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COOPERATIVE CONGESTION GAMES

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Cooperative congestion games

This paper studies a model for cooperative congestion games. There is an array of cooperative games V and a player's strategy is to choose a subset of the set V. The player gets a certain payoff from each chosen game. The paper demonstrates that if a payoff is the Shapley or the Banzhaf value, then the corresponding cooperative congestion game has a Nash equilibrium in pure strategies. The case is examined where each game in V has a coalition partition. The stability of the vector of coalition structures is determined, taking into account the transitions of players within a game and their migrations to other games. The potential function is defined for coalition partitions, and is used as a means of proving the existence of a stable vector of coalition structures for a certain class of cooperative game values.

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1 Introduction

There is a rich choice of sectors where economic actors can do business. Since the power and working capacity of economic agents are limited, each agent focuses on the sectors they prefer. Rosenthal (1973) suggested a congestion game to model this situation. There is a certain set of resources in the congestion game, and the payoff function for each resource depends on how many players have chosen it. It has been established that a congestion game has a Nash equilibrium in pure strategies. Holzmann (1997) found sufficient conditions under which a strong Nash equilibrium exists in the congestion game. If a resource payout depends not on the number of players, but on the sum of their weights, then the game is called a weighted congestion game. The existence of Nash equilibrium in such games was studied in Ackermann et al. (2009), Fanelli and Moscardelli (2011), and Harks et al. (2001). The price of anarchy for some classes of congestion games was found in Aland et al. (2011), Correa et al. (2008), and Chau and Sim (2003). A class of games in which a resource payout depends on the number of players and on the player's strategy was considered in Milchtaich (1996).

Economic actors with common objectives often form commercial alliances to maximize profit. If several agents have chosen the same resource, they can collaborate. A cooperative congestion game is suggested in this paper to model such a situation. Suppose there is a set of cooperative games V. Any player's strategy is to choose a subset of V. Players cooperate within each game. The payoff from the game $v, v \in V$ depends on the set of players it was chosen by. A significant distinction of the cooperative congestion game from the classical congestion model is that the payoff function of the game v depends on the set of players, not on their number.

A question of interest is the existence of an equilibrium in a cooperative congestion game. It is demonstrated that if a player's payoff in each cooperative game is the Shapley value or the Banzhaf value, then the corresponding cooperative congestion game has a Nash equilibrium in pure strategies. The existence of a Nash equilibrium in pure strategies is guaranteed by the defined potential function.

A player's payoff in a congestion game is the sum of the numbers of the form $c_j(k)$, where k is the number of players who have chosen the resource j. If there are n players and m resources in a congestion game, then there is a total of $m \cdot n$ parameters which express the players' payoff functions. The number of such parameters in a cooperative congestion game equals $m \cdot 2^n$. As demonstrated, payoff functions can be obtained for a congestion game by applying a special procedure to find characteristic functions in a cooperative congestion game with the Shapley value.

Unless the characteristic function of the cooperative game is monotonic, players would not always benefit from a grand coalition. In this case, a coalition partition can be formed in the cooperative game. A Nash-stable vector of coalition structures was defined taking into account solo migrations of players within the game and migrations of players into the coalitions from other games. To prove the existence of a Nash-stable vector of coalition structures, the definition of the potential function is reformulated in the paper. The potential function in this case is defined on the set of vectors of coalition partitions rather than on the set of the players' strategy profiles. This approach was used by Gusev and Mazalov (2019) to prove that a Nash-stable coalition partition exists in any cooperative game with respect to the Aumann-Drèze value.

Hart and Mas-Colell (1989) introduced the following definition of the potential function for a cooperative game,

$$\sum_{i \in N} (P(N, v) - P(N \setminus \{i\}, v)) = v(N).$$

This definition of the potential function is not bound to the existence of a Nash-stable coalition partition. The notions of potential functions defined by Monderer and Shapley (1996) are used.

The structure of the paper. The second section describes a cooperative congestion game. The third section proves the existence of a Nash equilibrium in the cooperative congestion game for some values of the cooperative game. A cooperative congestion game with a vector of coalition structures is described in the fourth section. After that, the axiomatic system for a cooperative congestion game with the Shapley value is suggested. Proofs are given in the Appendix.

