see, under the incomplete-information benchmark, it is better if the social planner has wrong constant expectations compared to receiving private and public signals. This happens because the noise term and the precision of signals that influence expectations about the state variable, $E(\theta_t)$, change every period. Therefore, it is better if $E(\theta_t)$ is equal to wrong constant in social-planner's mind, compared to the case that the social planner gets more precise signals then all other agents.

6. Conclusion

Polarization has increased during the last decades. There is a large literature in political economics, trying to understand how to solve problems polarization in networks, by spotting the key players who distort information, trying bring agents' action closer to pragmatic variables.

I examined whether a social planner, such as the manager of a social-media internet platform, can manipulate network dynamics so as to bring agents' actions closer to pragmatic viewpoints, thus increasing social welfare. Specifically, I examined if social planners can influence network dynamics by recommending people as network friends to online-platform users, and by pointing annoying behaviors by existing social-media friends. Importanty, I let network users to decide alone whom to make a new friend and whom to abolish as network friend. I also examined how my analysis changes if the social planner also has incomplete information or wrong priors concerning fundamental variables.

I built a dynamic network formation model, where each agent has strong incentives to coordinate their action with the actions of other agents and also to find out the truth about fundamental variables. Using simulations, I demonstrated that if the social planner is perfectly informed about fundamentals, then his policy will be to suggest agents create more indegree links, if their private signal precision is high compare with public signal, and agents, who have low private signal precision, create more outdegree links. This strategy is crucial for increasing social welfare. Using a static game example, I provided a formal proof of why social welfare goes up when nodes with indegree centrality increase.

One of the main explanations of this result is the following: if central agents share their information with others, their signals have the same characteristics as public information, and agents can decide on switching from one signal to another if the precision of the public signal increases. This characteristic of my analysis improves a feature of the standard Morris and Shin (2002) model who find that increasing the precision of public information can decrease social welfare. Specifically, in my model, I demonstrated that social planners try to organize the network structure as a combination of star networks. It is this structure that can increase social welfare.

Interesting is also the case where the social planner has imperfect imperfect knowledge about fundamentals. I demonstrated, that in both cases, this where the social planner has noisy information, and this where the social planner is sure about the wrong fundamental value, social-planner manipulation can increase social welfare. But the most interesting result is that with a social planner being sure about the wrong expectation, welfare improvement is higher than having noisy information about fundamentals, even if the public and private signals the social planner receives have lower noise than those of the agents. One of the key explanations can be that if the social planner fixes an expectation of the fundamental value, then it can be easier for him to organize the network dynamics. In the case of noisy signals, the changing signals of agents combined with the noisy signals of the social planner, bring some mess to the social planner's strategy.

To the best of my knowledge, this is the first paper where social planner tries to manipulate network in a dynamic setting, not by directly influencing agents' action, but by just trying to introduce agents to each other in a way that social welfare will increase. An appealing feature of the examined model is that it rationalizes decisions under incomplete information. Agents in the model make decisions to create new links or to delete some of the old links, depending on their value functions. The calculation of value functions is challenging, because of the complexity of the model. Our model can be still demanding even with a case of $N = 50$, but it can offer new venues for improvement.

Finally, future work can focus on extending the biased-sampling setup, focusing on some real-world challenges that social-network platforms suggest. We can use this method of creating and deleting links using the biases b_i, and endogenizing the strength of peer-induced assimilation bias.

7. Appendix

7.1 Expectations of the state of the world

Agent i's information set consists of her own private signal $x_{i,t}$, the set of her neighbors' private signals $\{x_{j,t}\}_{j\in N_i}$ and public signal y_t. Since all signals are random variables, centered at the θ, to predict the state of the world conditional on it's information set, agent i should consider the following probability density function[23] :

$$\begin{aligned} p\left(\theta_t|\mathcal{I}_{i,t}\right) &= p\left(\theta_t|x_{i,t},\{x_{j,t}\}_{j\in N_i},y_t\right) \propto p(x_i,\{x_{j,t}\}_{j\in N_i} y|\theta)p(\theta) \propto \\ &\propto \exp\left[-\frac{1}{2}\left(\beta_i(\theta_t-x_{i,t})^2+\sum_{j\in N_i}\beta_{j,t}(\theta_t-x_{j,t})^2+\alpha(\theta_t-y_t)^2\right)\right] \end{aligned}$$

Using standard derivations frequently used in the Bayesian statistics literature (see, for instance, Koop (2007)). So we get the following result:

$$\theta|_{(x_{i,t},\{x_{j,t}\}_{j\in N_i},y_t)} \sim \mathbf{N}\left(\frac{\alpha y+\beta_i x_i+\sum_j\beta_j x_j}{\alpha+\beta_i+\sum_j\beta_j},\frac{1}{\alpha+\beta_i+\sum_j\beta_j}\right)$$

For calculating value functions, we need to calculate $E(\theta^2|\mathcal{I}_i)$. Following the Hakobyan and Koulovatianos (2020) we will find the following:

$$E(\theta^2|\mathcal{I}_i)=\left(\frac{\alpha y+\beta_i x_i+\sum_j\beta_j x_j}{\alpha+\beta_i+\sum_j\beta_j}\right)^2+\frac{1}{\alpha+\beta_i+\sum_j\beta_j} \tag{16}$$

7.2 More detail examples: 6-Agent case

Let's consider the network structure which include 6 agents. The graph $\mathcal{G}$ is unweighted and undirected, so $M_t^{ij}=M_t^{ji}$ as demonstrated in the Figure 2.

[23]With absolutely non-informative prior where $p(\theta)\propto 1$.

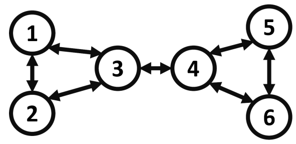

Figure 2. Example with 6 agents

In the Section 3.1 we define the information set as $\mathcal{I}_{i,t} = \left(y_t, x_{i,t}, \sum_j x_{j,t} \right)$. Let's begin from the first period $t = 1$, the information set will be the following $\mathcal{I}_{1,1} = (y_1,\, x_{1,1},\, x_{2,1},\, x_{3,1})$; $\mathcal{I}_{2,1} = (y_1,\, x_{1,1},\, x_{2,1},\, x_{3,1})$; $\mathcal{I}_{3,1} = (y_1,\, x_{1,1},\, x_{2,1},\, x_{3,1},\, x_{4,1})$; $\mathcal{I}_{4,1} = (y_1,\, x_{3,1},\, x_{4,1},\, x_{5,1},\, x_{6,1})$; $\mathcal{I}_{5,1} = (y_1,\, x_{4,1},\, x_{5,1},\, x_{6,1})$ and $\mathcal{I}_{6,1} = (y_1,\, x_{4,1},\, x_{5,1},\, x_{6,1})$. In Section 4 I introduce the algorithm of finding the equilibrium. The linear strategy which I define in equation *(10)* looks like the follows:

$$
\begin{aligned}
a_{1,1} &= \omega_{11}^1 x_{1,1} + \omega_{12}^1 x_{2,1} + \omega_{13}^1 x_{3,1} + w_{b_1}^1 b_1 + \left(1 - \left(\omega_{11}^1 + \omega_{12}^1 + \omega_{13}^1 + w_{b_1}^1\right)\right) y_1 \\
a_{2,1} &= \omega_{21}^1 x_{1,1} + \omega_{22}^1 x_{2,1} + \omega_{23}^1 x_{3,1} + w_{b_2}^1 b_2 + \left(1 - \left(\omega_{21}^1 + \omega_{22}^1 + \omega_{23}^1 + w_{b_2}^1\right)\right) y_1 \\
a_{3,1} &= \omega_{31}^1 x_{1,1} + \omega_{32}^1 x_{2,1} + \omega_{33}^1 x_{3,1} + \omega_{34}^1 x_{4,1} + w_{b_3}^1 b_3 + \left(1 - \left(\omega_{31}^1 + \omega_{32}^1 + \omega_{33}^1 + \omega_{34}^1 + w_{b_3}^1\right)\right) y_1 \\
a_{4,1} &= \omega_{43}^1 x_{3,1} + \omega_{44}^1 x_{4,1} + \omega_{45}^1 x_{5,1} + \omega_{46}^1 x_{6,1} + w_{b_4}^1 b_4 + \left(1 - \left(\omega_{43}^1 + \omega_{44}^1 + \omega_{45}^1 + \omega_{46}^1 + w_{b_4}^1\right)\right) y_1 \\
a_{5,1} &= \omega_{54}^1 x_{4,1} + \omega_{55}^1 x_{5,1} + \omega_{56}^1 x_{6,1} + w_{b_5}^1 b_5 + \left(1 - \left(\omega_{54}^1 x + \omega_{55}^1 + \omega_{56}^1 + w_{b_5}^1\right)\right) y_1 \\
a_{6,1} &= \omega_{64}^1 x_{4,1} + \omega_{65}^1 x_{5,1} + \omega_{66}^1 x_{6,1} + w_{b_6}^1 b_6 + \left(1 - \left(\omega_{64}^1 x + \omega_{65}^1 + \omega_{66}^1 + w_{b_6}^1\right)\right) y_1
\end{aligned}
$$

