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Sylvain Carre, Damien Klossner

BANKS AS LIQUIDITY MULTIPLIERS: ASSET LIQUIDITY PREMIA AND FUNDING LIQUIDITY RISK

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Banks hoard large amounts of liquid reserves while sustaining high levels of leverage, and the return on these reserves is often below the cost of their deposits. Why do banks engage in such a negative carry trade? Using a novel observation on global games, we build a model where banks manage their liquidity risk through their demand of liquid assets (safe public debt) and their private provision of debt to creditors. Banks value liquid reserves particularly since those allow them to “multiply liquidity”: buy public debt and issue more private debt, keeping liquidity risk constant. How much more is measured by the slope of an endogenous iso-risk curve. Liquidity multiplication is key to understand the rich interaction between leverage and liquidity buffer strategies and implies a theory for the negative carry trade puzzle.

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Sylvain Carre, ICEF, National Research University Higher School of Economics (HSE University), Russian Federation; Office S708, 11 Pokrovsky Boulevard, 109028 Moscow, Russian Federation; E-mail: scarre@hse.ru

*Damien Klossner***, Swiss National Bank; E-mail: damien.klossner@snb.ch

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1 Introduction

As any severe downturn, the global financial crisis came with a silver lining: economic theory as well as the regulatory and industry landscapes have been profoundly reshaped by ideas that (re)surfaced in the aftermath of 2007-2008. The notion of bank liquidity risk management is a case in point. Once a bit of a forgotten child, it is now essential in analysing the roots of the crisis and attempting to prevent similar events going forward. The tools needed for this management are widely agreed upon: academics, policy-makers and practicioners alike recognise the importance of complementing the control of bank leverage (especially short-term) with a strategy of building and maintaining sufficient liquidity buffers.

Yet, puzzles remain. Why do banks hold *such* large amounts of low-return liquid reserves? Specifically, since 2008, why do European banks routinely perform a negative carry trade, hoarding liquid government securities whose yields are lower than their own deposits?[1] This question is particularly intriguing as the amount of such securities held by a typical European bank significantly exceeds the levels mandated by the regulator.[2] Why do not banks curtail their short-term leverage to monitor their liquidity risk, rather than invest, at a loss, in additional liquid reserves? Answering these questions implies to understand how financial institutions optimise their levels of debt issuance and liquid reserves in the presence of liquidity risk; and then to consider the asset pricing implications of this joint choice.

In our stylised 3-dates model of financial intermediation, we abstract from the distinctions that exist between different liquid assets and consider a generic high-quality liquid asset (HQLA), modeled by a perfectly risk-free government bond. This bond provides *public* liquidity. We also abstract

[1]Empirical evidence is given in Section 2.

[2]For instance, the European Banking Authority (EBA) consistently reports that the large majority of banks hold more high-quality liquid assets than prescribed by Basel III's Liquidity Coverage Ratio, with an average of 134% in 2015 and 146% in 2018 instead of the mandated 100%. See eba.europa.eu, "EBA sees considerable improvement in the average LCR across EU banks", December 21, 2016 or EBA (2019).

from the various forms that short-term bank debt takes and consider a single type of demand-deposit contract. This contract provides *private* liquidity to investors. The two classes of agents in our model—banks and creditors—are risk neutral, but creditors have an extra valuation of the liquidity benefits provided by debt contracts. Bank debt, being subject to default, may be less liquid than the default-free public debt. We use global games theory to model liquidity risk and map banks' balance sheet decisions to their likelihood to fail. Taking this mapping into account as well as the liquidity demand of creditors, banks' demand for public debt and supply of private debt can be computed. One proceeds similarly for the creditors' demand and the prices of private and public securities arise through market clearing conditions.

Our main results are to propose a theory that explains the negative carry trade puzzle and to connect it to a quantity that we call a *marginal liquidity multiplier*. This multiplier arises endogenously and tractably from our global games model. It is key in understanding the rich interaction between leverage and liquid reserves decisions.

We now explain how we get there and detail our findings.

First, we make a new observation on global games that allows us to analyse the interaction between short-term leverage, liquid asset holdings, and liquidity risk. In order to obtain a tractable global games equilibrium with liquid reserves, the literature had so far either resorted to simplified payoffs or imposed an exogenous ceiling on the size of the liquidity buffer. We show that this is not necessary.

Moreover, the states where the bank fails due to a run are characterised in closed-form as a simple function of two choice variables: the *liquidity ratio* (the value of the numeraire good that can be realised through a liquidation of the bank's assets relative to its liabilities) and the *reserve ratio* (the value of the bank's liquid reserves relative to its liabilities). By issuing no more debt than the resources that can be raised in the short run, that is, by selecting a liquidity ratio of one, the bank can eliminate the coordination failure. We refer to the bank as being "fully liquid" in this case, as runs only occur when a liquidation of the bank's assets is efficient. Any deviation from this benchmark opens up the possibility of inefficient liquidations, and hence captures the *liquidity risk-taking* behaviour of the bank.

Define a bank's *iso-risk curves* as the combinations of debt levels and liquid asset holdings that hold the likelihood of runs constant. The slope of the iso-risk curve quantifies how many additional units of debt the bank can take on without changing its liquidity risk, when one unit of liquid asset is added to its buffer. We prove that this slope is strictly above one as long as a bank is not fully liquid, and equal to one otherwise. We call the slope of the iso-risk curve a *marginal liquidity multiplier*, since it quantifies the ability of a bank to "multiply liquidity": instead of having investors buying directly a unit of government debt, the bank can buy this unit and then issue to investors more than one unit of bank debt, without affecting the bank failure likelihood.

Next, we characterise the balance sheet decisions of banks. Similarly to corporate finance textbooks, the bank's initial equity value can be decomposed into the franchise value, minus the cost of inefficient liquidations, plus the aggregate liquidity spread captured by the bank, that is, the liquidity premium priced into its debt, minus the liquidity premium priced into its liquid asset holdings. Maximising the bank's equity value is a two-dimensional problem that can be conveniently represented as follows. In the first step, we ask what combination of debt and liquid reserves achieves a given level of liquidity risk at minimal cost. We show that, if the creditors' demand for liquidity is not satiated, the combination is unique; if the creditors' demand for liquidity is satiated, then banks do not want to hold government bonds as liquid reserves unless no liquidity premium is priced into this asset. In a second step, we determine the bank's optimal level of liquidity risk. We show that taking on liquidity risk is optimal if and only if the creditors' demand for liquidity is not satiated.

Consequently, general equilibria in our model can be divided into two liquidity regimes. If safe/liquid government bonds are abundantly available, banks do not take liquidity risk, the yields of government bonds and bank debt claims solely reflect their expected payoffs, and government bonds are held indifferently by banks or creditors. If, however, there is a macroeconomic shortage of public safe/liquid assets, then banks take on liquidity risk, a 'money premium' or 'convenience yield' gets priced into government bonds and bank debt, and banks hold the entire bond supply. This result is striking because we assume that creditors, not banks, have a preference for liquidity; in addition, they value government bonds more than bank debt, since the former are risk-free.

The marginal liquidity multiplier provides the missing piece to the puzzle. When the supply of liquid assets is scarce, banks are incentivised to issue short-term debt in order to take advantage of the low yield of money-like assets. In turn, taking liquidity risk allows banks to act as liquidity multipliers, using a unit of government debt to issue *more* than one unit of debt without increasing their liquidity risk. Borrowing terminology from *e.g.* DeAngelo and Stulz (2015), banks engage in *liquid-claims production*, using public liquid assets as input to produce private money-like instruments. We show that this marginal product depends crucially on the bank's liquidity risk appetite. Then, as banks compete with one another for the limited supply of government bonds, they bid up the price *above* the valuation of creditors due to their ability to earn a levered money premium by using a marginal investment in government bonds to increase their debt by more than one unit — without impacting their liquidity risk.

When the supply of liquid assets is scarce, it is (privately and socially) beneficial for banks to *intermediate* the supply of government bonds. Put simply, when liquidity is scarce, liquid government bonds are more valuable sitting on banks' balance sheets than held in the creditors' portfolios. The private debt issued by banks is an imperfect substitute for public liquid assets, since the former is not risk-free. The presence of a liquidity multiplier ensures that banks can nevertheless enhance the provision of liquidity in the economy through *liquidity tranformation*, that is, backing their debt partly with risky/illiquid assets. Hence, while our stylised model is not able match real-world quantities (after all, banks do not hold *all* safe/liquid government bonds), this result is in line with the empirical fact that banks hold a large share of safe and liquid securities with low return; and they do so because of their unique ability to "multiply liquidity". Studies of the liquidity premium, or convenience yield, priced into near-money assets such as Treasuries have typically supported their empirical analysis by a representative agent model (Krishnamurthy and Vissing-Jorgensen (2012), Greenwood, Hanson, and Stein (2015), Nagel (2016)). Our model suggests that a key driver of this convenience yield is the liquidity risk management strategy of the banking sector. Hence, it would be interesting to see if an empirical analysis including explicitly a financial and a non-financial sectors would allow to push the aforementioned literature even further.

Our paper provides a theory for the negative carry trade puzzle. Specifically, our framework allows us to capture analytically the spread between the yield of high-quality liquid assets and a bank's financing costs, and how this spread depends on the funding liquidity risk of the bank. We show that the premium of government bonds can be decomposed into two components: (i) the marginal value of liquidity to creditors and (ii) an additional term that reflects the ability of banks to earn a levered money premium. It immediately follows that the yield of high-quality liquid assets must be higher than a bank's financing costs in equilibrium. Why are government bonds nevertheless a profitable investment for banks? Because holding such assets allows banks to reduce their financing costs by substituting (expensive) equity for (cheap) short-term debt financing. Thus, our model explains why holding government bonds and financing these holdings by bank debt is a negative carry trade, and why it is nevertheless beneficial for banks to hold such assets. In fact, the existence of a marginal liquidity multiplier explains, not just the possibility, but the necessity of a negative spread between high-quality liquid assets and a typical bank's financing costs.

Finally, one important message of our paper is that *being able to multiply* *liquidity goes hand in hand with being exposed to liquidity risk.* Another way to look at this result is through the following thought experiment. Starting from a laissez-faire equilibrium, suppose that a constraining liquidity risk limit is imposed on one individual bank. Would the bank satisfy its risk limit by increasing its liquid reserves? The answer is no. A bank with low risk is characterised by a low marginal liquidity multiplier. Put differently, holding government bonds is less effective at reducing this bank's financing costs, because it has to back each unit of debt by a larger quantity of liquid reserves. Given the lower profitability of investing in government bonds, compared to other banks, the constrained bank instead satisfies its risk limit by reducing its use of short-term debt financing, while also *decreasing* its liquid reserves. More generally, this example illustrates how the balance sheets of banks and money premium of high-quality liquid assets are determined jointly. In our model, the leveraging ability of banks is a crucial driver of the money premium of sovereign bonds, which, in turn, is a key component of the liquidity risk management strategy of individual banks.

Related literature. So far, how have economic theorists thought about and modelled the interaction between short-term leverage, liquid asset holdings, and funding liquidity risk? Capturing formally a coordination failure among debt holders and isolating the resulting liquidity risk from the "fundamental" solvency risk is a highly non-trivial task, and we owe much to the seminal works of Rochet and Vives (2004), Goldstein and Pauzner (2005), Morris and Shin (2006) and Morris and Shin (2009).[3] These papers focus mostly on characterizing the behaviour of creditors as a function of the information they receive, given the characteristics of the leveraged institution. The natural follow-up of the research agenda was to endogenise the balance sheet of the leveraged institution, and to embed the partial equilibrium analysis of

[3] Global games were introduced by Carlsson and van Damme (1993). The aforementioned papers applied these techniques in settings where creditors fail to coordinate *simultaneously*. *Intertemporal* coordination failures have been studied in the dynamic debt runs literature, heralded by He and Xiong (2012).

creditors' behavior into a general equilibrium model. To do so, the literature has consistently taken one of the two following paths: (i) Some authors assume, as do Rochet and Vives, that creditors delegate their rollover decisions to fund managers, whose payoffs have a simplified *ad hoc* structure; (ii) Others add liquid reserves in the framework of Goldstein and Pauzner, but either do not model liquid asset holdings as a separate balance sheet item or impose restrictive assumptions of the allowable size of the liquidity buffer. The key issue is that there needs to be sufficient strategic complementarity between the creditors' payoffs to ensure that the crucial *single-crossing property* of Goldstein and Pauzner is met, which, in turn, implies the existence of an equilibrium in threshold strategies.

Besides the founding contributions mentioned above, a significant body of literature has incorporated the global games apparatus into banking papers. Szkup (2013) studies an optimal maturity problem and shows how to endogenise the face value promised by a bank to its creditors at the interim date. We incorporate his technology in our paper for two reasons. First, the face value exhibited by Szkup is the only dynamically consistent choice of the bank. Second, this choice improves the overall tractability of the model. Vives (2014) shows that the degree of strategic complementarity of investors' actions is a key quantity for characterising the sensititivity of equilibrium to changes in parameters and analysing policy effectiveness. Koenig (2015) shows that liquidity requirements may increase the default risk of banks: while an increase in cash holdings makes the bank more robust to withdrawals, it also raises solvency risk by reducing the bank's portfolio returns. Both Koenig and Vives build on Rochet and Vives (2004) and characterise iso-risk curves in this context. But they do not endogeneise the banks' decisions nor the prices.[4]

Several papers have embedded the global games approach into general equilibrium models of banking. Eisenbach (2017) views coordination risk as

[4]The same remark applies to Tourre (2016), who produces iso-risk curves in a model of dynamic debt runs.

a disciplining device for banks and shows that in the presence of aggregate risk this device will be too strong in bad times and too weak in good times. Eisenbach designs his payoff matrix specifically to ensure global strategic complementarity; and focuses on a different problem than ours. Kashyap, Tsomocos, and Vardoulakis (2017) build a complex model that captures coordination failures in the spirit of Goldstein and Pauzner. The richness of their framework allows them to study the impact of various capital and liquidity requirements. However, they assume that a bank's liquid reserves can never exceed their amount of sure outflows. This immediately ensures that the single-crossing property holds, but forces them to focus on the direct value of liquid reserves only: for those impatient agents that will certainly demand early repayment of their debt, the bank can use its reserves instead of carrying out a costly partial liquidation of its risky asset. This makes liquid reserves valuable for banks, but only inasmuch as they cover sure outflows; and does not capture their strategic value. By contrast, even though we do not assume no sure outflows at the interim date in our model, the bank still finds optimal to bid for expensive liquid reserves. Carletti, Goldstein, and Leonello (2018) study capital and liquidity regulation in a framework similar in spirit to ours. They consider an exogenous liquidity-return menu. In this approach, the bank holds a single asset, which is risky. Its interim liquidation is used as a proxy for the liquidity of a whole bank portfolio—that would include both risky and safe/liquid assets—and one posits a decreasing relationship between this variable and the final payoff of the bank's asset. Liu (2016) "provides a general-equilibrium treatment of Rochet and Vives (2004) and endogeneizes the interbank market in their model." As us, he considers a setup with a competitive banking sector, where liquid reserves and bank debt are substitutes in managing liquidity risk. Nevertheless, his work differs from ours in several respects. First, he uses simplified payoffs in the spirit of Rochet and Vives. Second, he focuses mostly on idiosyncratic risk while we only consider aggregate risk. In his model, banks with different shocks trade liquidity in the interbank market. Because the price of this liquidity must be anticipated and affects the willingness of creditors to run, multiple equilibria can resurface. In our model the value of reserves at the interim date derives trivially from our assumptions, the real endogenous object being their value at the initial date. Tying this object to the properties of an iso-risk curve and evidencing a liquidity multiplication mechanism is a specific endeavour of our paper. Liu (2019) extends his ideas to a dynamic setting.