2 Cooperative congestion model

The congestion model is a tuple $\langle N, M, \{S_i\}_{i \in N}, \{c_j\}_{j \in M} \rangle$, where N is the set of players, $M = \{1, 2, ..., m\}$ is the set of resources, S_i is the set of strategies of player i. An element of the set S_i $\forall i \in N$ is a non-empty subset of set M. The function $c_j : R \to R$ is the payoff from the resource $j, j \in M$. Denote $S = S_1 \times ... \times S_n, S_{-i} = S_1 \times ... \times S_{i-1} \times S_{i+1} \times ... \times S_n$.

The congestion game is a normal-form game, $\langle N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$,

$$u_i(s) = \sum_{j \in s_i} c_j(k_j(s)),$$

where $k_j(s)$ is the number of players who chose resource j from strategy profile $s, s \in S$. There exists a Nash equilibrium in the congestion game. This is guaranteed by the existence of a potential function

$$P(s) = \sum_{j \in \underset{i \in N}{\cup} s_i} \sum_{k=1}^{k_j(s)} c_j(k).$$

The cooperative congestion model is a tuple $\langle N, V, \{S_i\}_{i \in N}, \{\varphi(v)\}_{v \in V} \rangle$. The sets N and $S_i, i \in N$ have the same meaning as in the congestion game, $\forall i \in N$, $\forall s_i \in S_i : s_i \subseteq V$. The set $V = \{v_1, v_2, ..., v_m\}, v_j : 2^N \to R, v_j(\emptyset) = 0$ $\forall j \in \{1, 2, ..., m\}$ is an array of cooperative games and $\varphi(v)$ is the value of the cooperative game $v, v \in V$.

The cooperative congestion game with the value φ is a normal-form game $\langle N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N} \rangle$, in which the payoff functions have the form

$$h_i(s) = \sum_{v \in s_i} \varphi_i(N_v(s), v), i \in N,$$

where $N_v(s) = \{i | v \in s_i, i \in N\}$ is the set of players who chose game v from the strategy profile $s, s \in S$.

A cooperative game in a cooperative congestion model can be regarded as a resource. Player $i, i \in N$ chooses a set of cooperative games $s_i, s_i \in S_i$, that maximizes his/her payoff

$h_i(s)$. A significant distinction of the cooperative congestion game from the classical congestion game is the payoff from a resource. The payoff in the former depends on the set of players $N_v(s)$, while in the latter it depends only on the number of players $k_j(s)$.

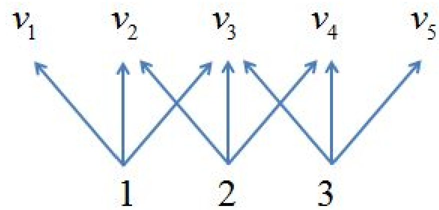

Fig. 1. Choice of cooperative games by players.

Example 1. Let $N = \{1, 2, 3\}, V = \{v_1, v_2, ..., v_5\}$ and φ be the value of the cooperative game defined for any game in V. Suppose the players' strategy profile has the form

$$s = (s_1, s_2, s_3) = (\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_3, v_4, v_5\}).$$

Then, the players' payoffs in the cooperative congestion game have the form:

$$h_1(s) = \varphi_1(\{1\}, v_1) + \varphi_1(\{1, 2\}, v_1) + \varphi_1(\{1, 2, 3\}, v_1),$$

$$h_2(s) = \varphi_2(\{1, 2\}, v_2) + \varphi_2(\{1, 2, 3\}, v_3) + \varphi_2(\{2, 3\}, v_4),$$

$$h_3(s) = \varphi_3(\{1, 2, 3\}, v_3) + \varphi_3(\{2, 3\}, v_4) + \varphi_3(\{3\}, v_5).$$

3 Existence of a Nash equilibrium

In this section we find the values φ for which there exists a Nash equilibrium in the cooperative congestion game.

Theorem 1. *Let the value φ of the cooperative game fulfill the following two conditions:*

1. Linearity property. For any games v, w and real numbers a, b the equality $\varphi(av+bw) = a \cdot \varphi(v) + b \cdot \varphi(w)$ is true.