We normalized the weights, so $\omega_{11} + \omega_{12} + \omega_{13} + w_{b_1} + w_{y_1} = 1$ and we will solve system of linear equations for $\omega_{11}; \omega_{12}; \omega_{13}; w_{b_1}$. Weight which agent 1 put on public signal we will find in the following way $w_{y_1} = 1 - (\omega_{11} + \omega_{12} + \omega_{13} + w_{b_1})$. If the agent has no information about the signal j, then $E_i x_j = E_i \theta$. Let's consider the optimal action from 1st agent side, which looks like the following equation:

$$
\begin{aligned}
a_1 \;=\;& (1-r_1)\,E_1(\theta_1) + (1-r_1)\,b_1 + r_1 \frac{q_1}{\#N_1}\left[\omega_{21}x_1 + \omega_{22}x_2 + \omega_{23}x_3 + w_{b_2}b_2 + (1-\omega_{21}-\omega_{22}-\omega_{23}-w_{b_2})\,y\right] \\
& + r_1 \frac{q_1}{\#N_1}\left[\omega_{31}x_1 + \omega_{32}x_2 + \omega_{33}x_3 + \omega_{34}E_1(\theta_1) + w_{b_3}b_3 + (1-\omega_{31}-\omega_{32}-\omega_{33}-\omega_{34}-w_{b_3})\,y\right] + \\
& + r_1 \frac{(1-q_1)}{\#N_{-1}}\left[\omega_{43}x_3 + \omega_{44}E_1(\theta_1) + \omega_{45}E_1(\theta_1) + \omega_{46}E_1(\theta_1) + w_{b_4}b_4 + (1-\omega_{43}-\omega_{44}-\omega_{45}-\omega_{46}-w_{b_4})\,y\right] + \\
& + r_1 \frac{(1-q_1)}{\#N_{-1}}\left[\omega_{54}E_1(\theta_1) + \omega_{55}E_1(\theta_1) + \omega_{56}E_1(\theta_1) + w_{b_5}b_5 + (1-\omega_{54}-\omega_{55}-\omega_{56}-w_{b_5})\,y\right] + \\
& + r_1 \frac{(1-q_1)}{\#N_{-1}}\left[\omega_{64}E_1(\theta_1) + \omega_{65}E_1(\theta_1) + \omega_{66}E_1(\theta_1) + w_{b_6}b_6 + (1-\omega_{64}-\omega_{65}-\omega_{66}-w_{b_6})\,y\right]
\end{aligned}
$$

In the second step, we need to rearrange the terms, and get the coefficient preceding the $E_1(\theta_1)$.

$$
\begin{aligned}
a_1 \;=\;& E_1(\theta_1)\left[(1-r_1) + r_1\left[\frac{q_1}{\#N_1}\omega_{34} + \frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
& (1-r_1)\,b_1 + r_1\frac{q_1}{\#N_1}\left[\omega_{21}+\omega_{31}\right]x_1 + r_1\frac{q_1}{\#N_1}\left[\omega_{22}+\omega_{32}\right]x_2 + r_1\left[\frac{q_1}{\#N_1}\left[\omega_{23}+\omega_{33}\right] + \frac{(1-q_1)}{\#N_{-1}}\omega_{43}\right]x_3 + \\
& + r_1\frac{q_1}{\#N_1}\left(w_{b_2}b_2 + w_{b_3}b_3\right) + r_1\frac{(1-q_1)}{\#N_{-1}}\left(w_{b_4}b_4 + w_{b_5}b_5 + w_{b_6}b_6\right) + \\
& + r_1\left(\frac{q_1}{\#N_1}\left[w_{2,y}+w_{3,y}\right] + \frac{(1-q_1)}{\#N_{-1}}\left[w_{4,y}+w_{5,y}+w_{6,y}\right]\right)y
\end{aligned}
$$

Let's use the algebra from Appendix A1 to calculate the $E_i(\theta_1)$.

$$
\begin{aligned}
E_1(\theta_1) \;=\;& E_2(\theta_1) = \frac{\beta_1x_1+\beta_2x_2+\beta_3x_3+\alpha y_1}{\beta_1+\beta_2+\beta_3+\alpha}; \quad E_3(\theta_1) = \frac{\beta_1x_1+\beta_2x_2+\beta_3x_3+\beta_4x_4+\alpha y_1}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\;; \\
E_4(\theta_1) \;=\;& \frac{\beta_3x_3+\beta_4x_4+\beta_5x_5+\beta_6x_6+\alpha y_1}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\;; \quad E_4(\theta_1) = E_5(\theta_1) = \frac{\beta_4x_4+\beta_5x_5+\beta_6x_6+\alpha y_1}{\beta_4+\beta_5+\beta_6+\alpha}\;;
\end{aligned}
$$

Now we can find the weights which agent 1 put in y, x_1, x_2, x_3 and b_1.

Weight for x_1

$$
\begin{aligned}
\omega_{11} \;=\;& \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_1) + r_1\left[\frac{q_1}{\#N_1}\omega_{34} + \frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
& + r_1\frac{q_1}{\#N_1}\left[\omega_{21}+\omega_{31}\right]
\end{aligned}
$$

Weight for x_2

$$\omega_{12} = \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_1)+r_1\left[\frac{q_1}{\#N_1}\omega_{34}+\frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+$$
$$+r_1\frac{q_1}{\#N_1}\left[\omega_{22}+\omega_{32}\right]$$

Weight for x_3

$$\omega_{13} = \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_1)+r_1\left[\frac{q_1}{\#N_1}\omega_{34}+\frac{(1-q_1)}{\#N_{-1}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+$$
$$+r_1\left[\frac{q_1}{\#N_1}\left[\omega_{23}+\omega_{33}\right]+\frac{(1-q_1)}{\#N_{-1}}\omega_{43}\right]$$

Weight for b_1

$$w_{b_1}b_1 = (1-r_1)\,b_1+r_1\frac{q_1}{\#N_1}\left(w_{b_2}b_2+w_{b_3}b_3\right)+r_1\frac{(1-q_1)}{\#N_{-1}}\left(w_{b_4}b_4+w_{b_5}b_5+w_{b_6}b_6\right)\Longleftrightarrow$$
$$\Longleftrightarrow\quad w_{b_1}=(1-r_1)+\frac{r_1}{b_1}\frac{q_1}{\#N_1}\left(w_{b_2}b_2+w_{b_3}b_3\right)+\frac{r_1}{b_1}\frac{(1-q_1)}{\#N_{-1}}\left(w_{b_4}b_4+w_{b_5}b_5+w_{b_6}b_6\right)$$

We can do the same Algebra from 2nd agent side.

$$\omega_{21} = \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_2)+r_2\left[\frac{q_2}{\#N_2}\omega_{34}+\frac{(1-q_2)}{\#N_{-2}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+$$
$$+r_2\frac{q_2}{\#N_2}\left[\omega_{11}+\omega_{31}\right]$$

$$\omega_{22} = \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_2)+r_2\left[\frac{q_2}{\#N_2}\omega_{34}+\frac{(1-q_2)}{\#N_{-2}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+$$
$$+r_2\frac{q_2}{\#N_2}\left[\omega_{12}+\omega_{32}\right]$$

$$\omega_{23} = \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha}\left[(1-r_2)+r_2\left[\frac{q_2}{\#N_2}\omega_{34}+\frac{(1-q_2)}{\#N_{-2}}\left[\omega_{44}+\omega_{45}+\omega_{46}+\omega_{54}+\omega_{55}+\omega_{56}+\omega_{64}+\omega_{65}+\omega_{66}\right]\right]\right]+$$
$$+r_2\left[\frac{q_2}{\#N_2}\left[\omega_{13}+\omega_{33}\right]+\frac{(1-q_2)}{\#N_{-2}}\omega_{43}\right]$$

$$w_{b_2} = (1-r_2) + \frac{r_2}{b_2}\frac{q_2}{\#N_2}\left(w_{b_1}b_1 + w_{b_3}b_3\right) + \frac{r_2}{b_2}\frac{(1-q_2)}{\#N_{-2}}\left(w_{b_4}b_4 + w_{b_5}b_5 + w_{b_6}b_6\right)$$