Finally, Weymuller (2015) also attempts to provide a solution to the carry trade puzzle. To that end, he builds a theory of banks as "safety multipliers" that relies on risk-aversion heterogeneity. Our paper provides an entirely distinct theory purely based on liquidity risk and the strategic value of liquid assets, where banks act as "liquidity multipliers".

2 Motivating Stylized Facts

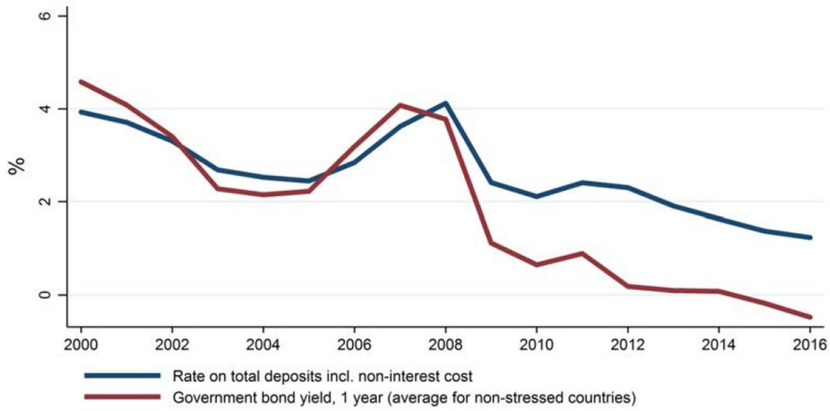

Figure 1: Rates on deposits and government bond yields. Source: Hoerova, Mendicino, Nikolov, Schepens, and van den Heuvel (2018).

Figure 1 illustrates the negative carry trade performed by European banks since 2008. For convenience we report the details given by Hoerova et al. (2018): "the blue line is the average interest rate on total deposits of househoulds and non-financial corporations in euro area banks. The red line is the average yield on 1-year sovereign bonds of non-stressed euro area countries, as a proxy for the yield on (level 1) HQLA assets."

Weymuller (2015) also evidences the negative carry trade phenomenon. In addition, he confirms that banks hold large shares of safe and liquid securities with low return: for European banks, from 1997 to 2013, this represented around 15% of their assets.

The fact that "the supply of genuinely high-quality liquid assets is not unlimited" (Hoerova et al. (2018)) means that among the two general equilibrium liquidity regimes we evidence, the most likely to occur in reality is the one were banks do take liquidity risk, the marginal liquidity multiplier is above one, and the premia on bank debt and liquid reserves are such that there is a negative carry trade.

3 Global Game Equilibrium with Liquid Reserves

We consider an augmentation of Goldstein and Pauzner (2005)'s model in which banks can hoard liquid reserves. This leads us to consider a global game with novel payoffs. In this section, we focus on this game only: there are two periods, $t = 1, 2$, and creditors simultaneously decide whether to withdraw their funds at $t = 1$ or $t = 2$ based on the balance sheet parameters of the bank and their private signals. The key result of this section is to show that existence and uniqueness of an equilibrium in threshold strategies still holds, under the standard assumptions. In the subsequent sections, we endogenise balance sheet decisions—banks' choices at $t = 0$— and analyse them in partial and general equilibrium.

There is a single, divisible good in the economy. A bank holds a risky asset and liquid reserves, and has demandable debt outstanding. This debt is held uniformly by a mass 1 continuum of uncoordinated creditors. They are risk-neutral and do not discount the future.

3.1 Bank's balance sheet

The bank's balance sheet in our model is as follows.

Asset side	Liability side
Single risky project	Demandable debt δ
Liquid reserves β	Equity

We now describe these elements in detail.

3.1.1 Risky Project

A bank owns a single project with random payoff Z in the final period $t = 2$, equal to $z > 0$ with probability $\varphi(\theta)$ and to 0 with probability $1-\varphi(\theta)$. We refer to θ interchangeably as the *state* or *fundamental*. $\varphi(\cdot)$ is strictly increasing with $\lim_{\theta\to-\infty}\varphi(\theta) = 0$.

At $t = 1$, the project can be abandoned and an amount $0 < \ell < z$ can be recovered, independent of θ. When the state θ is sufficiently low, the date-1 expected value of the project is lower than the liquidation value, $\varphi(\theta)z < \ell$. We denote by p^E the cutoff level for the success probability below which liquidation is efficient:

$$p^E := \ell/z. \tag{1}$$

Agents have prior $\theta \sim N(\mu_y, \tau_y^{-1})$.[5] Before taking their withdrawal decision, creditors receive private signals of precision τ_x about θ:

$$x_i = \theta + \tau_x^{-1/2}\varepsilon_i,$$

[5] In the next sections, this prior will be the posterior following a public signal about θ. The subscript y allows to distinguish the corresponding moments from those of the date-0 prior, μ and τ^{-1}.

where $(\varepsilon_i)_{i\in[0,1]}$ are independent standard Gaussian noise terms.[6]

3.1.2 Liquid Assets

Liquid assets are modeled as risk-free government bonds, paying one unit of the single good at $t = 2$. Let β be the quantity of bonds held by the bank. We call it *bond holdings*, *liquidity buffer* or *liquid reserves* interchangeably.

We restrict the bank from distributing it to its shareholders as a dividend at $t = 1$. Hence, the liquidity buffer is either used to repay early withdrawals at $t = 1$[7] or kept in the bank until $t = 2$. Moreover, liquid reserves are used in priority to meet time 1 withdrawals: only when they are exhausted does the bank start to liquidate its risky asset. We exclude other sources of funding at $t = 1$.[8]

3.1.3 Bank's debt

The bank has emitted a quantity δ of demandable debt. Let F_t denote the face value of an individual debt contract if withdrawn at $t = 1, 2$. At $t = 1$, creditors face a decision of whether to roll over their debt at face value F_2, or demand payment of F_1.

It is convenient to fix the value $F_1 = 1$ once and for all.[9] $F_2 > 1$ is fixed and known at the time of the withdrawal decisions. In section 4, F_2 will arise endogenously from the bank's shareholders maximisation program.[10]

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[6] Any noise satisfying the log-supermodularity condition of Athey (2002) (see the proof of Proposition 1) would be equally suitable.

[7] Bonds can be liquidated at time 1 for one unit of the single good (see section 4 for details).

[8] We can introduce a lender of last resort (LLR) in the model and retain existence, uniqueness and tractability of equilibrium. In bad times, the LLR injects liquidity at time 1 through a loan, the terms of which can be more or less stringent depending on the willingness of the LLR to subsidise banks. This analysis yields an array of results regarding the scope and efficiency of LLR interventions that we plan to output in a companion paper.

[9] This is without loss of generality: the model is invariant to the normalisation $F_1 \to 1$, $F_2 \to \frac{F_2}{F_1}$, and $\delta \to \delta F_1$.

[10] In particular $F_2 > 1$ will hold because if $F_2 \leq 1$, withdrawing early is a dominant action for creditors; and equity holders get nothing.

3.1.4 Liquidity and Reserve Ratios

Let $n \in [0,1]$ be the fraction of creditors who refuse to roll over their loans by demanding repayment at date 1. If those creditors cannot be repaid using liquid reserves only, the bank has to liquidate a fraction x of the project, where x satisfies

$$x\ell + \beta = n\delta.$$

The bank experiences early failure if n exceeds

$$\lambda := \frac{\ell + \beta}{\delta}.$$

which we refer to as the *liquidity ratio*. It is equal to the quantity of the numeraire good that can be realised at $t = 1$ relative to short-term liabilities.

Another important quantity is

$$\rho := \frac{\beta}{\delta},$$

the *reserve ratio* of the bank. It measures the quantity of liquid reserves relative to short-term liabilities. Since $\ell > 0$ and $\beta \geq 0$, we have:

$$0 \leq \rho < \lambda.$$

The following conditions will help us characterising the equilibrium.

Condition 1. $\delta < z + \beta$.

Condition 2. $\lambda < 1$.

These conditions are automatically satisfied once we endogenise the banks' choices.

Condition 1 says that equity holders receive a strictly positive payoff at maturity in the scenario where: (i) the project is successful and (ii) all creditors roll over. Condition 2 says that the bank is not able to raise sufficient resources in the short run to repay *all* of its debt. As we shall see, this guarantees that the decisions to withdraw at $t = 1$ feature enough strategic complementarity to ensure the existence of an equilibrium in threshold strategies.

If $n > \lambda$, so that the bank defaults at $t = 1$, we assume that the bank's assets are divided equally among early withdrawers, while creditors who rolled over get nothing. Absent default at $t = 1$, the bank repays all withdrawing creditors and continues to operate until $t = 2$. In the final period the bank receives the payoff

$$\left(1 - (\delta n - \beta)^{+}/\ell\right) Z + (\beta - \delta n)^{+}, \quad Z \in \{0, z\} \tag{2}$$

(the sum of the payoff of the project, potentially scaled down due to early withdrawals, and of the remaining liquid reserves). If this quantity exceeds the bank's remaining debt obligations, $(1 - n)\delta F_2$, creditors who chose to roll over receive the promised face value F_2, and equity holders receives the residual payoff (2) $-(1-n)\delta F_2$. Otherwise, equity holders exercise their limited liability option and the assets are divided equally among those creditors who chose to roll over.[11]

3.2 Global Game Assumptions

3.2.1 Vanishing Noise

Following much of the global games literature, we focus on the case of arbitrarily small signal noise: $\tau_x \to \infty$. This presents several benefits. First, it will ensure uniqueness of equilibrium in threshold strategies for the coordination game played by creditors at $t = 1$. Second, the equilibrium threshold can be expressed in closed-form. Third, the set of realizations of the state

[11] That is, debt contracts do not have a *cross-default* or *acceleration* provision in our model. If the bank defaults at $t = 1$, creditors who rolled over get nothing. Given the linearity of creditors' utility functions, this payoff structure is equivalent to assuming that the bank follows a *sequential service constraint*: it repays withdrawing creditors in full until its resources are exhausted, and the order in the queue in random.

θ which results in "partial liquidations" (i.e. banks are forced to rescale the risky project but can nevertheless avoid a liquidity-driven default) has measure zero. In section 4, this simplifies the expressions describing the payoffs of the banks' debt and equity claims.

3.2.2 Dominance Regions

Following Goldstein and Pauzner (2005), we assume that there is a region of extremely strong fundamentals in which success of the project is guaranteed, and the bank is able to liquidate the project at no cost: there exists a $\overline{\theta} \in \mathbb{R}$ such that for all $\theta > \overline{\theta}$, the project succeeds at $t = 2$ with probability one and can be rescaled at $t = 1$ at no cost (i.e. the liquidation value is $\overline{\ell} = z$).

Since $\overline{\theta}$ can be placed arbitrarily far from the mean, it eliminates the equilibrium where creditors ignore their signal and always run, without changing quantitatively the outcome of the game or the payoff of the claims issued by banks.[12]

Note that since $\varphi(\theta) \to 0$ as $\theta \to -\infty$, there exists a $\underline{\theta} \in \mathbb{R}$ such that for all $\theta < \underline{\theta}$ withdrawing early is the dominant action for creditor i regardless of his belief regarding the actions of other creditors. Hence, the model features a lower dominance region.

3.3 Payoffs

In models of global games such as Goldstein and Pauzner (2005), the bank can only be forced into default if the fraction of running creditors n is *sufficiently large*. Intuitively, this relates to the fact that withdrawal decisions are to a sufficiently large extent strategic complements. Introducing liquid reserves attenuates the degree of strategic complementarity. As a

[12] Hence, throughout the paper we assign positive but arbitrarily low probability to the interval $(\overline{\theta}, \infty)$, and carry out all payoff computations without explicit mention of the upper dominance region.

result, we must in general deal with *two* default thresholds: one above which too many creditors withdraw, and one *below which not enough creditors withdraw.*

We derive these thresholds below. We then describe the creditors' expected payoffs at time 1. We are then in a position to state the main result of section 3.

3.3.1 Banks' Default Condition at $t = 2$

Suppose the bank did not default at $t = 1$, i.e., $n < \lambda$. The payoff to equity holders at $t = 2$ is given by

$$E(n, F_2; Z) = \max\big\{0, \underbrace{(1 - (\delta n - \beta)^+/\ell)Z}_{\text{payoff of rescaled project}} + \underbrace{(\beta - \delta n)^+}_{\text{excess reserves}} - \underbrace{\delta F_2(1 - n)}_{\text{remaining debt}}\big\}. \tag{3}$$

The bank defaults at $t = 2$ if and only if the available resources per unit of outstanding debt,

$$\mathcal{Q}(n; Z) = \begin{cases} \frac{(1-(\delta n-\beta)^+/\ell)Z+(\beta-\delta n)^+}{\delta(1-n)} & \text{if} \quad n < \lambda \\ 0 & \text{if} \quad n \geq \lambda, \end{cases} \tag{4}$$

is lower than the face value, F_2. This condition always holds if the project fails ($Z = 0$).[13] Whether the bank is able to avoid default when the project succeeds ($Z = z$) depends on the fraction of creditors who refused to roll over at $t = 1$. An increase in n has two opposite effects on the expected payoff of rolling over. On the one hand, the bank has to liquidate a larger fraction of the project (a decrease in the numerator in (4)), which reduces the pool of assets backing the debt at the final date. On the other hand, as more creditors withdraw early the outstanding debt is reduced, so the remaining creditors can rely on a greater "share of the pie" (a decrease in

[13] This is because the ratio $\frac{\beta - n\delta}{(1-n)\delta}$ is strictly decreasing in n for $0 \leq n < \rho$ and equal to ρ for $n = 0$, with $\rho < \lambda < 1 < F_2$.

the denominator in (4)).

If $\lambda < 1$, then $\mathcal{Q}$ is strictly decreasing in n for $\rho \leq n < \lambda$. In words: once the withdrawals are so large that the bank is forced to liquidate the risky project, early withdrawals reduce the expected payoff of the remaining creditors. This means that there is a cutoff level $\overline{n}$ *above* which the bank defaults at $t = 2$ —even if the project is successul.

But in the region $0 \leq n \leq \rho$, early withdrawals reduce the outstanding debt without causing costly liquidations of the risky project. This has the effect of increasing the payoff that the remaining creditors get when the project succeeds. Hence, when the face value F_2 is sufficiently high, there is a cutoff level $\underline{n} > 0$ *below* which the bank defaults at $t = 2$ even if the project succeeds: see Figure 3.