2. For any player $i, i \in N$ the equality $\varphi_i(v_T) = \begin{cases} f(T), & i \in T; \\ 0, & i \notin T. \end{cases}$ is true, where v_T is a unanimity game.

Then, a cooperative congestion game with the value φ is a potential game, and the potential function has the form

$$P(s) = \sum_{v \in \bigcup\limits_{i \in N} s_i} \sum_{\substack{K \subseteq N_v(s) \\ K \neq \emptyset}} f(K) \lambda_K(v), \; \lambda_K(v) = \sum_{R \subseteq K} (-1)^{|K|-|R|} v(R).$$

See Appendix for proof.

Theorem 1 leads to the following corollaries.

Corollary 1. *A cooperative congestion game with the Shapley value is a potential game.*

Corollary 2. *A cooperative congestion game with the Banzhaf value is a potential game.*

See Appendix for proof of Corollaries 1 and 2.

Corollaries 1 and 2 indicate the following. Let there be a set of cooperative games $V = \{v_1, v_2, ..., v_m\}$, and the strategy $s_i, s_i \in S_i$ of player $i, i \in N$ be the choice of a subset of V. Suppose the game $v, v \in s_i$ gives player i the Shapley value. Then, the cooperative congestion game has a Nash equilibrium in pure strategies. If the payoff of the player $i, i \in N$ in the game $v, v \in s_i$ is the Banzhaf value, then the corresponding cooperative congestion game also has a Nash equilibrium in pure strategies. What matters is that the games from Corollaries 1 and 2 are potential games. Any assertion valid for potential games holds for the games in question.

What is interesting about Theorem 1 and its corollaries is that the characteristic functions from the set V can take arbitrary values. There are no monotonicity or superadditivity restrictions. It is usually assumed in a congestion game that the function $c_j, j \in M$ is monotone decreasing. If, however, c_j is not a monotone decreasing function, then the existence of a potential function guarantees the existence of an equilibrium. The monotonicity restriction is usually associated with the solution's application or self-evidence. Suppose, for instance, that c_j is monotone increasing, whereas the payoff functions for the rest of resources are monotone decreasing. In such a case, players would most probably choose the resource j. The more players choose j, the greater the payoff they get in this case. The proposition below shows the relationship between a congestion game and a cooperative congestion game with the Shapley value.

Proposition 1. *A congestion game is a cooperative congestion game with the Shapley value if* $v_j(K) = |K| \cdot c_j(|K|)\ \forall j \in M,\ \forall K \subseteq N, K \neq \emptyset$.

See Appendix for proof.

Proposition 1 proclaims that the class of congestion games is a subclass of cooperative congestion games with the Shapley value. Substituting the numerical values of characteristic functions into $h_i(s), i \in N, s \in S$, one can generate potential non-cooperative games.

4 Coalition partitions in a cooperative congestion game

4.1 Notations

Let us introduce some notation needed to define a cooperative congestion game with a vector of coalition structures. We denote by $\Pi(K)$ the set of coalition partitions of the set $K, K \subseteq N, \Pi(\emptyset) = \emptyset$, i.e.

$$\Pi(K) = \left\{ \{B_1, B_2, ..., B_l\} \,\middle|\, \bigcup_{j=1}^{l} B_j = K, B_j \cap B_g = \emptyset, 1 \leq j < g \leq l \right\}.$$

Let Π_N^m be the set of all vectors of coalition structures,

$$\Pi_N^m = \left\{ (\pi^1, \pi^2, ..., \pi^m) | \exists K \subseteq N : \pi^j \in \Pi(K), j = 1, 2, ..., m \right\}.$$