Using the same strategy we will find the optimal action for 3rd agent.

$$\begin{aligned}
a_3 \quad = \quad & E(\theta)\left[(1-r_3) + r_3\left[\frac{q_3}{\#N_3}\left[\omega_{45}+\omega_{46}\right] + \frac{(1-q_3)}{\#N_{-3}}\left[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}\right]\right]\right] + (1-r_3)\,b_3 + \\
& + r_3\frac{q_3}{\#N_3}\left[\omega_{11}+\omega_{21}\right]x_1 + r_3\frac{q_3}{\#N_3}\left[\omega_{12}+\omega_{22}\right]x_2 + r_3\frac{q_3}{\#N_3}\left[\omega_{13}+\omega_{23}+\omega_{43}\right]x_3 + \\
& + r_3\left[\frac{q_3}{\#N_3}\omega_{44} + \frac{(1-q_3)}{\#N_{-3}}\left[\omega_{54}+\omega_{64}\right]\right]x_4 + r_3\frac{q_3}{\#N_3}\left(w_{b_1}b_1 + w_{b_2}b_2 + w_{b_4}b_4\right) + \\
& + r_3\frac{(1-q_3)}{\#N_{-3}}\left(w_{b_5}b_5 + w_{b_6}b_6\right) + r_3\left(\frac{q_3}{\#N_3}\left[w_{1,y}+w_{2,y}+w_{4,y}\right] + \frac{(1-q_3)}{\#N_{-3}}\left[w_{5,y}+w_{6,y}\right]\right)y
\end{aligned}$$

$$\begin{aligned}
\omega_{31} \quad = \quad & \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3) + r_3\left[\frac{q_3}{\#N_3}\left[\omega_{45}+\omega_{46}\right] + \frac{(1-q_3)}{\#N_{-3}}\left[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
& + r_3\frac{q_3}{\#N_3}\left[\omega_{11}+\omega_{21}\right]
\end{aligned}$$

$$\begin{aligned}
\omega_{32} \quad = \quad & \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3) + r_3\left[\frac{q_3}{\#N_3}\left[\omega_{45}+\omega_{46}\right] + \frac{(1-q_3)}{\#N_{-3}}\left[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
& + r_3\frac{q_3}{\#N_3}\left[\omega_{12}+\omega_{22}\right]
\end{aligned}$$

$$\begin{aligned}
\omega_{33} \quad = \quad & \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3) + r_3\left[\frac{q_3}{\#N_3}\left[\omega_{45}+\omega_{46}\right] + \frac{(1-q_3)}{\#N_{-3}}\left[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
& + r_3\frac{q_3}{\#N_3}\left[\omega_{13}+\omega_{23}+\omega_{43}\right]
\end{aligned}$$

$$\begin{aligned}
\omega_{34} \quad = \quad & \frac{\beta_4}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha}\left[(1-r_3) + r_3\left[\frac{q_3}{\#N_3}\left[\omega_{45}+\omega_{46}\right] + \frac{(1-q_3)}{\#N_{-3}}\left[\omega_{55}+\omega_{56}+\omega_{65}+\omega_{66}\right]\right]\right] + \\
& + r_3\left[\frac{q_3}{\#N_3}\omega_{44} + \frac{(1-q_3)}{\#N_{-3}}\left[\omega_{54}+\omega_{64}\right]\right]
\end{aligned}$$

$$w_{b_3} = (1-r_3) + \frac{r_3}{b_3}\frac{q_3}{\#N_3}\left(w_{b_1}b_1 + w_{b_2}b_2 + w_{b_4}b_4\right) + \frac{r_3}{b_3}\frac{(1-q_3)}{\#N_{-3}}\left(w_{b_5}b_5 + w_{b_6}b_6\right)$$

Consider the optimal strategy from agent 4 side.

$$\begin{aligned} a_4 \quad = \quad & E(\theta)\left[(1-r_4) + r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right] + \frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right] + (1-r_4)\,b_4 + \\ & + r_4\frac{q_4}{\#N_4}\left[\omega_{55}+\omega_{65}\right]x_5 + r_4\frac{q_4}{\#N_4}\left[\omega_{56}+\omega_{66}\right]x_6 + r_4\frac{q_4}{\#N_4}\left[\omega_{34}+\omega_{54}+\omega_{64}\right]x_4 + \\ & + r_4\left[\frac{q_4}{\#N_4}\omega_{33} + \frac{(1-q_4)}{\#N_{-4}}\left[\omega_{13}+\omega_{23}\right]\right]x_3 + r_4\frac{q_4}{\#N_4}\left(w_{b_3}b_3 + w_{b_5}b_5 + w_{b_6}b_6\right) + \\ & + r_4\frac{(1-q_4)}{\#N_{-4}}\left(w_{b_1}b_1 + w_{b_2}b_2\right) + r_4\left(\frac{q_4}{\#N_4}\left[w_{3,y}+w_{5,y}+w_{6,y}\right] + \frac{(1-q_4)}{\#N_{-4}}\left[w_{1,y}+w_{2,y}\right]\right)y \end{aligned}$$

$$\begin{aligned} \omega_{43} \quad = \quad & \frac{\beta_3}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4) + r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right] + \frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right] + \\ & + r_4\left[\frac{q_4}{\#N_4}\omega_{33} + \frac{(1-q_4)}{\#N_{-4}}\left[\omega_{13}+\omega_{23}\right]\right] \end{aligned}$$

$$\begin{aligned} \omega_{44} \quad = \quad & \frac{\beta_4}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4) + r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right] + \frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right] + \\ & + r_4\frac{q_4}{\#N_4}\left[\omega_{34}+\omega_{54}+\omega_{64}\right] \end{aligned}$$

$$\begin{aligned} \omega_{45} \quad = \quad & \frac{\beta_5}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4) + r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right] + \frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right] + \\ & + r_4\frac{q_4}{\#N_4}\left[\omega_{55}+\omega_{65}\right] \end{aligned}$$

$$\begin{aligned} \omega_{46} \quad = \quad & \frac{\beta_6}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_4) + r_4\left[\frac{q_4}{\#N_4}\left[\omega_{31}+\omega_{32}\right] + \frac{(1-q_4)}{\#N_{-4}}\left[\omega_{11}+\omega_{12}+\omega_{21}+\omega_{22}\right]\right]\right] + \\ & + r_4\frac{q_4}{\#N_4}\left[\omega_{56}+\omega_{66}\right] \end{aligned}$$

$$w_{b_4} = (1-r_4) + \frac{r_4}{b_4}\frac{q_4}{\#N_4}\left(w_{b_3}b_3 + w_{b_5}b_5 + w_{b_6}b_6\right) + \frac{r_4}{b_4}\frac{(1-q_4)}{\#N_{-4}}\left(w_{b_1}b_1 + w_{b_2}b_2\right)$$