[INSERT FIGURE 3 ABOUT HERE]

The next lemma derives the expressions for the cutoff levels $\underline{n}$ and $\overline{n}$, as well as the payoff of the debt maturing at $t = 2$.

Lemma 1. *Define*

$$\overline{F}_2 \equiv \frac{z}{\delta - \beta}.$$

(i) If $F_2 \leq \overline{F}_2$, then the bank is solvent at $t = 2$ if the project is successful and if the proportion of running creditors satisfies $\underline{n} \leq n \leq \overline{n}$, where

$$\overline{n} = (\lambda - p^E F_2)/(1 - p^E F_2) \tag{5}$$

$$\underline{n} = (F_2 - (z + \beta)/\delta)^+ / (F_2 - 1), \tag{6}$$

with $\lim_{F_2 \to \overline{F}_2} \overline{n} = \lim_{F_2 \to \overline{F}_2} \underline{n} = \rho$.

(ii) If $F_2 > \overline{F}_2$, the bank always defaults at $t = 2$.

(All proofs are relegated to Appendix B).

3.3.2 Creditors' Payoffs at $t = 1$ under a Threshold Strategy

We now derive the creditors' payoffs assuming that they all play the following threshold strategy: withdraw if $x_i < x^*$ and roll over if $x_i > x^*$, for some $x^* \in \mathbb{R}$.

If a fraction n of creditors withdraw their funds at $t = 1$ and the project pays off $Z \in \{z, 0\}$, then the payoff from rolling over is equal to

$$D_2(n, F_2; Z) = \min \{F_2, \mathcal{Q}(n, Z)\}, \tag{7}$$

while the payoff from withdrawing is

$$D_1(n) = \min \left\{1, \frac{\ell + \beta}{\delta n}\right\}.$$

Conditional on θ, the payoff gain from rolling over rather than withdrawing early, is then

$$v(\theta, n) = \underbrace{\varphi(\theta) D_2(n, F_2; z) + (1 - \varphi(\theta)) D_2(n, F_2; 0)}_{\text{expected payoff of rolling over}} - \underbrace{D_1(n)}_{\text{payoff of withdrawing}}. \tag{8}$$

When the fundamental realises at θ, the probability that any creditor observes a private signal less than x^* is $\Phi\left(\tau_x^{1/2}(x^* - \theta)\right)$. Φ (resp. ϕ) is the cdf (resp. pdf) of the standard normal distribution. Since there is a continuum of creditors and the realisations of the signals are independent conditional on θ, the fraction of creditors who withdraw their funds is

$$n(\theta, x^*) = \Phi\left(\tau_x^{1/2}(x^* - \theta)\right). \tag{9}$$

Inserting (9) in (8) the payoff gain from rolling over when the true fundamental is θ, is given by $v(\theta, n(\theta, x^*))$.

After observing x_i, the posterior belief of creditor i is

$$\theta | x_i \sim N(m(x_i), (\tau_x + \tau_y)^{-1})$$

with $m(x_i) = (\tau_x + \tau_y)^{-1}(\tau_y \mu_y + \tau_x x_i)$. Thus, from the perspective of creditor i the expected payoff gain from rolling over rather than withdrawing is

$$\Delta(x_i, x^*) = \int_{\mathbb{R}} (\tau_x + \tau_y)^{1/2} v(u, n(u, x^*)))\phi\left(\frac{u - m(x_i)}{(\tau_x + \tau_y)^{-1/2}}\right) \mathrm{d}u. \qquad (10)$$

Upon reception of the private signal x_i, creditor i therefore rolls over if and only if $\Delta(x_i, x^*) \geq 0$, and x^* is a valid equilibrium threshold if and only if

$$\begin{aligned} \Delta(x_i, x^*) &> 0 \quad \text{for} \quad x_i > x^* \\ \Delta(x_i, x^*) &< 0 \quad \text{for} \quad x_i < x^*. \end{aligned}$$

The fact that there exists at least a candidate threshold x^* ($\Delta(x^*, x^*) = 0$) is standard. What remains to be shown for x^* to be a valid equilibrium threshold is that the function $x \mapsto \Delta(x, x^*)$ crosses zero only once. Proving this crucial single-crossing result allows us to establish the main result of section 3.

3.4 Existence and Uniqueness of Equilibrium with Liquid Reserves

Theorem 1. *The global game defined above has an equilibrium in threshold strategies, which is unique in the limit $\tau_x \to \infty$.*

When the noise of private signals vanishes, uncertainty about θ becomes arbitrarily small, while the belief of the pivotal agent —the creditor receiving signal x^*— about the proportion of creditors who choose to run, becomes uniform (or *Laplacian* in the terminology of Morris and Shin (2006)) over the unit interval. In this case, the indifference condition $\Delta(x^*, x^*) = 0$ can be derived by averaging over n the payoff gain (8) evaluated at $\theta = x^*$. This yields the functional form of the equilibrium run threshold:

Lemma 2. *When $\tau_x \to \infty$, the unique equilibrium run threshold satisfies*

$$x^*(F_2) = \varphi^{-1}\left(\frac{\int_0^1 D_1(n)\,\mathrm{d}n - \int_0^1 D_2(n, F_2; 0)\,\mathrm{d}n}{\int_0^1 D_2(n, F_2; z)\,\mathrm{d}n - \int_0^1 D_2(n, F_2; 0)\,\mathrm{d}n}\right), \qquad (11)$$

where the expressions for $\int_0^1 D_1(n)\,\mathrm{d}n$*,* $\int_0^1 D_2(n, F_2; z)\,\mathrm{d}n$*, and* $\int_0^1 D_2(n, F_2; 0)\,\mathrm{d}n$ *are given by* (50)–(52) *in the appendix.*

3.5 Discussion

Under certain conditions, it can be shown that an equilibrium in threshold strategies such as the one described in the previous section is the unique Bayesian Nash Equilibrium of such symmetric, binary action global games (Morris and Shin, 2006). One key condition is that the game features *global strategic complementarities* between players' actions, that is, that the payoff gain $v(\theta, n)$ is nondecreasing in n.

This property, however, is not satisfied in our model. First, the incentive to run is highest when the fraction of running creditors n reaches $\lambda < 1$: once the bank is already bankrupt at $t = 1$ additional outflows *reduce* the incentive to run since they lower the payoff of the debt contract at $t = 2$ (Goldstein and Pauzner, 2005). Liquid reserves introduce a second reason why the game does not feature global strategic complementarities: when the good state is sufficiently likely, additional outflows are initially "good news" for the remaining creditors because they reduce the debt claims on the assets at $t = 2$ without forcing the bank to liquidate the project below its fundamental value. Hence, the game does not feature *one-sided strategic complementarities* either, which would require that the function $n \mapsto v(\theta, n)$ is non-increasing when positively valued. However, this differential payoff function still features the single-crossing property used by Goldstein and Pauzner (2005), which guarantees existence of a monotonic pure strategy equilibrium (see the proof of Proposition 1 for details.)

[INSERT FIGURE 4 ABOUT HERE]

Figure 4 illustrates this situation graphically by plotting the payoff gain v as a function of n for different values of the state θ (we set F_2 equal to $\overline{F}_2$). For certain values of θ the function $v(\theta, \cdot)$ features two segments where it is *in*creasing in n.[14]

The key point is that—even though the payoffs of creditors feature these regions where they are strategic substitutes—these effects are too weak in our model to prevent the existence of an equilibrium in threshold strategies.

4 Endogenous Balance Sheet and General Equilibrium

Theorem 1 pins down the behaviour of creditors for any choices of the banks satisfying Conditions 1 and 2 and any realisation of the fundamental. This implies that banks can unambiguously determine the impact of their balance sheet decisions on their liquidity risk.

The two-dimensional balance sheet problem of a bank has the following economic interpretation. In a first step, we ask what combination of debt issuance and liquid reserves achieves a given level of liquidity risk at minimal cost. Clearly, it depends on the spreads of bank debt and the liquid asset, which are endogenous in our framework. In a second step, we ask what level of liquidity risk a bank should take on. The preferences of investors call for some amount of liquidity risk, but banks also need to make sure that inefficient liquidations are not too likely.

The remainder of the text makes formal these trade offs and their consequences in terms of banks' balance sheet decisions and asset pricing. To that end, we embed the coordination game described and solved in section 3 into a three-dates model: $t = 0, 1, 2$. There are two groups of agents, each of unit mass: banks and creditors. All agents are risk neutral with a discount rate of zero. But creditors value the liquidity benefits provided by bank debt and government bonds.

[14] A similar feature appears in Liu and Mello (2011).

4.1 Banks

4.1.1 Risky Project

Each bank owns a risky project as described in section 3. Projects are perfectly correlated across banks.[15]

4.1.2 Liquid Assets

There is a a positive supply B of risk-free government bonds. Bonds are issued at $t = 0$ at a price q_b, and pay one unit of the single good at $t = 2$. Proceeds from the issue at $t = 0$ are transferred in a lump-sum fashion to creditors. The bonds are retired at $t = 2$ using lump-sum taxes levied on creditors. The budget constraint of the government thus satisfies

$$T_0 = q_b B \tag{12}$$

$$T_2 = -B, \tag{13}$$

where T_0 denotes the date-0 transfer to creditors from the proceeds of the bond issuance, and T_2 the date-2 transfer (tax) from retiring the bonds.

At $t = 0$ and $t = 1$, banks and creditors can trade bonds on a centralised market, subject to no-short sale constraints. The liquid reserves β that were taken as given in section 3 are those held by a bank after the date-0 round of trading. The date-1 round allows a bank to obtain units of the single good in exchange of units of liquid reserves.

Let q_b (q_{b1}) denote the date-0 (date-1) government bond price.

[15] Shocks to the payoff of projects are macroeconomic shocks, rather than individual risks that might be diversified across loans or across banks. As pointed by Hellwig (2015): "Subdivision and diversification of individual specific risks are important in banking, but are nevertheless neglected here. In practice, the most serious banking problems tend to be tied to macroeconomic shocks." (p. 18)

4.1.3 Financing

At $t = 0$ each bank can raise funds by issuing short-term debt to creditors, and use the proceeds to finance the acquisition of liquid reserves and the distribution of an initial dividend

$$\delta q_d - \beta q_b \tag{14}$$

to its equity holders.

Recall that banks do not have access to any external source of funding in the interim period. Hence, if some creditors do not roll over their lending, then a bank must raise the resources necessary to repay withdrawing creditors by selling government bonds of the date-1 bond market (and, if its bond holdings are insufficient, through a liquidation of the risky project).

As we describe below, creditors get liquidity benefits from holding liquid assets. For simplicity, we assume that these liquidity benefits only depend on their holding of liquid assets at $t = 0$. Since creditors do not discount the future, this pins down the price at which banks can liquidate their government bond holdings at $t = 1$, namely $q_{b1} = 1$.

4.2 Creditors

A typical creditor has initial endowments in the single good ω_t in period t, $t = 0, 1, 2$. Creditors value short-term debt for their returns and, in addition, for the liquidity benefits they provide. We do not model these benefits explicitly but simply assume that they feel better if they hold "liquid" debt in the form of government bonds or bank debt. A typical creditor's preferences are given by the utility function

$$U = c_0 + \int_{\mathbb{R}} c_1(\theta)\,\mathrm{d}G(\theta) + \int_{\mathbb{R}} [\varphi(\theta)c_2(\theta, z) + (1-\varphi(\theta))c_2(\theta, 0)]\,\mathrm{d}G(\theta) + \underbrace{\Psi(\mathcal{L})}_{\text{liquidity benefits}}, \tag{15}$$

where

$$\mathcal{L} = b + \int_0^1 d(j) w(j) \, \mathrm{d}j. \tag{16}$$

Modeling liquidity benefits in reduced-form is a deliberate choice made for the purposes of tractability and interpretability. A liquidity benefits function Ψ can also be derived endogenously from the usual specification of a state-dependant creditors' utility whereby they potentially have a large valuation of date-1 consumption.[16]

The first three terms in (15) represent the creditor's utility from consuming the single good. c_0 denotes the level of consumption in period 0, $c_1(\theta)$ is the level of consumption in period 1 when the state realises at θ, and $c_2(\theta, Z)$ represents the level of consumption in period 2 conditional on the state θ and the payoff of the project Z.

The last term in (15) represents the liquidity benefits that the creditor gets from holding a quantity b of government bonds and $(d(j))_{j\in[0,1]}$ of bank debt. Liquidity benefits are a function of the creditor's liquid asset holdings at $t = 0$, where $\Psi(\cdot)$ is a continuously differentiable, strictly increasing concave function satisfying $\Psi(0) = 0$. We further assume that the creditors' demand for liquidity is eventually satiated: $\exists \overline{\mathcal{L}} \in \mathbb{R}_{++}$ such that $\Psi(\mathcal{L}) = \overline{\Psi}$ for all $\mathcal{L} \geq \overline{\mathcal{L}}$.

The creditor's government bond holdings b enter the function $\Psi(\cdot)$ with a weight of one, whereas the debt issued by bank j enters with a weight $w(j) < 1$:

$$w(j) = 1 - (1 - \lambda(j)) \int_{\mathbb{R}} \mathbf{1}_{(\lambda(j),1]}(n(j,\theta)) \, \mathrm{d}G(\theta). \tag{17}$$

$n(j, \theta)$ denotes the fraction of creditors withdrawing their funds from bank j in state θ. The term $\mathbf{1}_{(\lambda(j),1]}(n(j,\theta))$ thus indicates whether bank j fails due to a run in state θ given its liquidity ratio. Since $\lambda(j) < 1$, the

[16] One drawback of doing so is that the bank is subject to sure withdrawals which mechanically add a direct value to the liquidity buffer, as discussed in the Introduction. By contrast, here, there are no "impatient" agents and we can isolate the strategic value of reserves.

second term in (17) is strictly positive, and can be interpreted as a liquidity risk penalty that quantifies the disutility of (risky) bank debt being "less liquid" than (risk-free) government debt. The penalty is greater, the lower the bank's liquidity ratio and the higher the likelihood that it suffers a run.

The term $(1 - \lambda(j))$ ensures that the liquidity penalty vanishes in the limit $\lambda(j) \to 1$. This property is meaningful, because when λ approaches 1: (i) creditors are paid in full even in the event of an early default and (ii) only *efficient* runs occur in that case (see Proposition 2 below). Hence, we do not wish to attribute a positive liquidity penalty to a bank with $\lambda(j)$ approaching 1.

Each creditor maximises his expected utility, taking the actions of banks and other creditors as given. In the interim period $t = 1$, he faces two decisions: (1) whether to roll over his lending, and (2) whether to buy some of the government bonds liquidated by banks. In the initial period $t = 0$, he chooses his portfolio of bank debt and government bonds.

We assume that the period 1 endowment is sufficiently large to absorb the government bonds sold by banks, $\omega_1 > \beta$, and the endowment in period 2 always cover the lump-sum taxes, $\omega_2 > T_2$.