Take a set X in the subset $\Pi_N^m, X \subseteq \Pi_N^m$. Let $\pi \in X, \pi = (\pi^1, \pi^2, ..., \pi^m)$. Player i can be part of several coalition partitions π at the same time. To isolate elements of π, we use the notation $\pi = (\pi^j, \pi_{-j}) = (\pi^j, \pi^l, \pi_{-j,l})$. To isolate elements of the coalition structure $\pi^j, j = 1, 2, ..., m$, we use a similar notation $\pi^j = \{A, \pi^j_{-A}\} = \{A, B, \pi^j_{-A,B}\}$. The set of coalition structures in which player i is included is

$$Y_i(\pi) = \{\pi^j | \exists B \in \pi^j : i \in B, j = 1, 2, ..., m\}.$$

Let $\pi^j \in Y_i(\pi)$. Denote by $\pi^j(i)$ the coalition from π^j that contains player i and $\pi^j - i = \{\pi^j(i) \setminus \{i\}, \pi^j_{-\pi^j(i)}\}$. Two sets are introduced:

$$D_i(\pi^j) = \{\{\pi^j(i) \setminus \{i\}, A \cup \{i\}, \pi^j_{-\pi^j(i),A}\} | A \in \pi^j \text{ or } A = \emptyset\},$$

$$M_i(\pi) = \left\{ \rho = \left\{\pi^j - i, \{A \cup \{i\}, \pi^l_{-A}\}, \pi_{-j,l}\right\} \middle| \rho \in X, \pi^j \in Y_i(\pi), \pi^l \notin Y_i(\pi) \right\}.$$

Coalition structures from the set $D_i(\pi^j)$ are obtained from π^j by moving player i to another coalition within the game, or when player i forms a coalition of his/her own. The set of coalition structure vectors $M_i(\pi)$ appears from π when player i moves from one coalition partition to another. Player i can move from coalition partition π^j to π^l if such a move is allowed, and if player i is not a member of π^l, i.e. $\rho \in X, \pi^l \notin Y_i(\pi)$.

4.2 A stable vector of coalition structures

It used to be believed that if players have chosen one and the same game, they form a grand coalition within the game. In this section, the case is considered where a coalition partition of players exists in every cooperative game from V.

Definition 1. *The cooperative congestion game with a vector of coalition structures in our case will be a tuple (N, V, X, H), where $N = \{1, 2, ..., n\}$ is the set of players, $V = \{v_1, v_2, ..., v_m\}$ is the set of cooperative games and $X \subseteq \Pi_N^m, H : X \to R^n$.*

The set of players' strategy profiles S is given for a normal-form game. Knowing the S of the cooperative congestion game, the corresponding set X can be constructed to perform the transition from a normal-form game to a coalition partition game.

Example 2. Let $N = \{1, 2, 3\}$ and $V = \{v_1, v_2, ..., v_5\}$. Suppose that coalition partitions of players in each game have the form

$$\pi^1 = \{\{1\}, \{2, 3\}\}, \pi^2 = \{\{2\}, \{3\}\}, \pi^3 = \{\{1, 2\}, \{3\}\}, \pi^4 = \{\{1, 3\}\}, \pi^5 = \{\{1, 2, 3\}\}.$$

Let the payoff of player i from the cooperative game $v_j, j = 1, 2, ..., 5$ be $\varphi_i(\pi^j(i), v_j)$. Then, the payoffs of players in the cooperative congestion game with a vector of coalition structures have the following form:

$$H_1(\pi) = \varphi_1(\{1\}, v_1) + \varphi_1(\{1, 2\}, v_3) + \varphi_1(\{1, 3\}, v_4) + \varphi_1(\{1, 2, 3\}, v_5),$$

$$H_2(\pi) = \varphi_2(\{2, 3\}, v_1) + \varphi_2(\{2\}, v_2) + \varphi_2(\{1, 2\}, v_3) + \varphi_2(\{1, 2, 3\}, v_5),$$

$$H_3(\pi) = \varphi_3(\{2,3\}, v_1) + \varphi_3(\{3\}, v_2) + \varphi_3(\{3\}, v_3) + \varphi_2(\{1,3\}, v_4) + \varphi_3(\{1,2,3\}, v_5).$$

Player 1 is absent from the coalition partition for game v_2, but present in the coalition partitions of other games. The form of coalition structure vectors is defined by the set X.