The strategy from the 5th agent side

$$\begin{aligned}
a_5 &= E(\theta)\left[(1-r_5) + r_5\left[\frac{q_5}{\#N_5}\omega_{43} + \frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right] + \\
&\quad (1-r_5)\,b_5 + r_5\frac{q_5}{\#N_5}\left[\omega_{45}+\omega_{65}\right]x_5 + r_5\frac{q_5}{\#N_5}\left[\omega_{46}+\omega_{66}\right]x_6 + r_5\left[\frac{q_5}{\#N_5}\left[\omega_{44}+\omega_{64}\right] + \frac{(1-q_5)}{\#N_{-5}}\omega_{34}\right]x_4 + \\
&\quad + r_5\frac{q_5}{\#N_5}\left(w_{b_4}b_4 + w_{b_6}b_6\right) + r_5\frac{(1-q_5)}{\#N_{-5}}\left(w_{b_1}b_1 + w_{b_2}b_2 + w_{b_3}b_3\right) + \\
&\quad + r_5\left(\frac{q_5}{\#N_5}\left[w_{4,y}+w_{6,y}\right] + \frac{(1-q_5)}{\#N_{-5}}\left[w_{1,y}+w_{2,y}+w_{3,y}\right]\right)y
\end{aligned}$$

$$\begin{aligned}
\omega_{54} &= \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_5) + r_5\left[\frac{q_5}{\#N_5}\omega_{43} + \frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right] + \\
&\quad + r_5\left[\frac{q_5}{\#N_5}\left[\omega_{44}+\omega_{64}\right] + \frac{(1-q_5)}{\#N_{-5}}\omega_{34}\right]
\end{aligned}$$

$$\begin{aligned}
\omega_{55} &= \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_5) + r_5\left[\frac{q_5}{\#N_5}\omega_{43} + \frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right] + \\
&\quad + r_5\frac{q_5}{\#N_5}\left[\omega_{45}+\omega_{65}\right]
\end{aligned}$$

$$\begin{aligned}
\omega_{56} &= \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha}\left[(1-r_5) + r_5\left[\frac{q_5}{\#N_5}\omega_{43} + \frac{(1-q_5)}{\#N_{-5}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right] + \\
&\quad + r_5\frac{q_5}{\#N_5}\left[\omega_{46}+\omega_{66}\right]
\end{aligned}$$

$$w_{b_5} = (1-r_5) + \frac{r_5}{b_5}\frac{q_5}{\#N_5}\left(w_{b_4}b_4 + w_{b_6}b_6\right) + \frac{r_5}{b_5}\frac{(1-q_5)}{\#N_{-5}}\left(w_{b_1}b_1 + w_{b_2}b_2 + w_{b_3}b_3\right)$$

From the 6th agent side

$$
\begin{aligned}
a_6 \;=\;& E\left(\theta\right)\left[\left(1-r_6\right)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{\left(1-q_6\right)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\
&\left(1-r_6\right)b_6+r_6\frac{q_6}{\#N_6}\left[\omega_{45}+\omega_{55}\right]x_5+r_6\frac{q_6}{\#N_6}\left[\omega_{46}+\omega_{56}\right]x_6+r_6\left[\frac{q6}{\#N_6}\left[\omega_{44}+\omega_{54}\right]+\frac{\left(1-q_6\right)}{\#N_{-6}}\omega_{34}\right]x_4+\\
&+r_6\frac{q_6}{\#N_6}\left(w_{b_4}b_4+w_{b_5}b_5\right)+r_6\frac{\left(1-q_6\right)}{\#N_{-6}}\left(w_{b_1}b_1+w_{b_2}b_2+w_{b_3}b_3\right)+\\
&+r_6\left(\frac{q_6}{\#N_6}\left[w_{4,y}+w_{5,y}\right]+\frac{\left(1-q_6\right)}{\#N_{-6}}\left[w_{1,y}+w_{2,y}+w_{3,y}\right]\right)y
\end{aligned}
$$

$$
\begin{aligned}
\omega_{64} \;=\;& \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha}\left[\left(1-r_6\right)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{\left(1-q_6\right)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\
&+r_6\left[\frac{q6}{\#N_6}\left[\omega_{44}+\omega_{54}\right]+\frac{\left(1-q_6\right)}{\#N_{-6}}\omega_{34}\right]
\end{aligned}
$$

$$
\begin{aligned}
\omega_{65} \;=\;& \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha}\left[\left(1-r_6\right)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{\left(1-q_6\right)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\
&+r_6\frac{q_6}{\#N_6}\left[\omega_{45}+\omega_{55}\right]
\end{aligned}
$$

$$
\begin{aligned}
\omega_{66} \;=\;& \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha}\left[\left(1-r_6\right)+r_6\left[\frac{q_6}{\#N_6}\omega_{43}+\frac{\left(1-q_6\right)}{\#N_{-6}}\left[\omega_{11}+\omega_{12}+\omega_{13}+\omega_{21}+\omega_{22}+\omega_{23}+\omega_{31}+\omega_{32}+\omega_{33}\right]\right]\right]+\\
&+r_6\frac{q_6}{\#N_6}\left[\omega_{46}+\omega_{56}\right]
\end{aligned}
$$

$$
w_{b_6}=\left(1-r_6\right)+\frac{r_6}{b_6}\frac{q_6}{\#N_6}\left(w_{b_4}b_4+w_{b_5}b_5\right)+\frac{r_6}{b_6}\frac{\left(1-q_6\right)}{\#N_{-6}}\left(w_{b_1}b_1+w_{b_2}b_2+w_{b_3}b_3\right)
$$

7.3 Generalizing The Solution in Matrix Form

The matrix $\mathcal{A}$ represents the adjacency matrix plus identity matrix (shows all connections including self-loops) $\mathcal{A}=\mathbf{M}_t+eye\left(N\right)$. Let's demonstrate the matrix $\mathcal{A}$ for the example which I introduce in Figure 2. Matrix $\mathcal{A}$ looks like,

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix};$$

The expression which is multiplied by $E_i(\theta_t)$ includes the following 2 parts: $(1 - r_i)$ and the weights of the signals, which agent i doesn't observe.[24]

The weight of this signals can be not only the signals of agents who are not in agent i's network, but the same time the weight which agent i's neighbors put to their friends, which are not in agent i's friends.[25]

For finding the weight before $E_i(\theta_t)$ I will introduce the matrix B_i. As we need to find the signals which are not in agent i's network, we need to exclude the agent i's friends from the adjacency matrix.I take the row of matrix $\mathcal{A}$, build a new matrix $(N \times N)$ and repeated that row N times, for all agents $N = \{1, 2...6\}$. For example for 6 agents case, the matrix B looks like the following.

$$\begin{aligned} B_1 &= B_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}; B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}; \\ B_4 &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}; B_5 = B_6 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}; \end{aligned}$$

The next step will be introducing the matrix C_i, which will show all connections excluding agent i's connections.. So $C_i = A - B_i$. If there is negative elements in the matrix $C_i\,(C_{ij} < 0)$, we will replace to 0. For algoritm we will use $C_i\,(C_i < 0) = 0$. Which will find

[24] As we show in example (**??**), from the first agent side its look like the following $E_1(\theta_1)\,[(1 - r_1) + r_1 \frac{q_1}{\#N_1} \omega_{34} + \frac{(1-q_1)}{\#N_{-1}}\,[\omega_{44} + \omega_{45} + \omega_{46} + \omega_{54} + \omega_{55} + \omega_{56} + \omega_{64} + \omega_{65} + \omega_{66}]$.

[25] For example, if we look from the first agent side the weight ω_{34} is the weight which agent 3 puts to his friend 4, which is not in agent's i's network.

all negative elements and change it to 0.

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ so after replacing negative values } \quad C_1 = C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix};$$

$$C_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}; C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; C_5 = C_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

Agent i differ friends from non-friends by putting different weights to them. As you can see in the equation (1) the weight which agent i put to his friend is $\frac{q_i}{\#N_i}$, non-friends weight is equal to $\frac{(1-q_i)}{\#N_{-i}}$. We will take every row from matrix $\mathcal{A}$. Let's call it as $\mathcal{L}_i = \mathcal{A}(:,i)^T$, and replace the "1's" to $\frac{q_i}{\#N_i}$, and "0's" to $\frac{(1-q_i)}{\#N_{-i}}$. So in this way we will find the matrix which shows the weights from each agent side, we will call it as $\mathcal{D}_i = \mathcal{L}_i \circ \mathcal{A}$.[26]

$$\mathcal{D}_1 = \begin{bmatrix} \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & 0 & 0 & 0 \\ \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & 0 & 0 & 0 \\ \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & \frac{q_1}{\#N_1} & 0 & 0 \\ 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \end{bmatrix}; \mathcal{D}_2 = \begin{bmatrix} \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & 0 & 0 & 0 \\ \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & 0 & 0 & 0 \\ \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & \frac{q_2}{\#N_2} & 0 & 0 \\ 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \end{bmatrix};$$

$$\mathcal{D}_3 = \begin{bmatrix} \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & 0 & 0 & 0 \\ \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & 0 & 0 & 0 \\ \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & 0 & 0 \\ 0 & 0 & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} \\ 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \\ 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \end{bmatrix}; \mathcal{D}_4 = \begin{bmatrix} \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 \\ \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 \\ \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & 0 & 0 \\ 0 & 0 & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} \\ 0 & 0 & 0 & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} \\ 0 & 0 & 0 & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} \end{bmatrix};$$

$$\mathcal{D}_5 = \begin{bmatrix} \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 \\ 0 & 0 & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} \\ 0 & 0 & 0 & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} \\ 0 & 0 & 0 & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} & \frac{q_5}{\#N_5} \end{bmatrix}; \mathcal{D}_6 = \begin{bmatrix} \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 \\ 0 & 0 & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} \\ 0 & 0 & 0 & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} \\ 0 & 0 & 0 & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} & \frac{q_6}{\#N_6} \end{bmatrix};$$

[26] Please note that when we will consider network formation process, the only matrix which which will change and influence to decision send an invitation or cause an annoyances, is matrix $\mathcal{D}_i$.