4.3 Information Structure

The information structure of our 3-dates model is as in Szkup (2013). The state θ is drawn from a Gaussian distribution $N\left(\mu, \tau^{-1}\right)$. The prior distribution is common knowledge. We denote its cdf by G.

θ is realised at the beginning of period 1, the interim period $t = 1$, but it is not publicly revealed before period 2. Instead, at $t = 1$, banks and creditors observe a public signal θ_p of precision τ_p,

$$\theta_p = \theta + \tau_p^{-1/2} \varepsilon_p,$$

with Gaussian noise $\varepsilon_p \sim N(0, 1)$ independent of θ. Once the public signal

θ_p is observed, banks and creditors share the common posterior belief

$$\theta|\theta_p \sim N(m(\theta_p), \tau_y^{-1}),$$

where $\tau_y^{-1} = (\tau + \tau_p)^{-1}$ and $m(\theta_p) = \tau_y^{-1}(\tau\mu + \tau_p\theta_p)$. Banks then choose the face value F_2.

Subsequently, each creditor i observes a private signal x_i of precision τ_x,

$$x_i = \theta + \tau_x^{-1/2}\varepsilon_i$$

where $(\varepsilon_i)_{i\in[0,1]}$ are Gaussian noise terms that are independent and identically distributed according to $N(0,1)$, and are independent of θ and ε_p. Then, given F_2 and the information conveyed by the signals θ_p and x_i, each creditor i decides whether to roll over his lending or withdraw his funds.

In the presence of both a public and a private signal, we need to recast the definition of vanishing noise:

Definition 1. *With public and private signals,* noise vanishes *when $\tau_x, \tau_p \to \infty$ with $\frac{\tau_x}{\tau_p} \to \infty$.*

That $\frac{\tau_x}{\tau_p} \to \infty$ ensures that the existence and uniqueness result of section 3 applies. That $\tau_p \to \infty$ allows, for all intents not related to the global game equilibrium, to talk about "the state of the economy": the observable θ_p is arbitrarily close to the unobservable fundamental θ. This simplifies the expressions of the payoffs and their interpretation. The approach of Szkup (2013), detailed in section 5.1, is useful to pin down a dynamically consistent value of F_2; but the schedule F_2 disappears *ex post* of the expressions of the payoffs because of rational expectations: what matter are the run states, *i.e.* the states where the bank cannot find an F_2 that prevents withdrawals. *Overall, the advantage of the chosen information structure is that it provides a tractable and interpretable framework which is fully consistent game-theoretically.*

4.4 Timeline

[INSERT FIGURE 2 ABOUT HERE.]

Figure 2 summarises the timeline of the model. At $t = 0$, each bank raises funds by issuing short-term debt. These funds are used to purchase government bonds and to distribute an initial dividend to equity owners.

At the beginning of the interim period $t = 1$, the state θ realises. Banks and creditors observe a noisy public signal θ_p. Then, given the realization θ_p, banks offer a new debt contract with face value F_2, which matures at $t = 2$. In addition, before making their rollover decision, but after banks set F_2, creditors observes a private signals $(x_i)_{i\in[0,1]}$. Given F_2 and the information conveyed by the signals θ_p and x_i, creditor i decides whether to accept the new debt contract or withdraw his funds.

A bank satisfies early withdrawals by selling government bonds on a centralised market and by rescaling the risky project. If early withdrawals exceed the proceeds of the bond sale and the project's liquidation, the bank defaults at $t = 1$. Otherwise, the bank continues to operate until the final period $t = 2$.

At $t = 2$ the return of the project realises. If the payoff collected on the remaining assets is sufficient to meet the banks' debt obligations, creditors are paid F_2 and equity holders receive the residual. Otherwise, banks default and the assets are divided equally among creditors.

Equilibrium. Our equilibrium concept is standard; for conciseness we report it in Appendix A.1.

Remark. In section 3, we claimed that Conditions 1 and 2, which are necessary to obtain the threshold global games equilibrium, are automatically satisified once we endogenise the banks' choices. Indeed, when Condition 1 is violated, withdrawing early is a dominant actions for creditors and equity holders get nothing. When Condition 2 is violated, withdrawal decisions become strategic *substitutes* and one can exhibit a mixed equilibrium where *not enough* agents withdraw, i.e. the project is not fully liquidated in some states $\theta < \varphi^{-1}(p^E)$. However, since their debt can be sold at a premium, banks have no incentive to reduce their leverage so much that they create inefficient continuations, as this reduces both bank value and the benefits of debt issuance.

5 Results

We are now in a position to present the implications of Theorem 1 and our model. First, we use the approach of Szkup (2013) to pin down the choice of F_2 following the release of the public signal. This allows us to obtain a "pullback" of the ex post run threshold x^* into an ex ante run threshold x^R. When the fundamental realises below x^R, the bank cannot avoid liquidation, whatever the choice of F_2. This happens with probability $G(x^R)$, which is the measure of liquidity risk that the bank must take into account. We obtain a neat characterisation of x^R as a function of the liquidity and reserve ratios λ and ρ only. Importantly, we can compute explicitly the *coordination failure multiplier* which captures what fraction of runs are inefficient—purely due to the coordination failure. We show that the bank has the option to set this multiplier to one by selecting $\lambda \to 1$, but typically does not want to do so. We then go to the general equilibrium and asset pricing implications.

5.1 Ex-ante Probability of Runs

Under vanishing noise, the fraction of withdrawing creditors at date 1 is (arbitrarily close to) zero or one. This implies that equity is wiped out in case of a run, which occurs when $\theta < x^*(F_2)$. The bank therefore tries to avoid this outcome as long as it is able to by promising a higher face value F_2 (thereby decreasing x^*).

Moreover, when the public signal noise vanishes then the bank's uncertainty about the true state θ becomes vanishly small. This means that *even* *if it does not observe* θ, the bank can implement $x^*(F_2) > \theta$ with high certainty by setting the face value F_2 arbitrarily close to (but higher than) the solution to

$$\theta = x^*(F_2). \tag{18}$$

Since x^* is decreasing in F_2 for $F_2 < \overline{F}_2$, this equation has a unique solution for $\theta > x^*(\overline{F}_2)$. We denote it $F_2^*(\theta)$. The bank has no incentive to pick a face value above and bounded away from F_2^* as this increases financing costs without changing the run likelihood (zero).

For $\theta < x^*(\overline{F}_2)$, the proportion of running creditors converges to one in the limit of vanishing noise. The assets of the bank are so impaired that a run cannot be prevented, even if the bank transfers all the assets to creditors and equity holders are wiped out. Hence, a run occurs in this case, and no run occurs otherwise.

We are thus in a position to define and compute the ex-ante likelihood that a bank fails due to a run. (Technical details are given in Appendix A.2.)

Proposition 1. *(Ex-ante likelihood of runs)*

(i) When noise vanishes, the ex-ante likelihood of runs is equal to $G(x^R)$*, where* $x^R \equiv x^*(\overline{F}_2)$ *and* $G(\cdot)$ *is the cdf of the fundamental* θ*.*

(ii) The project's success probability at the run threshold, $p^R \equiv \varphi(x^R)$*, is equal to* $p^E \times \Gamma$*, where*

$$\Gamma(\lambda, \rho) = \frac{\pi_1(\lambda) - \pi_2(\rho)}{\pi_2(\lambda) - \pi_2(\rho)} \tag{19}$$

and

$$\pi_1(u) = u - u\log(u) \tag{20}$$

$$\pi_2(u) = 1 - \pi_1(1-u). \tag{21}$$

The success probability at the run threshold can be decomposed into the product of the efficient liquidation threshold, $p^E \equiv \ell/z$, and a multiplier

$\Gamma > 1$ that captures the effect of the coordination failure among creditors.[17] It is noteworthy that this multiplier only depends on the two ratios, λ and ρ, and a single function, π_1 (see Figure 5).

[INSERT FIGURE 5 ABOUT HERE]

Proposition 2. *(Coordination failure multiplier)*

(i) The coordination failure among creditors causes them to run below a threshold that is higher than the efficient level:

$$\Gamma > 1 + \frac{1-\lambda}{\lambda - \rho}.$$

(ii) In the limit $\lambda \nearrow 1$, only efficient runs occur.

$$\lim_{\lambda \nearrow 1} \Gamma = 1.$$

For any $\rho < 1$, we define by continuity: $\Gamma(1, \rho) \equiv 1$.

For $\varphi(\theta) \in (p^E, p^R)$ the coordination failure among creditors compels them to run on the bank and force a liquidation of the project below its fundamental value, which is equal to $\varphi(\theta)z$. The spread $\varphi(\theta)z - \ell$ is the loss caused by the forced liquidation of the project.

Banks can reduce the risk of forced liquidations by selecting a higher liquidity ratio. In fact, as shown by the second result of Proposition 2, banks can completely eliminate the risk of inefficient liquidations by issuing no more short-term debt than the amount of funds that can be realised in the short-run. We will see, however, that such a "safe policy" is not optimal for banks as long as creditors get liquidity benefits at the margin from holding liquid assets.

[17] The definition of p^R in (19) holds for (λ, ρ) such that $p^R < \varphi(\bar{\theta}) < 1$ from the assumption of the existence of an upper dominance region.

The following result confirms that our model is well-behaved and captures properly the trade offs faced by the banks in determining their time-0 balance sheet:

Proposition 3. *(Comparative statics) The ex-ante probability of runs,* $G(x^R)$*, is*

(i) increasing in the quantity of debt δ*;*

(ii) decreasing in the liquidity buffer β*;*

(iii) decreasing in the expected value, μ*, and increasing (decreasing) in the volatility,* $\tau^{-1/2}$*, of the fundamental as long as the likelihood of runs is below (above)* $1/2$.

By adjusting its capital structure and the size of its liquid reserves, the bank can control ex ante in which states of the world it will fail due to a run; the more debt the bank takes on and the less liquid assets it aquires at $t = 0$, the larger is the rollover risk it faces at $t = 1$. In section 5.3 we analyse how a bank manages its rollover risk, taking into account the effect of its choices on the risk profile and pricing of its claims.

5.2 Portfolio and Financing Decisions at $t = 0$

Since agents within each class are identical ex-ante, we focus on symmetric equilibria in which all banks select the same capital structure and liquidity buffer, and all creditors choose identical portfolios. All derivations are made in the case of vanishing noise as described in Definition 1.

5.2.1 Creditors

At $t = 0$ a typical creditor chooses an investment level for the government bond as well as for the debt contracts issued by banks so as to maximise his expected utility. In so doing, he must respect the period 0 budget constraint

$$c_0 + q_b b + \int_0^1 q_d(j) d(j) \, \mathrm{d}j = \omega_0 + T_0 \tag{22}$$

and the state-dependent budget constraints

$$c_1(\theta) = \omega_1 - q_{b1}(b_1 - b) + \int_0^1 [D_1(n(j,\theta))d(j) \times \mathbf{1}_{(-\infty, x^R(j))}(\theta)] \, \mathrm{d}j \qquad (23)$$

$$c_2(\theta, Z) = \omega_2 + T_2 + b_1 + \int_0^1 [D_2(n(j,\theta), F_2^*(j,\theta); Z)d(j) \times \mathbf{1}_{(x^R(j), \infty)}(\theta)] \, \mathrm{d}j \qquad (24)$$

in period 1 and 2, where $x^R(j)$ denotes the equilibrium run threshold played bank j's creditors, $n(j,\theta)$ is the associated mass of creditors withdrawing their funds,

$$n(j,\theta) = \mathbf{1}_{(-\infty, x^R(j))}(\theta), \qquad (25)$$

and $F_2^*(j,\theta)$ is the date-2 face value set by bank j (as defined by (18)).

The constraint for $t = 0$ says that consumption and portfolio investment need to be financed by the initial endowment and the transfer from the government. The constraint for $t = 1$ says that consumption and additional purchases of government bonds need to be financed by the date 1 endowment and the funds obtained by withdrawing funds at $t = 1$. The constraint for the period $t = 2$ equates the creditor's consumption to the sum of his date 2 endowment and the portfolio payoff, net of taxes.

Substituting the constraints (22)–(24) into the creditor's utility (15) and differentiating with respect to the creditor's holdings of government bonds b and of bank j's debt $d(j)$, we obtain the first-order conditions:

$$q_b \geq 1 + \Psi'(\mathcal{L}) \qquad (26)$$

$$q_d(j) \geq \overbrace{\int_{-\infty}^{x^R(j)} D_1(n=1) \, \mathrm{d}G(\theta)}^{\text{payoff of withdrawing}} + \overbrace{\int_{x^R(j)}^{\infty} \mathbb{E}_\theta \left[D_2(n=0, F_2^*(j,\theta); Z)\right] \mathrm{d}G(\theta)}^{\text{payoff of rolling over}}$$
$$+ \underbrace{\Psi'(\mathcal{L})[1 - (1-\lambda(j))G(x^R(j))]}_{\text{marginal liquidity benefits}}, \qquad (27)$$

where the expectation inside the second integral is taken with respect to the realization of Z. Each condition holds as a strict equality when the creditor holds a strictly positive quantity of the respective asset.

5.2.2 Banks

We now solve for the ex-ante choices of an individual bank, taking as given the behavior of all other banks. At $t = 0$, a bank chooses its capital structure and the size of its liquid reserves to maximise shareholders' value, taking prices as given:

$$\begin{aligned}\max_{\beta,\delta\geq 0} V &= q_d\delta - q_b\beta + \int_{x^R}^{\infty} \varphi(\theta)E(n=0, F_2^*(\theta); z)\,\mathrm{d}G(\theta) \\ &= q_d\delta - q_b\beta + \int_{x^R}^{\infty} \varphi(\theta)\max\left\{0, z+\beta-\delta F_2^*(\theta)\right\}\mathrm{d}G(\theta) \quad (28)\end{aligned}$$

where the second line follows from (3) (evaluated at $n = 0$). The first term represents the initial dividend paid by the bank to equity holders. The second term is the expected payoff to equity holders at $t = 2$.

Price-taking has a clear meaning when considering the behavior of banks on the market for government bonds. But when an individual bank decides to issue a quantity δ of debt or to hold an amount β of government bonds, this decision impacts the payoff profile of its debt and hence its market value $q_d(\beta, \delta)$.