Definition 2. *The vector $\pi, \pi \in X$ is said to be Nash-stable in the game (N, V, X, H) if $\forall i \in N, \forall \pi^j \in Y_i(\pi)$ the following inequalities hold:*

$$H_i(\pi^j, \pi_{-j}) \geq H_i(\rho^j, \pi_{-j}) \ \forall \rho^j \in D_i(\pi^j),$$

$$H_i(\pi) \geq H_i(\rho) \ \forall \rho \in M_i(\pi).$$

Depending on the set X, any player can be present in several games at the same time. Each player can only be part of one coalition from the coalition partition $\pi^j, j = 1, 2, ..., m$. The first inequality from Definition 1 indicates that there is no benefit for any player from moving to another coalition within the game they are in. What follows from the second inequality is that solo players would not gain from changing one cooperative game for another on their own. The next theorem is an extension of Theorem 1.

Theorem 2. *Let the value φ satisfy the restrictions from Theorem 1. Then, there exists in a Nash-stable vector of coalition partitions in the game (N, V, X, H) with the payoff functions*

$$H_i(\pi) = H_i(\pi^1, \pi^2, ..., \pi^m) = \sum_{\pi^j \in Y_i(\pi)} \varphi_i(\pi^j(i), v_j), i \in N.$$

See Appendix for proof.

The proof of Theorem 2 is based on the existence of the following potential function

$$P(\pi) = \sum_{j=1}^{m} \sum_{B \in \pi^j} \sum_{\substack{K \subseteq B \\ K \neq \emptyset}} f(K) \lambda_K(v_j).$$

Suppose players in each cooperative game from V have formed a coalition partition and a player's payoff in the coalition partition depends on the coalition they belong to. Then, if the payoff is the Shapley value or the Banzhaf value, there exists a stable vector of coalition structures in the cooperative congestion game.

4.3 Numerical example

Let $N = \{1, 2, 3\}$ and $V = \{v_1, v_2\}$. The values v_1 and v_2 are given in Table 1.

Set X has the form

$$X = \{(\pi^1, \pi^2) | \pi^1 \in \Pi(K), \pi^2 \in \Pi(N \setminus K), K \subseteq N\}.$$

The coalition partitions from X tell us that any player from N can be present only in one game from V. Take a cooperative congestion game (N, V, X, H) with the Shapley value and a vector of coalition partitions. Let us find the vector of stable coalition structures. According

Table 1: Values of v_1, v_2.

K	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
v^1	1	2	9	8	1	2	8
v^2	1	8	7	10	4	1	6

to Theorem 2, such a vector exists. To find it, we calculate the highest value of the potential function

$$P(\pi) = \sum_{j=1}^{m} \sum_{B \in \pi^j} \sum_{\substack{K \subseteq B \\ K \neq \emptyset}} \frac{\lambda_K(v_j)}{|K|}$$

in Table 2. The potential function for some π is calculated as follows,

$$P\big(\{\{1\}\}, \{\{2,3\}\}\big) = \lambda_{\{1\}}(v_1) + \lambda_{\{2\}}(v_2) + \lambda_{\{3\}}(v_2) + \frac{\lambda_{\{2,3\}}(v_2)}{2},$$

$$P\big(\{\{2,3\}\}, \{\{1\}\}\big) = \lambda_{\{2\}}(v_1) + \lambda_{\{3\}}(v_1) + \frac{\lambda_{\{2,3\}}(v_1)}{2} + \lambda_{\{1\}}(v_2),$$

$$P\big(\{\{1\}, \{2,3\}\}, \emptyset\big) = \lambda_{\{1\}}(v_1) + \lambda_{\{2\}}(v_1) + \lambda_{\{3\}}(v_1) + \frac{\lambda_{\{2,3\}}(v_1)}{2},$$

$$P\big(\emptyset, \{\{1\}, \{2,3\}\}\big) = \lambda_{\{1\}}(v_2) + \lambda_{\{2\}}(v_2) + \lambda_{\{3\}}(v_2) + \frac{\lambda_{\{2,3\}}(v_2)}{2}.$$