After describing matrix C_i and $\mathcal{D}_i$, in our paper we need a Hadamard product of this two matrices $\mathcal{E}_i = D_i \circ C_i$.

$$\mathcal{E}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q_1}{\#N_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \\ 0 & 0 & 0 & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} & \frac{(1-q_1)}{\#N_{-1}} \end{bmatrix}; \quad \mathcal{E}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{q_2}{\#N_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \\ 0 & 0 & 0 & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} & \frac{(1-q_2)}{\#N_{-2}} \end{bmatrix};$$

$$\mathcal{E}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q_3}{\#N_3} & \frac{q_3}{\#N_3} \\ 0 & 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \\ 0 & 0 & 0 & 0 & \frac{(1-q_3)}{\#N_{-3}} & \frac{(1-q_3)}{\#N_{-3}} \end{bmatrix}; \quad \mathcal{E}_4 = \begin{bmatrix} \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 & 0 \\ \frac{(1-q_4)}{\#N_{-4}} & \frac{(1-q_4)}{\#N_{-4}} & 0 & 0 & 0 & 0 \\ \frac{q_4}{\#N_4} & \frac{q_4}{\#N_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$\mathcal{E}_5 = \begin{bmatrix} \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & \frac{(1-q_5)}{\#N_{-5}} & 0 & 0 & 0 \\ 0 & 0 & \frac{q_5}{\#N_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad \mathcal{E}_6 = \begin{bmatrix} \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & \frac{(1-q_6)}{\#N_{-6}} & 0 & 0 & 0 \\ 0 & 0 & \frac{q_6}{\#N_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

So we solve the first part of optimal action, now we need to get how agent i's signal depends on signals which he gets. Let's introduce the matrix F_i, For every agent i I take the i-th row from the matrix $\mathcal{A}$ and build a new matrix F_i where other rows are 0s.

$$F_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; F_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$F_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad F_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad F_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix};$$

For finding the last part of optimal action[27] , I will Introduce the matrix $G_i = \mathcal{A} - C_i - F_i$.

The matrix G_i shows the signals which agent i knows which other people gets.

[27]For example, if we look from the first agent side we need to find the following part. $r_1 \frac{q_1}{\#N_1} [\omega_{21} + \omega_{31}] x_1 + r_1 \frac{q_1}{\#N_1} [\omega_{22} + \omega_{32}] x_2 +$
$r_1 \left[\frac{q_1}{\#N_1} [\omega_{23} + \omega_{33}] + \frac{(1-q_1)}{\#N_{-1}} \omega_{43} \right] x_3$. So we need to get the weights before agent private signal.

$$G_1 = \begin{bmatrix} 0&0&0&0&0&0\\ 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}; \quad G_2 = \begin{bmatrix} 1&1&1&0&0&0\\ 0&0&0&0&0&0\\ 1&1&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}; G_3 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&0&0 \end{bmatrix};$$

$$G_4 = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&1&0&0&0\\ 0&0&1&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1 \end{bmatrix}; \quad G_5 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&1&1\\ 0&0&0&0&0&0\\ 0&0&0&1&1&1 \end{bmatrix}; \quad G_6 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&0&0&0 \end{bmatrix};$$

The private information which agent i gets, can be part of linear strategy not only his friends, but from neighbors of his friends, so we need to multiply by elements the matrix G_i and D_i. The matrix $H_i = D_i \circ G_i$.

$$H_1 = \begin{bmatrix} 0&0&0&0&0&0\\ \frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&0&0&0\\ \frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&\frac{q_1}{\#N_1}&0&0&0\\ 0&0&\frac{(1-q_1)}{\#N_{-1}}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}; \quad H_2 = \begin{bmatrix} \frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&0&0&0\\ 0&0&0&0&0&0\\ \frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&\frac{q_2}{\#N_2}&0&0&0\\ 0&0&\frac{(1-q_2)}{\#N_{-2}}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix};$$

$$H_3 = \begin{bmatrix} \frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&0&0&0\\ \frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&0&0&0\\ 0&0&0&0&0&0\\ 0&0&\frac{q_3}{\#N_3}&\frac{q_3}{\#N_3}&0&0\\ 0&0&0&\frac{(1-q_3)}{\#N_{-3}}&0&0\\ 0&0&0&\frac{(1-q_3)}{\#N_{-3}}&0&0 \end{bmatrix}; \quad H_4 = \begin{bmatrix} 0&0&\frac{(1-q_4)}{\#N_{-4}}&0&0&0\\ 0&0&\frac{(1-q_4)}{\#N_{-4}}&0&0&0\\ 0&0&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}\\ 0&0&0&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4}&\frac{q_4}{\#N_4} \end{bmatrix};$$

$$H_5 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\frac{(1-q_5)}{\#N_{-5}}&0&0\\ 0&0&0&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5}\\ 0&0&0&0&0&0\\ 0&0&0&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5}&\frac{q_5}{\#N_5} \end{bmatrix}; \quad H_6 = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&\frac{(1-q_6)}{\#N_{-6}}&0&0\\ 0&0&0&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}\\ 0&0&0&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}&\frac{q_6}{\#N_6}\\ 0&0&0&0&0&0 \end{bmatrix};$$

Before getting the weights, lets analyze the expectation side. As we explain in the previous section the $E_t(\theta_t)$ depends on the private and public signals which every agents have.

$$E_1(\theta_1) = E_2(\theta_1) = \frac{\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \alpha y_1}{\beta_1 + \beta_2 + \beta_3 + \alpha}; \quad E_3(\theta_1) = \frac{\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \alpha y_1}{\beta_1 + \beta_2 + \beta_3 + \beta_4 + \alpha};$$

$$E_4(\theta_1) = \frac{\beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \alpha y_1}{\beta_3 + \beta_4 + \beta_5 + \beta_6 + \alpha}; \quad E_4(\theta_1) = E_5(\theta_1) = \frac{\beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6 + \alpha y_1}{\beta_4 + \beta_5 + \beta_6 + \alpha};$$

I will write the algorithm for finding the optimal weights before private signals and biases. The optimal weight of public signal can be find using the optimal weight of private signals and biases. Now let consider the precision side. Let introduce β which is a vector of all private signal precision. We will use command *repmat* which will repeat this vector. For example in our example $N = 6$; we will $\beta = repmat(\beta, N, 1)$, which repeat the row β six time.

$$\beta = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \end{bmatrix}$$

For getting the precision which is known from all agents side, let's multiply every element of the matrix β with matrix A. $\beta_{new} = \beta \circ \mathcal{A}$.

$$\beta_{new} = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & 0 & 0 \\ 0 & 0 & \beta_3 & \beta_4 & \beta_5 & \beta_6 \\ 0 & 0 & 0 & \beta_4 & \beta_5 & \beta_6 \\ 0 & 0 & 0 & \beta_4 & \beta_5 & \beta_6 \end{bmatrix};$$

Now we need to sum the row of β_{new} and plus α, which is precision of public signal. The precision of public signal is common knowledge.

$$norm = \text{sum}\left(\beta_{new}\ , 2\right) + \alpha \circ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Longleftrightarrow norm = \begin{bmatrix} \beta_1 + \beta_2 + \beta_3 + \alpha \\ \beta_1 + \beta_2 + \beta_3 + \alpha \\ \beta_1 + \beta_2 + \beta_3 + \beta_4 + \alpha \\ \beta_3 + \beta_4 + \beta_5 + \beta_6 + \alpha \\ \beta_4 + \beta_5 + \beta_6 + \alpha \\ \beta_4 + \beta_5 + \beta_6 + \alpha \end{bmatrix};$$