Suppose an individual bank considers choosing the plan (β, δ), which might differ from the *equilibrium* plan, say $(\bar{\beta}, \bar{\delta})$. Since each bank is infinitesimally small, its debt makes up a negligeable part of the portfolio held by creditors; hence, the choice of (β, δ) does not impact $\Psi'(\mathcal{L})$, which is determined by $(\bar{\beta}, \bar{\delta})$. Thus, given a conjecture on $\mathcal{L}$—which must turn out to be correct in equilibrium—a bank knows the creditors' marginal valuation of liquidity. From this, we obtain:

Lemma 3. *The market value of a bank's debt given the plan (β, δ), is*

$$\begin{aligned} q_d(\beta,\delta) \quad &= \int_{-\infty}^{x^R} D_1(n=1)\,\mathrm{d}G(\theta) + \int_{x^R}^{\infty} \mathbb{E}_\theta[D_2(n=0, F_2^*(\theta); z)]\,\mathrm{d}G(\theta) \\ &\quad +\Psi'(\mathcal{L})[1-(1-\lambda)G(x^R)] \end{aligned} \tag{29}$$

where

$$\mathcal{L} = B - \bar{\beta} + \bar{\delta}\Bigg[1 - \Bigg(1 - \underbrace{\frac{\ell + \bar{\beta}}{\bar{\delta}}}_{\substack{\textit{system-wide}\\ \textit{liquidity ratio}}}\Bigg) \underbrace{G\left(x^*\left(\frac{z}{\bar{\delta}-\bar{\beta}}\right)\right)}_{\textit{system-wide run likelihood}}\Bigg]. \tag{30}$$

$x^R(\beta, \delta)$ is bank-specific while $\mathcal{L}$ results from the choices of all banks.

The banks' debt market value consists of three components. The first two represent the date-0 expected payoff achieved by creditors by following the strategy of rolling over the debt when $x_i \geq x^R$, and withdrawing their funds in the interim period otherwise. The third component represents the bank's conjecture about the liquidity benefits priced into the its debt. We dub this term the *bank debt liquidity premium S_d*.

We refer to the spread $S_b \equiv q_b - 1 \geq \Psi'(\mathcal{L})$ as the *government bond liquidity premium.* S_b is bounded below by the marginal liquidity benefits. As we explain below, when the government bond supply is scarce, the previous inequality is strict.

Using these definitions, the problem of the bank at $t = 0$ can be rewritten as follows

Lemma 4. *Let $H(p) \equiv G(\varphi^{-1}(p))$ denote the CDF of the project's success probability $p = \varphi(\theta)$. The bank's initial choice of capital structure and liquid* *reserves is a solution to*

$$\max_{\beta,\delta\geq 0} V = \underbrace{\ell H(p^E) + \int_{p^E}^{1} pz\,\mathrm{d}H(p)}_{\text{economic value of project}} - \underbrace{\int_{p^E}^{p^R} (pz-\ell)\,\mathrm{d}H(p)}_{\text{cost of inefficient liquidations}} + \underbrace{S_d\delta - S_b\beta}_{\text{net liquidity benefits}} \tag{31}$$

where

$$S_b = q_b - 1 \tag{32}$$

$$S_d = \Psi'(\mathcal{L})[1-(1-\lambda)G(x^R)], \tag{33}$$

represent the government bond and bank debt liquidity premium, respectively, and where $\mathcal{L}$ is the (risk-adjusted) equilibrium quantity of liquid assets held by creditors as defined by (30).

The first two terms give the economic value of the project if the efficient liquidation policy is implemented. The third term represents the cost of taking on liquidity risk. For $\lambda < 1$ the success probability at the run threshold is higher than under the efficient liquidation policy: $p^R > p^E$. For $p \in (p^E, p^R)$ the bank is forced to liquidate the project below its expected payoff, incurring a loss of $pz - \ell$. The last two terms represent the *net* liquidity benefits captured by the bank; that is, the liquidity premium priced into its debt, net of the liquidity premium priced into its government bond holdings.

Taking on more debt allows the bank to better capture the premium paid for liquid claims, but it raises the risk of inefficient liquidations. Holding more liquid reserves reduces the risk of inefficient liquidations and, by improving the risk profile of the bank's debt, increases the liquidity premium paid by creditors. These benefits have to be balanced against the cost of holding liquid reserves. We now analyse these trade-offs in details.

5.3 Optimal Choice of an Individual Bank

One way to interpret the bank's problem (31) is through the following two-step approach. In the first step, the bank chooses the size of its liquid reserves so as to maximise the initial equity value under a *risk constraint*:

$$\max_{\beta,\delta\geq 0} V \qquad \text{subject to} \qquad p^R = p^T$$

In the second step, the bank decides how much liquidity risk it should tolerate. This approach has a graphical interpretation. Define $\hat{\delta}(\beta, p^T)$ to be the p^T-level *iso-risk curve*; for a given β this function returns the value of δ that achieves an ex-ante likelihood of runs equal to $H(p^T)$. It is defined implicitly by the equation $p^T = p^R(\beta, \delta)$. At the first step, the bank moves along a given iso-risk curve and chooses the optimal position (β, δ) *on* the curve. At the second step, the bank chooses the optimal position p^T *of* the curve. Figure 6d displays the iso-risk curves in the (λ, ρ) plane (colored lines) and how the optimal choice of the bank varies with the risk constraint $p^R = p^T$ (solid black line).

5.3.1 Choice of Liquid Reserves

Suppose that the bank targets a run probability equal to $H(p^T)$. The slope of the iso-risk curve quantifies how much additional debt the bank can take on following a marginal increase in its liquid reserves without changing its liquidity risk:

$$\frac{\partial\hat{\delta}}{\partial\beta} = -\frac{\partial p^R/\partial\beta}{\partial p^R/\partial\delta}. \tag{34}$$

Proposition 4. *(i) A marginal increase in liquid reserves has a stronger impact on the run probability than the corresponding marginal reduction in debt:*

$$\frac{\partial\hat{\delta}}{\partial\beta} > 1. \tag{35}$$

(ii) To completely avoid the risk of inefficient liquidations, a marginal increase in debt must be accompanied by an equally-sized increase in liquid reserves.

$$\lim_{p^T \searrow p^E} \frac{\partial \hat{\delta}}{\partial \beta} = 1. \tag{36}$$

Result (i) of Proposition 4 reflects that the fact that liquid reserves have a *strategic value*: in the absence of a coordination failure problem, one should expect liquid reserves to be simply "negative debt", i.e. one-to-one moves in δ and β keep the bank failure probability constant. Instead, an increase in liquid reserves allows the bank to expand its short-term liability side more than one-to-one. Recall that the run threshold is determined by the indifference condition of the pivotal agent: she computes expected payoffs of withdrawing and rolling over seeing n as uniform over $[0, 1]$. Under this distribution, there are many n-states in which an additional unit of liquid reserves is particularly useful, as it allows to economise on liquidation of the risky asset at the discount ℓ.[18] Hence, an increase in β more than offsets the adverse impact of an increase in δ in terms of run likelihood. Consistent with this explanation, Result (ii) of Proposition 4 confirms that the effect vanishes precisely when the strategic problem vanishes: as λ approaches one, only efficient runs occur and the bank has effectively annihilated the coordination failure.

Hence, a bank that has decided to take on liquidity risk acts as a *liquidity multiplier*: instead of having investors buying directly a unit of government debt, the bank can buy this unit and then issue to investors more than one unit of private debt, without affecting the bank failure likelihood. A similar balance sheet operation cannot be performed by a bank that wishes or is constrained to be fully liquid.

Based on these comments, we dub $\partial \hat{\delta} / \partial \beta$ a *marginal liquidity multiplier.*

[18]In equilibrium, these states do not materialise, because $n \in \{0; 1\}$ in the limit of vanishing noise, but they are key to the determination of the bank failure states. In fact, in those states where the bank do not fail, liquid reserves remain intact at date 1 on the equilibrium path.

We now characterise the bank's demand for government bonds, when it is subject to the risk constraint $p^R(\delta, \beta) = p^T$. Since $p^R(\hat{\delta}(\beta, p^T), \beta) = p^T$ for any choice β, varying the size of the liquid reserves only impacts the last two terms in (31). Thus, at the first step the bank chooses its bond holdings so as to maximise the net liquidity benefits,

$$\max_{\beta} \; S_d\big(\hat{\delta}(\beta, p^T), \beta\big)\hat{\delta}(\beta, p^T) - S_b \beta. \tag{37}$$

Consider first the case where liquidity is abundant: $\mathcal{L} \geq \overline{\mathcal{L}}$ so that $\Psi'(\mathcal{L}) = 0$. Then, (33) implies that $S_d = 0$. Two cases are possible. If $S_b > 0$, the maximum in (37) obtains at $\beta = 0$. If $S_b = 0$, the bank's demand is indeterminate: $\beta \in [0, \infty)$.

Next, consider the case where liquidity is scarce: $\mathcal{L} < \overline{\mathcal{L}}$ so that $\Psi'(\mathcal{L}) > 0$. Differentiating (37) with respect to β, we obtain

$$S_b = \Psi'(\mathcal{L})\Bigg[\underbrace{\big(1 - H(p^T)(1-\lambda)\big)\frac{\partial \hat{\delta}}{\partial \beta}}_{\substack{\text{liquidity premium} \\ \text{earned on new debt}}} + \underbrace{H(p^T)\left(1 - \lambda \frac{\partial \hat{\delta}}{\partial \beta}\right)}_{\substack{\text{lower risk penalty} \\ \text{on existing debt}}}\Bigg]. \tag{38}$$

The first-order condition (38) equates the cost and benefit associated to a small move $(d\beta, \partial\hat{\delta}/\partial\beta \times d\beta)$ along the iso-risk curve. The marginal cost is given by the government bond liquidity premium. The marginal benefit consists of two components. First, the bank collects the bank debt liquidity premium on the incremental debt. Second, because this transaction increases the bank's liquidity ratio, it has the added benefit of reducing the risk penalty on the existing debt.[19] An increase in the spread S_b has the expected impact of reducing $\beta(p^T)$: banks hold fewer liquid reserves when the return of liquid

[19]The risk penalty (per unit of debt) is equal to: $H(p^R)(1-\lambda)$. The change in the risk penalty is thus:

$$-H(p^R)\left(\frac{\partial \lambda}{\partial \beta} + \frac{\partial \lambda}{\partial \delta}\frac{\partial \hat{\delta}}{\partial \beta}\right) = -H(p^R)\frac{1}{\delta}\left(1 - \lambda\frac{\partial \hat{\delta}}{\partial \beta}\right).$$

assets decreases.

The marginal liquidity multiplier, $\partial\hat{\delta}/\partial\beta$, is decreasing in β:[20] a bank holding large liquid reserves (but that has issued a lot of debt) must expand its liquid reserves when it incrementally increases its outstanding debt, compared to a bank with few liquid reserves and little debt. Hence, the *maximal* premium paid by banks for government bonds can be derived by evaluating the RHS of (38) at $\beta = 0$. It is equal to:

$$\overline{S}_b(p^T) \equiv \Psi'(\mathcal{L})\left[1 + (1 - H(p^T))\left(\frac{\partial}{\partial\beta}\hat{\delta}(0, p^T) - 1\right)\right]. \tag{39}$$

Therefore, when liquidity is scarce, two cases are possible. If $S_b \geq \overline{S}_b(p^T)$, then the return on liquid assets is so low that the bank does not want to hold any liquid reserve. Next, if $\Psi'(\mathcal{L}) < S_b < \overline{S}_b(p^T)$, then the bank's bond demand is given by the unique solution to (38). Note that as $S_b \searrow \Psi'(\mathcal{L})$, its demand grows unbounded. To see why, rewrite equation (38) as:

$$S_b = \underbrace{\Psi'(\mathcal{L})}_{\text{base spread}} + \underbrace{\Psi'(\mathcal{L})(1 - H(p^T))\left(\partial\hat{\delta}/\partial\beta - 1\right)}_{\text{incremental spread}}. \tag{40}$$

Since $\partial\hat{\delta}/\partial\beta > 1$ the second term is strictly positive when $\Psi'(\mathcal{L}) > 0$. The bank's ability to use a marginal unit invested in government bonds to back an additional $\partial\hat{\delta}/\partial\beta > 1$ units of debt means that it can get a higher return out of a marginal investment in government bonds than simply the marginal liquidity benefit $\Psi'(\mathcal{L})$.

Consider now the impact of a decline in S_b. Because it is less costly to hold government bonds, the bank can capture higher net liquidity benefits through the acquisition of additional bonds and the simultaneous issuance of new debt—with the latter being equal to some factor > 1 of the newly-

[20]Since $\hat{\delta}$ describes level curves of Γ, $\hat{\delta}$ being concave is equivalent to Γ being quasi-convex. A convenient necessary and sufficient condition is that the determinant of the bordered Hessian matrix $\mathcal{BH}_\Gamma$ of Γ is everywhere negative. Producing explicitly the parameter-free surface $(\det(\mathcal{BH}_\Gamma))$ reveals this holds true.

acquired bonds. It is optimal for the bank to expand its balance sheet in this way until the total liquidity premium priced into the newly-issued debt no longer covers the government bond liquidity premium. When S_b approaches $\Psi'(\mathcal{L})$ the cutoff level above which it is no longer profitable for the bank to expand its balance sheet becomes arbitrarily large.

The following Lemma summarises these results.

Lemma 5. *Suppose the bank targets an ex-ante likelihood of runs equal to $H(p^T)$.*

(i) (Abundant liquidity) If $\mathcal{L} \geq \overline{\mathcal{L}}$, then the bank's bond demand is

$$\beta(p^T) = \begin{cases} 0 & \textit{if} \quad S_b > 0 \\ [0, \infty) & \textit{if} \quad S_b = 0 \end{cases} \tag{41}$$

(ii) (Scarce liquidity) If $\mathcal{L} < \overline{\mathcal{L}}$, then the bank's bond demand is

$$\beta(p^T) = \begin{cases} 0 & \textit{if} \quad S_b \geq \overline{S}_b(p^T) \\ \textit{unique solution to } (38) & \textit{if} \quad \Psi'(\mathcal{L}) < S_b < \overline{S}_b(p^T) \\ \infty & \textit{if} \quad S_b = \Psi'(\mathcal{L}) \end{cases}, \tag{42}$$

where $\mathcal{L}$ represents is the quantity of liquid assets held by creditors as defined by (30) and $\overline{S}_b(p^T)$ denotes the maximal government liquidity premium as defined by (39).

5.3.2 Optimal Liquidity Risk-Taking

The previous section derived the bank's choice of liquid reserves $\beta(p^T)$ —and the associated level of debt issuance $\hat{\delta}\big(\beta(p^T), p^T\big)$— for a given level of liquidity risk p^T. We now analyse how much liquidity risk banks tolerate in equilibrium. As before, we need to differentiate between two cases, depending on whether public liquidity is abundant or scarce.

Consider the case where liquidity is abundant: $\mathcal{L} \geq \overline{\mathcal{L}}$. In the absence of liquidity benefits priced into bank debt, the problem of the bank boils down to minimising the losses due to inefficient liquidations. Hence, the bank targets the ex-ante likelihood of runs $p^T = p^E$.