Table 2: Values of potential function

π	$P(\pi \times V)$	π	$P(\pi \times V)$
$\big(\emptyset, \{\{1\}, \{2\}, \{3\}\}\big)$	16	$\big(\{\{1\}, \{2\}\}, \{\{3\}\}\big)$	10
$\big(\emptyset, \{\{1\}, \{2,3\}\}\big)$	9	$\big(\{\{1,2\}\}, \{\{3\}\}\big)$	12 1/2
$\big(\emptyset, \{\{2\}, \{1,3\}\}\big)$	14	$\big(\{\{1\}, \{3\}\}, \{\{2\}\}\big)$	18
$\big(\emptyset, \{\{3\}, \{1,2\}\}\big)$	16 1/2	$\big(\{\{1,3\}\}, \{\{2\}\}\big)$	13 1/2
$\big(\emptyset, \{\{1,2,3\}\}\big)$	9 5/6	$\big(\{\{2\}, \{3\}\}, \{\{1\}\}\big)$	12
$\big(\{\{1\}\}, \{\{2\}, \{3\}\}\big)$	16	$\big(\{\{2,3\}\}, \{\{1\}\}\big)$	7 1/2
$\big(\{\{1\}\}, \{\{2,3\}\}\big)$	9	$\big(\{\{1\}, \{2\}, \{3\}\}, \emptyset\big)$	12
$\big(\{\{2\}\}, \{\{1\}, \{3\}\}\big)$	10	$\big(\{\{1\}, \{2,3\}\}, \emptyset\big)$	7 1/2
$\big(\{\{2\}\}, \{\{1,3\}\}\big)$	8	$\big(\{\{2\}, \{1,3\}\}, \emptyset\big)$	7 1/2
$\big(\{\{3\}\}, \{\{1\}, \{2\}\}\big)$	18	$\big(\{\{3\}, \{1,2\}\}, \emptyset\big)$	14 1/2
$\big(\{\{3\}\}, \{\{1,2\}\}\big)$	18 1/2	$\big(\{\{1,2,3\}\}, \emptyset\big)$	8 1/2

The highest value of the potential function is 18 1/2. Hence,

$$\pi^* = (\pi^1, \pi^2) = \big(\{\{3\}\}, \{\{1,2\}\}\big)$$

is a Nash-stable vector of coalition structures. Players' payoffs have the form

$$H_1(\pi^*) = 1\ 1/2, H_2(\pi^*) = 8\ 1/2, H_3(\pi^*) = 9.$$

Any player's payoff would not be augmented by the player's departure from the coalition they belong to.

A stable array of cooperative games can be found without finding the potential function maximum. Instead, one can map the sequence of the best responses and find its limit. The existence of a potential function guarantees that the sequence of best responses converges to a stable vector of coalition structures.

5 Axiomatic system

This section suggests the axiomatic system for a cooperative congestion game with the Shapley value. Denote by v_0 a cooperative game in which the worth of any coalition equals zero, i.e. $v_0(K) = 0 \; \forall K \subseteq N$. Let us list some axioms.

Decomposition (D). *Let* $s, s', s'' \in S$. *If* $s_i \setminus \{v_0\} = (s'_i \cup s''_i) \setminus \{v_0\} \; \forall i \in N$ *and* $\left(\bigcup_{i \in N} s'_i\right) \cap \left(\bigcup_{i \in N} s''\right) \in \{\emptyset, v_0\}$, *then*

$$h(s_1, s_2, ..., s_n) = h(s'_1, s'_2, ..., s'_n) + h(s''_1, s''_2, ..., s''_n) \; \forall i \in N.$$

The (D) axiom is an analog of the additivity property. This axiom implies that a player's payoff in one cooperative game has no effect on his/her payoff in another cooperative game. E.g., let $N = \{1, 2, 3\}$ and h possess the decomposition property, then

$$h(\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3, v_4\}) = h(\{v_1, v_2\}, \{v_1\}, \{v_2\}) + h(\{v_0\}, \{v_3\}, \{v_3, v_4\}).$$

Linearity* (L). *Let* $s \in S, s_i \in \{v, v_0\} \; \forall i \in N$ *and* $v = \alpha v_1 + \beta v_2$. *Then*

$$h(s) = \alpha h(s') + \beta h(s''),$$

where $s', s'' \in S$ *and* $(s_i = v) \Leftrightarrow (s'_i = v_1$ *and* $s''_i = v_2)$, $(s_i = v_0) \Leftrightarrow (s'_i = s''_i = v_0)$.