Using the command $repmat(norm, 1, N)$ gives us the matrix R_{norm} which repeat the column of the matrix $norm$ N times.

$$R_{norm} = \begin{bmatrix} \beta_1+\beta_2+\beta_3+\alpha & . & . & . & . & \beta_1+\beta_2+\beta_3+\alpha \\ \beta_1+\beta_2+\beta_3+\alpha & . & . & . & . & \beta_1+\beta_2+\beta_3+\alpha \\ \beta_1+\beta_2+\beta_3+\beta_4+\alpha & . & . & . & . & \beta_1+\beta_2+\beta_3+\beta_4+\alpha \\ \beta_3+\beta_4+\beta_5+\beta_6+\alpha & . & . & . & . & \beta_3+\beta_4+\beta_5+\beta_6+\alpha \\ \beta_4+\beta_5+\beta_6+\alpha & . & . & . & . & \beta_4+\beta_5+\beta_6+\alpha \\ \beta_4+\beta_5+\beta_6+\alpha & . & . & . & . & \beta_4+\beta_5+\beta_6+\alpha \end{bmatrix};$$

After defining the R_{norm} , we can find a new precision matrix $P = \beta_{new} \circ \div R_{norm}$.

$$P = \begin{bmatrix} \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha} & 0 & 0 & 0 \\ \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\alpha} & \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\alpha} & 0 & 0 & 0 \\ \frac{\beta_1}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & \frac{\beta_2}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & \frac{\beta_3}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & \frac{\beta_4}{\beta_1+\beta_2+\beta_3+\beta_4+\alpha} & 0 & 0 \\ 0 & 0 & \frac{\beta_3}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_4}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_5}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_6}{\beta_3+\beta_4+\beta_5+\beta_6+\alpha} \\ 0 & 0 & 0 & \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha} \\ 0 & 0 & 0 & \frac{\beta_4}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_5}{\beta_4+\beta_5+\beta_6+\alpha} & \frac{\beta_6}{\beta_4+\beta_5+\beta_6+\alpha} \end{bmatrix};$$

So after defining all matrices which we need to find the optimal weights, we will describe a large matrix $\mathcal{Z}$ which will combine $\mathcal{Z}_1$,$\mathcal{Z}_2$,$\mathcal{Z}_3$. At first, Let rearrange and put together the expression before $E_i(\theta)$.So let's introduce each elements of $\mathcal{Z}$.

For introducing $\mathcal{Z}_1$ at first i will introduce N blocks of matrices.

$$\mathcal{Z}_1 = \begin{bmatrix} \mathcal{Z}_1^1 \\ \mathcal{Z}_1^2 \\ \mathcal{Z}_{\dot{1}} \\ \mathcal{Z}_{\dot{1}} \\ \mathcal{Z}_1^{N-1} \\ \mathcal{Z}_1^N \end{bmatrix} = \begin{bmatrix} repmat\left(reshape\left(E_1^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_2^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_{.}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_{.}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_{N-1}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(E_N^T, 1, N*N\right), N, 1\right) \end{bmatrix}$$

So in matrix $\mathcal{Z}$ the $\mathcal{Z}_1$ represent a block of constant multiply to $E_i\left(\theta_t\right)$. We need to multiply by elements this matrix with precision matrix. So we need to rearrange the precision matrix and the matrix which represent the the weight on conformity(r_i). The new precision matrix looks like $P_{new} = reshape\left(P^T, N*N, 1\right)$. The matrix $R_{new} = reshape\left(repmat\left(R^T, 1, N\right)^T, N*N, 1\right)$, where $R = \begin{bmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \end{bmatrix}$.

Now we will introduce the block $\mathcal{Z}_2$, which consists of N blocks matrices.

$$\mathcal{Z}_2 = \begin{bmatrix} \mathcal{Z}_2^1 \\ \mathcal{Z}_2^2 \\ \mathcal{Z}_2^{\cdot} \\ \mathcal{Z}_2^{\cdot} \\ \mathcal{Z}_2^{N-1} \\ \mathcal{Z}_2^N \end{bmatrix} = \begin{bmatrix} repmat\left(reshape\left(H_1^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_2^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_{\cdot}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_{\cdot}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_{N-1}^T, 1, N*N\right), N, 1\right) \\ repmat\left(reshape\left(H_N^T, 1, N*N\right), N, 1\right) \end{bmatrix}$$

The third block $\mathcal{Z}_3$ is the identity matrix $\mathcal{Z}_3 = repmat(eye(N), N, N)$, which represent the private signal set, the first row represent x_1, the second row represent the x_2 and so on.

$$\mathcal{Z}_3 = repmat(\text{eye}(N), N, N), \text{where eye}(N) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ . & 1 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 1 & . \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So let introduce Matrix $\mathcal{X}$,which is in $(N*N)*(N*N)$ matrix.

$$\mathcal{X} = \boldsymbol{eye}(N*N) - R.{*}P_{new}.{*}\mathcal{Z}_1 - R.{*}\mathcal{Z}_2.{*}\mathcal{Z}_3$$

So the matrix of weights looks like the following

$$\mathcal{X} * \begin{bmatrix} \omega_{1,1} \\ . \\ \omega_{1,N} \\ \omega_{2,1} \\ . \\ \omega_{2,N} \\ . \\ \omega_{N-1,1} \\ \omega_{N-1,N-1} \\ \omega_{N,1} \\ . \\ \omega_{N,N} \end{bmatrix} = (\mathcal{I} - R).{*}P_{new}$$

Therefore, the optimal weights matrix will be,

$$\mathcal{W}_x^* = (\mathcal{X}_x)^{-1}(\mathbf{1} - R)P_{new} \tag{17}$$

Now let's find the optimal weight for bias. At first I will introduce the row-vector of

$b = \begin{bmatrix} b_1 & b_2 & . & . & . & b_N \end{bmatrix}$. We will use a command $b_{new} = repmat(\beta, N, 1)$.

$$b^*_{new} = b_{new} \circ \mathcal{A} \circ \begin{bmatrix} 0 & 1 & . & . & . & 1 \\ 1 & 0 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & . & . & . & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_2 & b_3 & 0 & 0 & 0 \\ . & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & b_{N-2} & 0 & b_N \\ 0 & 0 & 0 & b_{N-2} & b_{N-1} & 0 \end{bmatrix}$$

$$b_{+-} = \begin{bmatrix} -1 & . & -1_{N_{1,m}} & 1_{N_{1,k}} & . & 1_{N_{1,\text{end}}} \\ . & . & . & . & . & . \\ -1_{N_{m,1}} & . & -1_{N_{m,m}} & 1_{N_{m,k}} & . & 1_{N_{m,end}} \\ 1_{N_{k,1}} & . & 1_{N_{k,m}} & -1_{N_{k,k}} & . & -1_{N_{k,end}} \\ . & . & . & . & . & . \\ 1_{N_{N,1}} & . & 1_{N_{N,m}} & -1_{N_{\text{N},\text{k}}} & . & -1_{N_{N,N}} \end{bmatrix}$$

$$\mathcal{X}_b = \begin{bmatrix} 1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & 1 & 0 \\ 0 & . & . & . & . & 1 \end{bmatrix} + \begin{bmatrix} \frac{r_1}{b_1}\frac{q_1}{\#N_1} \\ \frac{r_2}{b_2}\frac{q_2}{\#N_2} \\ \frac{r_.}{b_.}\frac{q_.}{\#N_.} \\ \frac{r_.}{b_.}\frac{q_.}{\#N_.} \\ \frac{r_{N-1}}{b_{N-1}}\frac{q_{N-1}}{\#N_{N-1}} \\ \frac{r_N}{b_N}\frac{q_N}{\#N_N} \end{bmatrix} \circ (b^*_{new} \circ b_{+-}) + \begin{bmatrix} \frac{r_1}{b_1}\frac{(1-q_1)}{\#N_{-1}} \\ \frac{r_2}{b_2}\frac{(1-q_2)}{\#N_{-2}} \\ \frac{r_.}{b_.}\frac{(1-q_.)}{\#N_{-.}} \\ \frac{r_.}{b_.}\frac{(1-q_.)}{\#N_{-.}} \\ \frac{r_{N-1}}{b_{N-1}}\frac{(1-q_{N-1})}{\#N_{-(N-1)}} \\ \frac{r_N}{b_N}\frac{(1-q_N)}{\#N_{-N}} \end{bmatrix} \circ (b^*_{new} \circ b_{+-})$$

The optimal weights $\mathcal{W}_b$

$$\mathcal{W}^*_b = (\mathcal{X}_b)^{-1} * \begin{bmatrix} (1-r_1) \\ (1-r_.) \\ (1-r_.) \\ (1-r_.) \\ (1-r_N) \end{bmatrix} \tag{18}$$

The expressions (17) and (18) will give us the optimal weights for private signal and bias, and we can find the optimal weight for public signal.

$$\mathcal{W}_y = \mathbf{1} - \mathcal{W}_x - \mathcal{W}_b \tag{19}$$

7.4 Calculating the value functions

The value functions are equal to $E(u_i(a^*,\theta))$. Using equation (10), we find the optimal action a_i^* of each agent, and then we put the optimal action a_i^* into the expected utility function and find the expected utility from everyone's side.