Next, consider the case where liquidity is scarce: $\mathcal{L} < \overline{\mathcal{L}}$ so that $\Psi'(\mathcal{L}) > 0$. Substituting $\beta = \beta(p^T)$ and $\delta = \hat{\delta}(\beta(p^T), p^T)$ in the bank's problem (31) and differentiating with respect to p^T, we obtain the following optimality condition:

$$\underbrace{(p^T z - \ell)h(p^T)}_{\substack{\text{increase in the cost of}\\ \text{inefficient liquidations}}} = \Psi'(\mathcal{L})\Bigg(\underbrace{\Big[1 - H(p^T)(1-\lambda)\Big]\frac{\partial\hat{\delta}}{\partial p^T}}_{\substack{\text{liquidity premium}\\ \text{earned on new debt}}} - \underbrace{\Big[(1-\lambda)h(p^T)\delta + H(p^T)\lambda\frac{\partial\hat{\delta}}{\partial p^T}\Big]}_{\substack{\text{higher risk penalty}\\ \text{on existing debt}}}\Bigg), \tag{43}$$

where $\beta = \beta(p^T)$, $\delta = \hat{\delta}(\beta, p^T)$, $\lambda = (\ell + \beta)/\delta$, $\partial\hat{\delta}/\partial p^T$ is evaluated at $(\beta(p^T), p^T)$, and $h(\cdot)$ denotes the pdf of p.

The level of liquidity risk chosen by the bank balances the marginal increase in the cost due to inefficient liquidations to liquidity premium collected on the incremental debt, net of the decline in the bank debt liquidity premium. The latter is due to the higher probability that the bank defaults due to a run, as well as the higher losses of creditors in default.

As $p^T \searrow p^E$ (which requires that $\lambda \nearrow 1$), the LHS of (43) approaches zero, while the RHS of (43) approaches $\Psi'(\mathcal{L})(1 - H(p^T))\partial\hat{\delta}/\partial p^T \geq 0$. Thus, we reach the following conclusion.

Lemma 6. *A bank chooses to take on liquidity risk if, and only if, short-term debt provides liquidity benefits to creditors at the margin:* $\Psi'(\mathcal{L}) > 0$.

[INSERT FIGURE 6 ABOUT HERE]

Figure 6 provides a convenient graphical representation of the optimal ex ante choices of a bank. Panel 6a shows how bank equity value varies with the liquidity ratio λ for a given reserve ratio ρ. Panel 6b confirms that the optimal choice (red dot) involves a positive probability of inefficient runs. In Panel 6d, colored lines are the iso-risk curves in the (λ, ρ) plane, and the solid black line illustrates how the bank's optimal choice varies as the risk constraint $p^R = p^T$ changes, i.e. the optimal choice for each iso-risk curve. When λ is large, liquidity risk is low and the bank is not ready to bid for reserves ($\rho = 0$): this corresponds to the region $\overline{S}_b(p^T) < S_b$ in Panel 6c. When λ is intermediate, the bank finds it optimal to participate in the bond market ($\rho > 0$): $\overline{S}_b(p^T) > S_b$. When λ is too low, bank equity is wiped out in any case, so there is no point in buying reserves.

5.4 General Equilibrium

Recall that in equilibrium creditors enjoy the liquidity benefits $\Psi(\mathcal{L})$, where

$$\mathcal{L} = B - \bar{\beta} + \bar{\delta}\left[1 - \left(1 - \frac{\ell + \bar{\beta}}{\bar{\delta}}\right) G\left(x^*\left(\frac{z}{\bar{\delta} - \bar{\beta}}\right)\right)\right], \tag{30}$$

where $(\bar{\beta}, \bar{\delta})$ denotes the equilibrium choice of liquid reserves and capital structure.

Consider the case $\mathcal{L} \geq \overline{\mathcal{L}}$. Since banks are not remunerated for taking liquidity risk ($S_d = 0$), they issue no more debt than the amount of funds that can be realised in the short-run. Setting $\bar{\delta} = \ell + \bar{\beta}$ into (30), creditors do not value liquidity at the margin in equilibrium if $B \geq \overline{\mathcal{L}} - \ell$.

Next, consider the case $B < \overline{\mathcal{L}} - \ell$. Then, banks are incentivised to take liquidity risk in order to capture the premium paid by creditors for liquid assets. Since banks can earn a levered liquidity premium—by issuing debt equal to a multiple of their liquid reserves— they are able to bid up the price of government bonds above the valuation of creditors. In turn, competition among banks for the limited supply of government bonds compels them to do so. As a result, banks hold the entire stock of government bonds, $\beta(p^T) = B$. Government bonds trade at $q_b = 1 + S_b$ in the initial period $t = 0$, where S_b follows from the first-order condition of the bank's portfolio choice, equation (38). Finally, banks issue $\hat{\delta}(B, p^T) > \ell + B$ of debt, where the success probability of the run threshold, $p^T > p^E$, satisfies (43).

The next Proposition summarises these results.

Proposition 5.

(i) (Abundant liquidity) If the government bond supply satisfies $B \geq \overline{\mathcal{L}} - \ell$, then:

(a) Banks hold $\bar{\beta} \in [0, B]$ bonds, with the remainder being absorbed by creditors;

(b) Banks are not exposed to liquidity risk: $\bar{\delta} = \ell + \bar{\beta}$ and $p^R = p^E$;

(c) The bank debt and government bond liquidity premia are equal to zero.

(ii) (Scarce liquidity) If the government bond supply satisfies $B < \overline{\mathcal{L}} - \ell$, then:

(a) Banks hold the entire stock of government bonds: $\bar{\beta} = B$.

(b) Banks take on liquidity risk, $p^R > p^E$, where p^R satisfies (43) with $\bar{\delta} = \hat{\delta}(B, p^R)$ and $\partial\hat{\delta}/\partial p^T$ being evaluated at (B, p^R);

(c) The bank debt and government bond liquidity premia are strictly positive and given by (33) and (38), respectively.

5.5 Liquidity Premia

When liquidity is scarce, banks act as liquidity multipliers by issuing debt equal to a multiple of their liquid reserves. This holds true when considering the *aggregate* as well as the *marginal* debt-to-reserve ratio; $1/\lambda$ and $\partial\hat{\delta}/\partial\beta$ are both greater than one. But it is the latter property that enables banks to bid up the price of risk-free bonds above the creditors' valuation, and makes banks the *natural buyers* of risk-free bonds in this economy.

Because banks are the marginal bond investors, the first-order condition of their portfolio choice becomes a pricing equation for government bonds. As mentioned earlier, this implies that the government bond liquidity premium can be decomposed into a base spread, which is equal to the marginal liquidity benefits to creditors, and an incremental spread, which arises from the ability of banks to lever the premium paid for liquid claims, recall (40).

Consequently, the government bond and bank debt liquidity premia satisfy the ranking:

$$S_b \geq \Psi'(\mathcal{L}) \geq S_d \tag{44}$$

The spread $S_b - \Psi'(\mathcal{L})$ reflects the ability of banks to act as liquidity multipliers. The spread $\Psi'(\mathcal{L}) - S_d$ reflects the penalty for the bank's liquidity risk. Note that the ranking (44) confirms that *unit by unit*, the bank earns less on private debt issuance than it spends on public liquidity. But of course, due to the liquidity multiplier mechanism, the bank can issue more than one unit of private debt for each unit of public liquidity.

The spread $S_b - \Psi'(\mathcal{L})$ also provides the answer to the question of *how* an individual bank takes on liquidity risk. By this we mean the following. Consider an equilibrium allocation. Now suppose one bank, say bank j, is initially not allowed to take any liquidity risk. Next, remove this risk constraint. How does bank j adjust its funding and portfolio mix? Does it economise on liquid reserves, issue more debt, or both?

When the risk constraint $p^T = p^E$ is imposed on bank j, it can increase its debt by at most one unit for each additional unit of bond holdings. Since bank j is unable to earn a levered liquidity premium, it cannot cover the costs of holding liquid reserves. As a result, it is optimal for bank j to issue little debt, and not hold any liquid reserves: $\beta = 0$ and $\delta = \ell$, or, equivalently, $\lambda = 1$ and $\rho = 0$.

As the risk constraint is gradually relaxed, the quantity of debt issued by bank j increases and the marginal liquidity multiplier, $\partial\hat{\delta}/\partial\beta$, raises above one, eventually reaching the point where the bank is willing and able to pay the spread S_b to accumulate liquid reserves. Interestingly, this means that banks holding larger liquid reserves are actually those more exposed to liquidity risk, since only banks with such an exposure are willing to bid aggressively for liquid assets. This is well represented in Figures 6c and 6d: the large values of λ, that involve low liquidity risk, correspond to the region $\overline{S}_b(p^T) < S_b$ in Figure 6c, where the bank is not ready to bid for liquid reserves. Correspondingly, $\rho = 0$ at the right of Figure 6d.[21]

Remark (Welfare). As discussed above, when public liquidity is scarce ($B < \overline{\mathcal{L}} - \ell$), banks take on liquidity risk. But this choice fully internalises its impact on the incentives of creditors to roll over. As a result, it is not too difficult to show that the equilibrium is constrained-optimal, in the sense that for a given $\mathcal{L}$, a social planner whose only tools are a liquidity ratio requirement and reserve ratio requirement would not seek to alter the banks' private choices. Hence, the presence of a coordination failure is not sufficient *per se* to justify regulations aiming at curtailing the short-term leverage of banks, and/or increasing their liquid reserves. Rather, it is the interaction of the coordination failure with various distortions that affect the banks' choice of funding and portfolio mix which can justify a policy intervention.

There is, however, clear scope to improve welfare by increasing the supply of public liquidity, assuming this is feasible. The policy intervention can also take place ex post, in the form of the form of emergency liquidity assistance by a lender of last resort. We discuss this in our concluding remarks.

6 Conclusion

This paper suggests a solution to the negative carry trade puzzle that relies crucially on an endogenous *marginal liquidity multiplier*. This multiplier is key to understand how banks decide between controlling their debt issuance and hoarding a large liquidity buffer, two strategies that are substitute in managing liquidity risk. When the government is not willing or able to supply a large quantity of risk-free bonds, the economy features liquidity risk, the marginal liquidity multiplier is above one, and return on government bonds—all held by banks—is dominated by the cost of banks' deposits. Our analysis indicates that banks should be important drivers of

[21] As mentioned above, the fact that it is also zero at the left is because for those low values of λ, bank equity is wiped out in any case, so there is no point in buying reserves.

the liquidity premium priced into sovereign securities. This hints at the fact that empirical analyses of the convenience yields of near-money assets such as Treasuries might benefit from distinguishing between the financial and non-financial sectors.

To obtain these results, we have incorporated liquid reserves in a framework à la Goldstein and Pauzner (2005) and showed that we retain the existence of an equilibrium in threshold strategies, unique in the limit of vanishing noise. This allowed us to map unambiguously a bank's balance sheet decisions to a probability of failure and in turn to determine their optimal joint choice of leverage and liquidity buffer. That this probability is known in closed-form implies that the key object of our analysis, the marginal liquidity multiplier, appears analytically as the slope of an endogenous iso-risk curve. This slope quantifies the ability of a bank to act as a liquidity multiplier by buying public debt and issuing more private debt without altering its risk profile. Interestingly, the ability to multiply liquidity goes hand in hand with being exposed to liquidity risk. Therefore, (being ready to bid for and) hoarding expensive reserves and being subject to significant liquidity risk are two sides of the same coin.

We would like to mention three important avenues for future research. The first one is to introduce emergency liquidity assistance in the model. As mentioned earlier, we advanced significantly in this direction by adding a lender of last resort (LLR) to our framework, while retaining existence, uniqueness, and tractability of equilibrium. A LLR is desirable ex post in bad times, but distorts ex ante incentives to manage liquidity risk, which can imply that LLR intervention must be coupled with ex ante regulation. A full-fledged welfare analysis of the different regulatory tools is still on the agenda. Second, it could be interesting to add heterogeneity across banks at the interim date, as in Liu (2016). Third, investigating the consequences of imperfect competition among banks could also deliver new and interesting results.

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A Additional Material

A.1 Formal equilibrium concept

The model combines a competitive equilibrium among banks choosing their capital structure and liquidity buffer with an equilibrium in threshold strategies for the rollover game played among the banks' creditors. An equilibrium is a collection $(\delta, \beta, b, d, q_b, q_d(\cdot), x^R(\cdot), F_2^*(\cdot), T_0, T_2)$ such that

1. The government's budget constraint holds: (T_0, T_2) satisfy (12) and (13);
2. In the interim period $t = 1$
   (a) x^R is an equilibrium threshold of the rollover game: it is given by (11);
   (b) F_2^* is consistent with maximization of the banks' equity value: it solves (18);
   (c) The government bond market clears: $b_1 = \beta$ if $\theta < x^R$ and $b_1 = 0$ if $\theta > x^R$.
3. In the initial period $t = 0$
   (a) For each creditor $i \in [0, 1]$ the portfolio (b, d) (with d invested uniformly across banks) maximise his expected utility: it satisfies (26) and (27);
   (b) For each bank $j \in [0, 1]$ the choice (β, δ) maximises its expected profit given the resulting equilibrium threshold at the rollover stage: they solve problem (28) subject to the market valuation $q_d(\cdot)$ given by (29);
   (c) Markets clear: $b + \beta = B$ and $d = \delta$.

A.2 Optimal Face Value F_2^*

This section gives details on how we amend the result of Szkup (2013) to our framework, as in section 5.1.

As the public signal noise vanishes, the expected equity payoff at the beginning of the interim period—a function of the public signal θ_p—converges to the following function of the true state θ:

$$\int_{\mathbb{R}} \tau_y^{1/2} \varphi(u) E\left(n(u, x^*(F_2)), F_2; z\right) \phi\left(\frac{u - m(\theta_p)}{\tau_y^{-1/2}}\right) \mathrm{d}u \;\to\; \varphi(\theta) \times E(n = 0, F_2; z)$$
$$= \varphi(\theta) \times \max\left\{0, z + \beta - \delta F_2\right\} \times \mathbf{1}_{(x^*(F_2), \infty)}(\theta).$$

The bank does not observe θ. This fact combined with the assumption of vanishing noise implies that the indicator function in this expression will be identically zero or identically one (with arbitrarily large probability) depending on the choice of F_2. Given that the payoff is zero if the indicator is zero, it is optimal to select a F_2 that implements the indicator being identically one with arbitrarily large probability. Furthermore, given the term $z + \beta - \delta F_2$, it is optimal to select the minimal such F_2.

Hence, for all $\theta > x^* \left(\frac{z+\beta}{\delta}\right)$, it is optimal for the bank to set the face value F_2 arbitrarily close to (but higher than) the solution to

$$\theta = x^*(F_2). \tag{45}$$

In our framework with liquid reserves, we need to deal with an additional region that is not present in Szkup (2013): if $x^*(\overline{F}_2) \leq \theta \leq x^* \left(\frac{z+\beta}{\delta}\right)$, then in the limit of vanishing noise the bank becomes indifferent between preventing a run, or not. Indeed, to prevent a run the bank has to promise a face value $F_2 > \frac{z+\beta}{\delta}$; but if it successfully rolls over its debt, then at $t = 2$ its assets are insufficient to meet its credit obligations in full, so equity holders get nothing. But because of the existence of an upper dominance region, there is always a probability—however small—that the bank gets a positive payoff at the final date. Hence, even if $x^*(\overline{F}_2) \leq \theta \leq x^* \left(\frac{z+\beta}{\delta}\right)$, the bank's choice of F_2 is $F_2^*(\theta)$. In words: the bank selects the lowest face value that allows to avoid a run with probability 1; the benefit of doing so is arbitrarily small, but positive.

Finally, when $\theta < x^*(\overline{F}_2)$, even if the bank promises a face value $F_2 \geq \overline{F}_2$ the proportion of running creditors converges to one in the limit of vanishing noise. Recall that x^* is decreasing in F_2 only as long as $F_2 < \overline{F}_2$. Beyond that point, increasing F_2 has no impact, because the bank has effectively transferred all its assets to creditors in all states of the world.

B Proofs

B.1 Lemma 1

Default decision when the risky project fails ($Z = 0$).
The debt payoff at $t = 2$ is equal to $\min \{F_2, \mathcal{Q}(n; Z)\}$. Hence, if the project fails,

then the debt pays off *at most*

$$\mathcal{Q}(n;0) = \frac{\beta - n\delta}{(1-n)\delta}.$$

Note that $\mathcal{Q}(0;0) = \rho$, $\mathcal{Q}(\rho;0) = 0$, and

$$\mathcal{Q}'(n;0) = -\frac{\delta-\beta}{\delta}\frac{1}{(1-n)^2} = -\frac{1-\rho}{(1-n)^2} < 0. \tag{46}$$

Since $F_2 > 1 > \rho \geq \mathcal{Q}(n;0)$, the bank always defaults.
Default decision when the risky project succeeds ($Z = z$).
We first show that the ratio of the available assets at $t = 2$ per unit of outstanding debt, $\mathcal{Q}$, is strictly increasing for $0 \leq n < \rho$, strictly decreasing for $\rho \leq n < \lambda$, and equal to 0 for $n \geq \lambda$.

(i) $n \geq \lambda$: Early withdrawals exceed the proceeds of the bond sale and the liquidation of the entire project: $\mathcal{Q}(n,0) = 0$. Hence, the bank defaults at $t = 1$.

(ii) $\rho \leq n < \lambda$: Liquid reserves are insufficient to cover the early withdrawals, so the bank is forced to liquidate a fraction $(n\delta - \beta)/\ell$ of the project. At $t = 2$ the debt pays off at most

$$\mathcal{Q}(n;z) = \frac{1-(n\delta-\beta)/\ell}{(1-n)\delta}z.$$

Note that $\mathcal{Q}(\rho;z) = z/(\delta-\beta) \equiv \overline{F}_2$, $\mathcal{Q}(\lambda;z) = 0$ and

$$\mathcal{Q}'(n;z) = -\frac{\delta-(\ell+\beta)}{\delta}\frac{1}{(1-n)^2} = -\frac{1-\lambda}{(1-n)^2} < 0.$$

since $\lambda < 1$ by Assumption 2.

(iii) $n < \rho$: Liquid reserves are sufficient to cover the withdrawals. The project is intact and the bank is still left with $\beta - n\delta > 0$ bonds. At $t = 2$ the debt pays off at most

$$\mathcal{Q}(n;z) = \frac{z+\beta-n\delta}{(1-n)\delta}.$$

Note that $\mathcal{Q}(0;z) = (z+\beta)/\delta$, $\mathcal{Q}(\rho;z) = z/(\delta-\beta) \equiv \overline{F}_2$ and

$$\mathcal{Q}'(n;z) = \frac{z+\beta-\delta}{\delta}\frac{1}{(1-n)^2} > 0. \tag{47}$$

since $z+\beta > \delta$ by Assumption 1.

The bank defaults on its debt if, and only if, $F_2 > \mathcal{Q}(n;z)$. From the above arguments, $\mathcal{Q}(n;z)$ reaches its maximal value when a mass ρ of creditors withdraw, at which point it is equal to $\overline{F}_2 \equiv z/(\delta-\beta)$. Therefore, there are three cases to consider.

(i) $F_2 \leq (z+\beta)/\delta$: There is a unique cutoff level for the mass of running creditors, above which the bank defaults at $t=2$. It satisfies the equation:

$$[1-(n\delta-\beta)/\ell]z = (1-n)\delta F_2.$$

Solving for n and using the definitions $p^E = \ell/z$ and $\lambda = (\ell+\beta)/\delta$, yields (5).

(ii) $(z+\beta)/\delta < F_2 \leq \overline{F}_2$: As in case (i), if the mass of running creditors is sufficiently high, the bank is left with too few assets to repay its debt in full at $t=2$. The cutoff level which triggers default is again given by (5). The new aspect is that the face value is so high that the credit obligations exceeds the bank's revenue *even if all creditors roll over*. Hence, there is a unique cutoff level *below* which the bank defaults at $t=2$. It satisfies the equation:

$$z+\beta-n\delta = (1-n)\delta F_2.$$

Solving for n gives (6).

(iii) $F_2 > \overline{F}_2$: The bank always default at $t=2$, irrespective of the fraction of running creditors.

B.2 Proposition 1

The steps of the proof have become standard since the seminal papers of Goldstein and Pauzner (2005) and Morris and Shin (2006): existence follows from *(i)* $n \mapsto v(n,\theta)$ satisfies a single-crossing property, *(ii)* this implies a single-crossing property for $\theta \mapsto v(\theta, n(\theta, x^*))$ and *(iii)* due to a result of Athey (2002), this translates into the desired single-crossing property for $x_i \mapsto \Delta(x_i, x^*)$ as long as the integration noise is log-supermodular, which is indeed the case with Gaussian noise. Uniqueness follows from the uniform convergence argument of Morris and Shin (2006) (Chapter 3, Appendix A).

We must, however, account for a key specificity of our framework: liquid reserves. Those create an additional region where withdrawing decisions are strategic substitutes rather than complements. This means that the function $n \mapsto v(n, \theta)$ can now be successively increasing, decreasing and increasing again. Nevertheless, we are able to prove that point (i) still holds. This novel result then allows us to use the rest of the proof technology described above. Hence, we must establish:

$\forall\theta$, the function $n \mapsto v(n, \theta)$ satisfies the single-crossing property. Specifically, we show that it crosses zero at most once from above except for a unique value of θ_0 of θ; and for $\theta = \theta_0$ the function $n \mapsto v(n, \theta)$ is constant equal to zero and then strictly negative and therefore also satisfies the single-crossing property in the sense of Definition 2 in Athey (2002).

Recall that

$$v(\theta, n) = \varphi(\theta) D_2(n, F_2; z) + (1 - \varphi(\theta)) D_2(n, F_2; 0) - D_1(n)$$

Since $v(\theta, n) = -\min\{1, \lambda/n\} < 0$ for all $n \geq \lambda$, the function $n \mapsto v(n, \theta)$ never crosses zero in the interval $[\lambda, 1)$ (observation (i)). For $n \in [0, \underline{n})$ and $n \in (\overline{n}, \lambda)$, we have

$$\frac{\partial v}{\partial n} = \varphi(\theta) \mathcal{Q}'(n; z) + (1 - \varphi(\theta)) \mathcal{Q}'(n; 0). \tag{48}$$

For $n \in (\overline{n}, \lambda)$, v is strictly decreasing in n since $\mathcal{Q}'(n; 0) < 0$ and $\mathcal{Q}'(n; z) < 0$ in this interval (observation (ii)).

$n \mapsto v(\theta, n)$ is constant over $[\underline{n}, \overline{n}]$ (observation (iii)), because over this interval, F_2 is repaid in full in the good state at time 2, and $F_1 = 1$ is paid in full at time 1 in case of early withdrawal.

From observations (i), (ii) and (iii) we conclude that it is enough to show that $n \mapsto v(\theta, n)$ does not cross zero over $[0, \underline{n})$ to obtain the desired single-crossing property. (Notice that if $\underline{n} = 0$ the proof already terminates.)

For $n \in [0, \underline{n})$, the sign of the derivative in (48) is a priori unclear since

$\mathcal{Q}'(n;0) < 0$ but $\mathcal{Q}'(n;z) > 0$. Let

$$\theta_0 = \varphi^{-1}\left(\frac{\delta - \beta}{z}\right)$$

be the unique value such that $v(\theta_0, 0)(= \varphi(\theta_0)\frac{z+\beta}{\delta} + (1-\varphi(\theta_0))\frac{\beta}{\delta} - 1) = 0$.

Using the expressions obtained in (46) and (47), we now notice that $n \mapsto v(\theta_0, n)$ is uniformly zero over $[0, \underline{n}]$:

$$\begin{aligned}\frac{\partial}{\partial n} v(\theta_0, n) &= \frac{\delta - \beta}{z} \times \frac{z+\beta-\delta}{\delta}\frac{1}{(1-n)^2} + \left(1 - \frac{\delta-\beta}{z}\right) \times -\frac{1-\rho}{(1-n)^2} \\ &= \left[\frac{\delta-\beta}{\delta} - \frac{\delta-\beta}{\delta}\right]\frac{z+\beta-\delta}{z(1-n)^2} \\ &= 0.\end{aligned}$$

For $n \in [0, \underline{n})$, we have:

$$\frac{\partial v}{\partial \theta} = \varphi'(\theta)(\mathcal{Q}(n;z) - \mathcal{Q}(n;0)) = \varphi'(\theta)\frac{z}{(1-n)\delta} > 0.$$

This implies that for $n \in [0, \underline{n})$, if $\theta < \theta_0$, then $v(\theta, n) < 0$; and if $\theta > \theta_0$, then $v(\theta, n) > 0$. Hence, for $\theta \neq \theta_0$, $v(\theta, n)$ never crosses 0 for $n \leq \underline{n}$.

This concludes the proof that $n \mapsto v(n, \theta)$ has the desired single-crossing property.

B.3 Proposition 2

The indifference condition under the Laplacian belief is given by

$$\int_0^1 D_1(n)\,\mathrm{d}n = \varphi(x^*)\int_0^1 D_2(n, F_2; z)\,\mathrm{d}n + (1-\varphi(x^*))\int_0^1 D_2(n, F_2; 0)\,\mathrm{d}n \quad (49)$$

Solving for x^* yields (11). We now compute each term in turn.

Under the Laplacian belief, the expected payoff of withdrawing is equal to

$$\int_0^1 D_1(n)\,\mathrm{d}n = \int_0^\lambda \mathrm{d}n + \lambda\int_\lambda^1 \frac{1}{n}\,\mathrm{d}n = \lambda - \lambda\log(\lambda). \quad (50)$$

Conditional on the success of the project, the expected payoff of rolling over is equal to

$$\int_0^1 D_2(n, F_2; z)\,\mathrm{d}n = \int_0^{\underline{n}} \frac{z+\beta-n\delta}{(1-n)\delta}\,\mathrm{d}n + F_2\int_{\underline{n}}^{\overline{n}} \mathrm{d}n + \int_{\overline{n}}^{\lambda} \frac{1-(n\delta-\beta)/\ell}{(1-n)\delta} z\,\mathrm{d}n.$$

Calculating the integrals:

$$\begin{aligned}
\int_0^{\underline{n}} \frac{z+\beta-n\delta}{(1-n)\delta}\,\mathrm{d}n &= \underline{n} - \frac{z-(\delta-\beta)}{\delta}\log(1-\underline{n}) \\
F_2\int_{\underline{n}}^{\overline{n}} \mathrm{d}n &= F_2(\overline{n}-\underline{n}) \\
\int_{\overline{n}}^{\lambda} \frac{1-(n\delta-\beta)/\ell}{(1-n)\delta} z\,\mathrm{d}n &= \frac{z}{\ell}\left[\lambda-\overline{n}-(1-\lambda)\log\left(\frac{1-\overline{n}}{1-\lambda}\right)\right].
\end{aligned}$$

Therefore:

$$\int_0^1 D_2(n, F_2; z)\,\mathrm{d}n = \underline{n} - \frac{z-(\delta-\beta)}{\delta}\log(1-\underline{n}) + F_2(\overline{n}-\underline{n}) + \frac{z}{\ell}\left[\lambda-\overline{n}-(1-\lambda)\log\left(\frac{1-\overline{n}}{1-\lambda}\right)\right]. \tag{51}$$

Finally, conditional on the failure of the project, the expected payoff of rolling over is

$$\int_0^1 D_2(n, F_2; 0)\,\mathrm{d}n = \rho\int_0^{\rho} \frac{1}{1-n}\,\mathrm{d}n - \int_0^{\rho} \frac{n}{1-n}\,\mathrm{d}n = \rho + (1-\rho)\log(1-\rho). \tag{52}$$

B.4 Proposition 1

Since creditors run on the bank (with probability one) when θ realises below $x^R \equiv x^*(\overline{F}_2)$, the ex-ante likelihood of runs is equal to $G(x^R) = \Phi\left(\tau^{1/2}(x^R-\mu)\right)$. We denote by $p^R \equiv \varphi(x^R)$ the associated success probability of the project. To derive the functional form of p^R, set $\underline{n} = \overline{n} = \rho$ in (51) and use the definitions

$$\begin{aligned}
\pi_1(u) &= u - u\log(u) && (53) \\
\pi_2(u) &= 1-\pi_1(1-u) = u + (1-u)\log(1-u), && (54)
\end{aligned}$$

to obtain:

$$\begin{aligned}\int_0^1 D_2(n,\overline{F}_2;z)\,\mathrm{d}n &= \rho-\frac{z-(\delta-\beta)}{\delta}\log(1-\rho)+\frac{z}{\ell}\left[\lambda-\rho-(1-\lambda)\log\left(\frac{1-\rho}{1-\lambda}\right)\right]\\ &= \left(1-\frac{z}{\ell}\right)[\rho+(1-\rho)\log(1-\rho)]+\frac{z}{\ell}[\lambda+(1-\lambda)\log(1-\lambda)]\\ &= \frac{z}{\ell}[\pi_2(\lambda)-\pi_2(\rho)]+\pi_2(\rho).\end{aligned}$$

Substituting this expression into (49) and using the definition of the functions π_1 and π_2, the indifference condition becomes

$$\begin{aligned}\pi_1(\lambda) &= p^R\left(\frac{z}{\ell}[\pi_2(\lambda)-\pi_2(\rho)]+\pi_2(\rho)\right)+(1-p^R)\pi_2(\rho)\\ &= \pi_2(\rho)+p^R\frac{1}{p^E}[\pi_2(\lambda)-\pi_2(\rho)].\end{aligned}$$

Solving for p^R yields

$$p^R=p^E\times\frac{\pi_1(\lambda)-\pi_2(\rho)}{\pi_2(\lambda)-\pi_2(\rho)}.$$

B.5 Proposition 2

Lower bound. We want to show that

$$\frac{\pi_1(\lambda)-\pi_2(\rho)}{\pi_2(\lambda)-\pi_2(\rho)}\geq\frac{1-\rho}{\lambda-\rho},$$

with a strict inequality if $\lambda<1$. Cross-multiplying and simplifying, we see that this condition is equivalent to

$$(\lambda-\rho)\pi_1(\lambda)\geq\pi_2(\lambda)(1-\rho)-\pi_2(\rho)(1-\lambda).$$

Both sides converge to zero as $\rho\to\lambda$. We now show that the LHS decreases faster than the RHS as ρ increases. This implies that the inequality is strict for all $\rho\in[0,\lambda)$.