I introduce some matrices that can in decreasing the size of equations.

$$\bar{x}\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_N \end{bmatrix};\ \bar{b}=\left(\begin{bmatrix} b_1 \\ . \\ . \\ . \\ b_N \end{bmatrix}\circ\begin{bmatrix} 1 \\ 1 \\ 1_m \\ -1_k \\ -1_{\text{end}} \end{bmatrix}\right);\ \bar{\omega}_y=\begin{bmatrix} \omega_{1,y} \\ \omega_{2,y} \\ . \\ . \\ \omega_{N,y} \end{bmatrix};\ \bar{\omega}_b=\begin{bmatrix} \omega_{1,b} \\ \omega_{2,b} \\ . \\ . \\ \omega_{N,b} \end{bmatrix};\overline{\overline{\beta}}=\begin{bmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ \beta_N \end{bmatrix};$$

$$\begin{bmatrix} E_1(\theta) \\ E_2(\theta) \\ . \\ . \\ E_N(\theta) \end{bmatrix}=\left(\frac{\left[\mathcal{A}\circ\begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_N \end{bmatrix}.*\begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix}+(\alpha*y)\circ\begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}\right]\circ\div}{\mathcal{A}\circ\begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix}+\alpha\circ\begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}}\right)$$

$$E^*(\theta^2)=\begin{bmatrix} E_1(\theta^2) \\ E_2(\theta^2) \\ . \\ . \\ E_N(\theta^2) \end{bmatrix}=\left(\frac{\left[\mathcal{A}\circ\begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_N \end{bmatrix}.*\begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix}+(\alpha*y)\circ\begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}\right]\circ\div}{\mathcal{A}\circ\begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix}+\alpha\circ\begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}}\right)^2+\left(\frac{\begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}\circ\div}{\mathcal{A}\circ\begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_N \end{bmatrix}+\alpha\circ\begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}}\right)$$

where matrix $\mathcal{A}$ represents the adjacency matrix plus identity matrix $\mathcal{A}=\boldsymbol{M}_t+eye(N)$. The operation "$\circ$" denotes element-by-element multiplication, $\circ\div$ denotes element-by-element division, and $\overset{2}{\circ}$ each element in the matrix are squared.

The calculations are summarized by,

$$\begin{bmatrix} E(u_i(a,\theta)) \\ . \\ . \\ . \\ . \\ E(u_N(a,\theta)) \end{bmatrix} = - \begin{bmatrix} a_1^2 \\ . \\ . \\ . \\ . \\ a_N^2 \end{bmatrix} - \begin{bmatrix} (1-r_1) \\ . \\ . \\ . \\ . \\ (1-r_N)\end{bmatrix} \circ \begin{bmatrix} E_1(\theta^2) \\ E_2(\theta^2) \\ . \\ . \\ E_N(\theta^2)\end{bmatrix} - 2\circ \bar{b} \circ \begin{bmatrix}(1-r_1)\\ . \\ . \\ . \\ (1-r_N)\end{bmatrix} \circ \begin{bmatrix} E_1(\theta) \\ E_2(\theta) \\ . \\ . \\ E_N(\theta)\end{bmatrix} -$$

$$-\begin{bmatrix}(1-r_1)\\.\\.\\.\\(1-r_N)\end{bmatrix}\circ\begin{bmatrix}b_1^2\\.\\.\\.\\b_N^2\end{bmatrix} + 2\begin{bmatrix}(1-r_1)\\.\\.\\.\\(1-r_N)\end{bmatrix}\circ\begin{bmatrix}a_1\\.\\.\\.\\a_N\end{bmatrix}\circ\begin{bmatrix}E_1(\theta)\\E_2(\theta)\\.\\.\\E_N(\theta)\end{bmatrix} + 2\begin{bmatrix}(1-r_1)\\.\\.\\.\\(1-r_N)\end{bmatrix}\circ\bar{b}\circ\begin{bmatrix}a_1\\.\\.\\.\\a_N\end{bmatrix} -$$

$$-\begin{bmatrix}r_1\\.\\.\\r_N\end{bmatrix}\circ\begin{bmatrix}(diag(\mathcal{D}_1))^T*((\mathcal{G}_1\circ\mathcal{W})*\bar{x})_\circ^2\\.\\.\\(diag(\mathcal{D}_N))^T*((\mathcal{G}_1\circ\mathcal{W})*\bar{x})_\circ^2\end{bmatrix}-$$

$$-\begin{bmatrix}r_1\\.\\.\\.\\r_N\end{bmatrix}\circ\left[\begin{bmatrix}diag(D_1)^T\\.\\.\\.\\diag(D_N)\end{bmatrix}\circ\begin{bmatrix}0&1&.&.&.&1\\1&0&.&.&.&1\\.&.&.&.&.&.\\.&.&.&.&.&.\\1&.&.&.&1&0\end{bmatrix}\right]*\left[\bar{\omega}_b\circ\bar{b}+y\circ\bar{\omega}_{1,y}\right]_\circ^2-$$

$$-2\begin{bmatrix}r_1\\.\\.\\.\\r_N\end{bmatrix}\circ\begin{bmatrix}diag(D_1)^T*\left(((\mathcal{G}_1\circ\mathcal{W})*\bar{x})\circ\left[\bar{\omega}_b\circ\bar{b}+y\circ\bar{\omega}_{1,y}\right]\right)\\.\\.\\.\\diag(D_N)^T*\left(((\mathcal{G}_N\circ\mathcal{W})*\bar{x})\circ\left[\bar{\omega}_b\circ\bar{b}+y\circ\bar{\omega}_{1,y}\right]\right)\end{bmatrix}-$$

$$-\begin{bmatrix}r_1\\.\\.\\.\\r_N\end{bmatrix}\circ\begin{bmatrix}diag(D_1)^T*\left(\left(\left(\begin{bmatrix}1&.&.&.&1\end{bmatrix}*(\mathcal{C}_1\circ\mathcal{W})^T\right)^T\right)^2_.\right)\\.\\.\\.\\diag(D_N)^T*\left(\left(\left(\begin{bmatrix}1&.&.&.&1\end{bmatrix}*(\mathcal{C}_N\circ\mathcal{W})^T\right)^T\right)^2_.\right)\end{bmatrix}\circ\begin{bmatrix}E_1(\theta^2)\\E_2(\theta^2)\\.\\.\\E_N(\theta^2)\end{bmatrix}-$$

$$-\begin{bmatrix}r_1\\.\\.\\.\\r_N\end{bmatrix}\circ\begin{bmatrix}diag(\mathcal{D}_1)^T*\left((\mathcal{C}_1\circ\mathcal{W})^2_\circ*\begin{bmatrix}\frac{1}{\beta_1}\\.\\.\\\frac{1}{\beta_N}\end{bmatrix}\right)\\.\\.\\diag(\mathcal{D}_N)^T*\left((\mathcal{C}_N\circ\mathcal{W})^2_\circ*\begin{bmatrix}\frac{1}{\beta_1}\\.\\.\\\frac{1}{\beta_N}\end{bmatrix}\right)\end{bmatrix}-$$

$$
-2\circ\begin{bmatrix} r_1\\ .\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix} diag(\mathcal{D}_1)^T*(((G_1\circ\mathcal{W})*\bar{x})\circ((\mathcal{C}_1\circ\mathcal{W})*\mathbf{1}))\\ .\\ .\\ diag(\mathcal{D}_N)^T*(((\mathcal{G}_N\circ\mathcal{W})*\bar{x})\circ((\mathcal{C}_N\circ\mathcal{W})*\mathbf{1}))\end{bmatrix}\circ\begin{bmatrix} E_1(\theta)\\ E_2(\theta)\\ .\\ .\\ E_N(\theta)\end{bmatrix}-
$$