Differenting both sides with respect to ρ, we obtain the condition

$$\begin{aligned}-\pi_1(\lambda) &< -\pi_2(\lambda)+\log(1-\rho)(1-\lambda)\\ -\lambda\log(\lambda) &> (1-\lambda)[\log(1-\lambda)-\log(1-\rho)].\end{aligned}$$

For all $\rho \leq \lambda < 1$, the LHS is stricty positive while the RHS is nonpositive.

Limit $\lambda \to 1$. The result follows immediately from plugging the limit

$$\lim_{\lambda \nearrow 1} \pi_2(\lambda) = 1$$

into the expression of Γ given in (19).

B.6 Proposition 3

Impact of δ and β. The comparative statics in δ and β can be deduced from section B.7, where we establish that

$$-\frac{\partial_\beta \Gamma}{\partial_\delta \Gamma} > 1. \tag{55}$$

This indicates that $\partial_\beta \Gamma$ and $\partial_\delta \Gamma$ have opposite signs. Moreover, they cannot cross zero simultaneously, because we would obtain from the expression of the ratio j_1 in section B.7 that at the corresponding (λ, ρ):

$$\begin{aligned} \partial_\lambda \Gamma + \partial_\rho \Gamma &= 0 \\ \partial_\lambda \Gamma \lambda + \partial_\rho \Gamma \rho &= 0, \end{aligned}$$

hence $\lambda = \rho$ (because $\partial_\rho \Gamma$ is positive), which is not possible. Because the partial derivatives involved in the ratio (55) are continuous, we deduce that they must be of constant sign. The result then obtains by checking the sign of one partial derivative at any point.

Impact of ex-ante distribution of fundamental. μ and τ do not impact the equilibrium threshold but only the distribution of the fundamental:

$$\begin{aligned} \frac{\partial G(x^R)}{\partial \mu} &= -\tau^{1/2} \phi \left(\frac{x^R - \mu}{\tau^{-1/2}} \right) \\ \frac{\partial G(x^R)}{\partial \tau^{-1/2}} &= -\tau (x^R - \mu) \phi \left(\frac{x^R - \mu}{\tau^{-1/2}} \right). \end{aligned}$$

B.7 Proposition 4

We know that

$$\frac{\partial \hat{\delta}}{\partial \beta}(\beta; p^T) = -\frac{\partial_\beta \Gamma}{\partial_\delta \Gamma}(\beta, \hat{\delta}(\beta; p^T)) \equiv j_1(\beta, \hat{\delta}(\beta; p^T)). \tag{56}$$

Now:

$$\begin{aligned} j_1(\beta, \delta) &= -\frac{\partial_\lambda \Gamma \partial_\beta \lambda + \partial_\rho \Gamma \partial_\beta \rho}{\partial_\lambda \Gamma \partial_\delta \lambda + \partial_\rho \Gamma \partial_\delta \rho}\left(\frac{\ell + \beta}{\delta}, \frac{\beta}{\delta}\right) \\ &= \frac{\partial_\lambda \Gamma + \partial_\rho \Gamma}{\partial_\lambda \Gamma \lambda + \partial_\rho \Gamma \rho}\left(\frac{\ell + \beta}{\delta}, \frac{\beta}{\delta}\right) \\ &\equiv j_2\left(\frac{\ell + \beta}{\delta}, \frac{\beta}{\delta}\right). \end{aligned}$$

Hence, the first part of the Proposition will follow from the fact that $j_2(\lambda, \rho) > 1$ for all $0 < \rho < \lambda < 1$. Brute force calculation shows that

$$j_2(\lambda, \rho) - 1 = \frac{-(\lambda - 1)\log(1-\lambda)(\lambda - \rho - \log \lambda) + \log \lambda((\lambda - 1)(\lambda - \rho) - (\rho - 1)\log(1-\rho))}{\log(1-\lambda)(-\lambda \log \lambda + (\lambda - \rho)(\lambda - \log(1-\rho))) + \lambda \log \lambda(-\lambda + \rho + \log(1-\rho))}.$$

Call $N(\lambda, \rho)$ and $D(\lambda, \rho)$ the numerator and denominator of this fraction, respectively. We have

$$\partial_{\rho\rho} D = \log(1-\lambda)\left(-\frac{1}{1-\rho} + \frac{\lambda - 1}{(1-\rho)^2}\right) - \frac{\lambda \log \lambda}{(1-\rho)^2} > 0.$$

Since $\partial_\rho D(\lambda, 0) = 0$, we obtain $\partial_\rho D > 0$ for $0 < \rho < \lambda$, and since $D(\lambda, \lambda) = 0$, we obtain in turn $D(\lambda, \rho) < 0$ for all $0 < \rho < \lambda < 1$. And

$$\partial_{\rho\rho} N = \frac{\log \lambda}{1 - \rho} < 0.$$

From a straightforward one-variable function analysis, $\partial_\rho N(\lambda, \lambda) = (\lambda - 1)\log(1 - \lambda) - \log \lambda(\lambda + \log(1-\lambda)) \geq 0$, so we obtain $\partial_\rho N > 0$ for $0 < \rho < \lambda$. Since $N(\lambda, \lambda) = 0$, we obtain in turn $N(\lambda, \rho) < 0$ for all $0 < \rho < \lambda < 1$.

Hence, $j_2 - 1 = N/D > 0$, which concludes the proof of the first part of the Proposition. The second part is proven by remarking that a probability of run equal to p^E only obtains for $\lambda = 1$; and to maintain $\lambda \equiv \frac{\ell+\beta}{\delta} = 1$, we must move δ and β one-to-one. Therefore, indeed, $\partial_\beta \hat{\delta}(\beta; p^T)$ approaches 1 as p^T approaches

p^E.

B.8 Lemma 3

A bank's debt market valuation (29) is nearly identical to the first-order condition (27) of the creditors' portfolio choice. The only difference is that, from the perspective of an individual bank, the aggregate (risk-adjusted) liquid asset holdings of creditors is a constant, which depends on the equilibrium choice $(\bar{\delta}, \bar{\beta})$ made by all other banks:

$$\begin{aligned} \mathcal{L} &= b + \int_0^1 d(j) w(j) \, \mathrm{d}j \\ &= b + d[1 - (1-\lambda) G(x^*(\overline{F}_2))] \\ &= B - \bar{\beta} + \bar{\delta} \left[1 - \left(1 - \frac{\ell + \bar{\beta}}{\bar{\delta}} \right) G \left(x^* \left(\frac{z}{\bar{\delta} - \bar{\beta}} \right) \right) \right]. \end{aligned}$$

The second equality uses symmetry ($d(j) = d$ and $w(j) = w$), the definition of w in (17), and the equilibrium mass of running creditors in (25). The third equality is obtained by using the definition of λ and $\overline{F}_2$, and imposing market clearing ($b + \hat{\beta} = B$ and $d = \hat{\delta}$).

B.9 Lemma 4

From the arguments of Section 3.3.2, F_2^* is strictly decreasing in θ for $\theta > F_2^{*-1}((z+\beta)/\delta) \equiv \hat{\theta}$, and $F_2^*(\theta) \geq (z+\beta)/\delta$ for all $\theta \leq \hat{\theta}$. Therefore, using the payoff definitions

$$\begin{aligned} D_2(n, F_2; Z) &= \min\{F_2, \mathcal{Q}(n, Z)\} \\ D_1(n) &= \min\{1, (\ell + \beta)/(\delta n)\}, \end{aligned}$$

substituting the equilibrium fraction of withdrawing creditors into (29) and multiplying the result by δ, we obtain:

$$\begin{aligned}q_d(\delta,\beta)\delta &= S_d\delta + (\ell+\beta)H(p^R) \\ &+(z+\beta)\int_{p^R}^{\varphi(\hat{\theta})} p\,\mathrm{d}H(p) + \beta\int_{p^R}^{\varphi(\hat{\theta})}(1-p)\,\mathrm{d}H(p) \\ &+\delta\int_{\varphi(\hat{\theta})}^{1} pF_2^*(\varphi^{-1}(p))\,\mathrm{d}H(p) + \beta\int_{\varphi(\hat{\theta})}^{1}(1-p)\,\mathrm{d}H(p). \end{aligned} \tag{57}$$

Recall that the equity value of the bank at $t=0$ is given by:

$$V = q_d\delta - q_b\beta + \int_{\varphi(\hat{\theta})}^{1} p\left[z+\beta-\delta F_2^*(\varphi^{-1}(p))\right]\,\mathrm{d}H(p) \tag{58}$$

Substituting (57) into (58) and symplifying, we obtain

$$V = \ell H(p^R) + z\int_{p^R}^{1} p\,\mathrm{d}H(p) + S_d\delta - (q_b-1)\beta. \tag{59}$$

C Figures

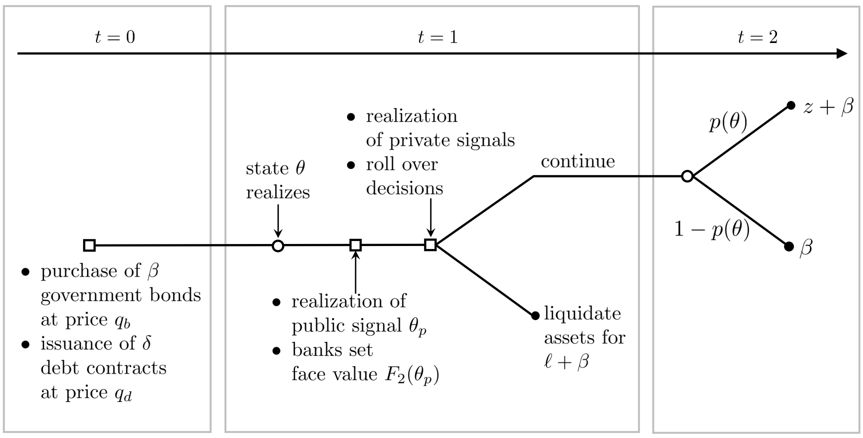

Figure 2: Timeline

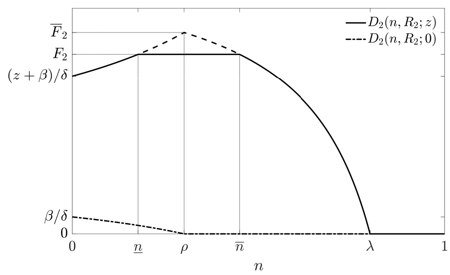

Figure 3: Payoff of debt contract in the final period

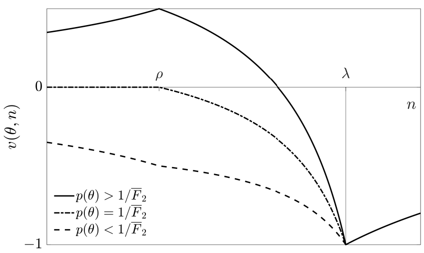

Figure 4: Single-crossing property of creditors' payoff gain

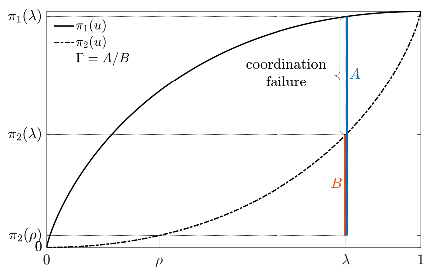

Figure 5: Coordination failure multiplier

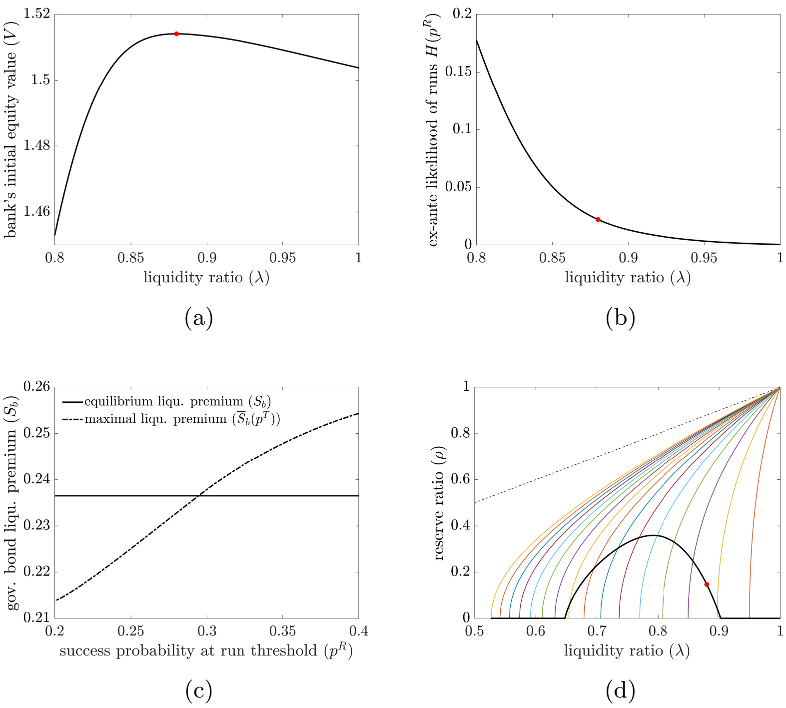

Figure 6: Optimal choice of capital structure and liquid reserves

Карр, С.Ж.П, Клосснер Д.

Банки как мультипликаторы ликвидности: премия за ликвидность активов и риск ликвидности финансирования [Электронный ресурс] : препринт WP9/2021/03 / С.Ж.П. Карр, Д. Клосснер ; Нац. исслед. ун-т «Высшая школа экономики». – Электрон. текст. дан. (1,84 Мб). – М. : Изд. дом Высшей школы экономики, 2021. – (Серия WP9 «Исследования по экономике и финансам»). – 67 с. (На англ. яз.)

Банки накапливают большие объемы ликвидных резервов, поддерживая высокий уровень левериджа, и доходность этих резервов часто ниже стоимости их депозитов. Почему банки проводят такую отрицательную кэрри-трейд? Используя новое наблюдение о глобальных играх, мы строим модель, в которой банки управляют риском ликвидности через спрос на ликвидные активы (безопасный государственный долг) и частное предоставление долга кредиторам. Банки особенно ценят ликвидные резервы, поскольку они позволяют им «умножать ликвидность»: покупать государственный долг и выпускать больше частных долгов, сохраняя постоянный риск ликвидности. Насколько больше – измеряется наклоном кривой эндогенного изо-риска. Умножение ликвидности является ключом к пониманию разнообразного взаимодействия между стратегиями кредитного плеча и буфера ликвидности и подразумевает теорию загадки отрицательного кэрри-трейда.

Карр Сильвен Жан Паскаль, МИЭФ, Национальный исследовательский университет «Высшая школа экономики»; Российская Федерация, 109028, Москва, Покровский бульвар, д. 11, каб. S708; E-mail: scarre@hse.ru

Дэмиен Клосснер, Швейцарский национальный банк; E-mail: damien.klossner@snb.ch

Препринты Национального исследовательского университета «Высшая школа экономики» размещаются по адресу: http://www.hse.ru/org/hse/wp

Препринт WP9/2021/03
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(*на английском языке*)

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