$$
-2\circ\begin{bmatrix} r_1\\ .\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix} diag(D_1)^T*\left(\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y*y\right)\circ(\mathcal{C}_1\circ\mathcal{W}*\mathbf{1})\right)\\ .\\ .\\ .\\ diag(D_N)^T*\left(\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y*y\right)\circ(\mathcal{C}_N\circ\mathcal{W}*\mathbf{1})\right)\end{bmatrix}\circ\begin{bmatrix} E_1(\theta)\\ E_2(\theta)\\ .\\ .\\ E_N(\theta)\end{bmatrix}-
$$

$$
+2\circ\circ\begin{bmatrix} r_1\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix} a_1\\ .\\ .\\ a_2\end{bmatrix}\circ\begin{bmatrix} diag(D_1)^T*\left((\mathcal{G}_1\circ\mathcal{W})*\bar{x}+(\mathcal{C}_1\circ\mathcal{W})*E(\theta)+\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y\circ\bar{y}\right)\right)\\ .\\ .\\ diag(D_N)^T*\left((\mathcal{G}_N\circ\mathcal{W})*\bar{x}+(\mathcal{C}_N\circ\mathcal{W})*E(\theta)+\left(\bar{\omega}_b\circ\bar{b}+\bar{\omega}_y\circ\bar{y}\right)\right)\end{bmatrix}-
$$

$$
-2\circ\begin{bmatrix} r_1\\ .\\ .\\ r_N\end{bmatrix}\circ\begin{bmatrix} diag(D_1)^T*\sum_{i=1}^{N}(\mathcal{M}(:,i)\circ\mathcal{M}*\mathbf{1})\\ .\\ .\\ diag(D_N)^T*\sum_{i=1}^{N}(\mathcal{M}(:,i)\circ\mathcal{M}*\mathbf{1})\end{bmatrix}\circ E(\theta)^2_\circ
$$

where $\mathcal{M}=\mathcal{C}_{1,new}\circ\mathcal{W}_{\text{private}}$ and $\mathcal{M}=\mathcal{M}|_{\{(M_1(:,i-1,i-2,\ldots i-N)=0\}}$.So when we take the first row from the matrix $\mathcal{M}$, we replace 0's in their place. The next step, when we take the second row, we will keep the matrix which we get before(with the first row equal to 0's) and we change and put the second row = 0.

8. Proof of the static-game result

As we can see in Figures 3a, 3b, and 3c, the final period graphs looks like combination of star networks with ring networks. Therefore, for simplification, I compare "central agent", "central sender" and "central receiver" cases. Below I describe the optimal private signal weight.

Star network joint with ring network. The central agent sends and gets the private signals of others

$$
a_i=w_1\frac{x_i+x_{i-1}+x_{i+1}}{3}+w_2x_c+(1-w_1-w_2)y
$$

$$a_c = \omega_1 x_c + \omega_2 \sum_{i=1}^{N-1} \frac{x_i}{N-1} + (1-\omega_1-\omega_2)y$$

$$w_c^* = \begin{bmatrix} \omega_1 \\ \omega_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -r \\ 0 & \frac{1}{N-1} & -\frac{r}{N-1} & 0 \\ 0 & -\frac{\phi_i}{4}\frac{(1-r)(N-4)}{(N-1)} - \frac{r}{(N-1)^2} & \frac{1}{3} - \frac{\phi_i}{4}\frac{r(N-4)}{(N-1)} - \frac{2}{3}\frac{r}{N-1} & 0 \\ \frac{r}{N-1} & -\frac{\phi_i}{4}\frac{r(N-4)}{(N-1)^2} & -\frac{\phi_i}{4}\frac{r(N-4)}{(N-1)} & 1-\frac{r(N-2)}{(N-1)} \end{bmatrix}^{-1} \begin{bmatrix} (1-r)\frac{\phi_c}{N} \\ (1-r)\frac{\phi_c}{N} \\ (1-r)\frac{\phi_i}{4} \\ (1-r)\frac{\phi_i}{4} \end{bmatrix} \tag{20}$$

One centralized agent who gets signals from others without showing his own signal We consider network structure where central agents gets signals from other agents in the networks, but didn't share his own signal.

Linear Strategy for central agents will look like the following way.

$$a_c = \omega_1 x_c + \omega_2 \frac{\sum x_i}{N-1} + (1-\omega_1-\omega_2)y$$

Linear strategy for other agents will look like the following way.

$$a_i = w_1 \frac{x_i + x_{i+1} + x_{i-1}}{3} + (1-\xi)y$$

$$w_r^* = \begin{bmatrix} \omega_1 \\ \omega_2 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -r \\ -\frac{\phi_i}{3}\frac{r}{N-1} & -\frac{\phi_i}{3}\frac{r(N-4)}{(N-1)^2} - \frac{r}{(N-1)^2} & \frac{1}{3} - \frac{\phi_i}{3}\frac{r(N-4)}{(N-1)} - \frac{2}{3}\frac{r}{N-1} \end{bmatrix}^{-1} \begin{bmatrix} (1-r)\frac{\phi_c}{N} \\ (1-r)\frac{\phi_c(N-1)}{N} \\ (1-r)\frac{\phi_i}{3} \end{bmatrix} \tag{21}$$

Optimal weights can be provide by request. We use this matrices to analyze the social welfare.

Central agent sends information without getting any signal Let's look at the linear strategy of agents.

$$a_i = w_1 \frac{x_i + x_{i+1} + x_{i-1}}{3} + w_2 x_c + (1 - \omega_1 - \omega_2) y$$

$$a_c = \omega_c x_c + (1 - \xi) y$$

$$\omega = \frac{v}{v+h}((1-r) + r\omega_1) + r\omega_2 \tag{22}$$

So we can find the optimal weights ω_1, w_1 and w_2.

$$w_s^* = \begin{bmatrix} \omega_1 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -r\frac{\beta}{\alpha+\beta} & -r \\ 0 & \frac{1}{3} - \frac{\phi_i}{3}\frac{r(N-4)}{(N-1)} - \frac{2}{3}\frac{r}{N-1} & 0 \\ -\frac{r}{N-1} & -\frac{\phi_i}{4}\frac{r(N-4)}{(N-1)} & 1 - \frac{r(N-2)}{(N-1)} \end{bmatrix}^{-1} \begin{bmatrix} (1-r)\frac{\beta}{\alpha+\beta} \\ (1-r)\frac{\phi_i}{4} \\ (1-r)\frac{\phi_i}{4} \end{bmatrix} \tag{23}$$

The optimal weights are available in online Appendix, using these optimal weights we can calculate the welfare W_c^*,W_r^* and W_s^*

The optimal welfare for central agent case is described by the following equations. Please notice, that in this example precision is equal for every agents and agents doesn't distinguish weights between friends and non-friends.

$$W_c^* = \frac{1}{N}\left(\frac{\omega_1^{2*}}{\beta} + \frac{\omega_2^{2*}}{(N-1)\,\beta} + \frac{(1-\omega_1^* - \omega_2^*)^2}{\alpha}\right) + \frac{N-1}{N}\left(\frac{w_1^{2*}}{3\beta} + \frac{w_2^{2*}}{\beta} + \frac{(1-w_1^* - w_2^*)^2}{\alpha}\right) \tag{24}$$

The welfare W_r^* from central reciever side describes by the following equations.

$$W_r^* = \frac{1}{N}\left(\frac{\omega_1^{2*}}{\beta} + \frac{\omega_2^{2*}}{(N-1)\,\beta} + \frac{(1-\omega_1^* - \omega_2^*)^2}{\alpha}\right) + \frac{N-1}{N}\left(\frac{w_1^{2*}}{3\beta} + \frac{(1-w_1^*)^2}{\alpha}\right) \tag{25}$$

The welfare W_r^* from central sender side describes by the following equations.

$$W_s^* = \frac{1}{N}\left(\frac{\omega_1^{2*}}{\beta} + \frac{(1-\omega_1^*)^2}{\alpha}\right) + \frac{N-1}{N}\left(\frac{w_1^{2*}}{3\beta} + \frac{w_2^{2*}}{\beta} + \frac{(1-w_1^* - w_2^*)^2}{\alpha}\right) \tag{26}$$

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