The (L) axiom is analogous to the linearity property. For example, let $N = \{1, 2, 3\}$ and h has the (L) property and $v = \alpha v_1 + \beta v_2$. Then,

$$h(\{v\}, \{v\}, \{v_0\}) = \alpha \cdot h(\{v_1\}, \{v_1\}, \{v_0\}) + \beta \cdot h(\{v_2\}, \{v_2\}, \{v_0\}).$$

Symmetry* (S). *Let* $s_i = s_k$ *and players* i, k *are symmetrical for any* $v, v \in s_i \cup s_k$, *i.e.* $v(K \cup \{i\}) = v(K \cup \{k\}) \; \forall K \subseteq N \setminus \{i, k\}$. *Then,*

$$h_i(s_i, s_k, s_{-i,k}) = h_k(s_i, s_k, s_{-i,k}) \; \forall s_{-i,k} \in S_{-i,k}.$$

If two players i, k have chosen the same games, and in each of the chosen games i and k are symmetrical, then they get identical payoffs.

Null Player* (NP). *Let* $\forall v \in s_i$ *player* i *is a null player for* v, *i.e.* $v(K \cup \{i\}) = v(K) \; \forall K \subseteq N \setminus \{i\}$. *Then,*

$$h_i(s_i, s_{-i}) = 0 \; \forall s_{-i} \in S_{-i}.$$

If a player is the null player in any chosen game, then his/her payoff is 0.

Efficiency* (E). *Let* $s = (s_1, s_2, ..., s_n) \in S$*, then*

$$\sum_{i \in N} h_i(s_1, s_2, ..., s_n) = \sum_{v \in \bigcup\limits_{i \in N} s_i} v(N_v(s)).$$

The sum of players' payoffs for each profile $s, s \in S$ is the sum of the values of the grand coalitions of the chosen games.

Theorem 3. *Let* V *be the set of all cooperative games for* N *players,* $S_i = 2^V$ $\forall i \in N$*. The value* $h : S \to R^n$ *satisfies the axioms (D), (L), (S), (NP), (E) iff*

$$h_i(s) = \sum_{v \in s_i} \sum_{\substack{K \subseteq N_v(s) \\ i \in K}} \frac{\lambda_K(v)}{|K|}.$$

See Appendix for proof.

The Shapley value on a set of cooperative games is determined uniquely by the classical additivity, null player, symmetry and efficiency axioms. As the cooperative congestion game is a normal-form game, the respective axioms are revised, and now only the payoff functions of the cooperative congestion game with the Shapley value satisfy these new revised axioms.

Conclusions

The new results expand the theory of congestion games. In the proposed cooperative congestion model, the resource is a game. Within the game $v, v \in V$, players can form a grand coalition or a coalition partition. The paper offers a transition from the strategic to the coalitional form of the game. With this transition, the definition of the potential function is revised for coalition partitions. The derived potential functions guarantee that an equilibrium exists in the cooperative congestion game with the Shapley value or the Banzhaf value.

Finding a stable vector of coalition structures is a complicated computational problem. The total number of partitions of the set N is $O(n^n)$. In order to reduce the complexity, one cooperative game can be decomposed into simpler cooperative games, where a stable partition is easier to find. Another option is to apply dynamic methods of solving discrete optimization problems, see Yeh (1986).

The potential game theory is quite advanced. The definition of a potential game is given according to the players' payoff functions. The broadest class of potential games has been described by Voorneveld (2000). Having formulated a suitable definition of the potential function, one can prove the existence of a stable coalition partition in cooperative congestion games.

There are weighting, continuous congestion games. The price of anarchy in such games has been studied. In the future, such notions can also be introduced for cooperative congestion games, and their properties can be investigated.

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Appendix

Proof of Theorem 1. Let $v \in V$. Let v be represented as a linear combination of unanimity games,

$$v = \sum_{\substack{K \subseteq N \\ K \neq \emptyset}} \lambda_K(v) \cdot v_K.$$

Considering propositions 1 and 2, the following sequence of equalities is true,

$$\varphi_i(v) = \varphi_i \left(\sum_{\substack{K \subseteq N \\ K \neq \emptyset}} \lambda_K(v) \cdot v_K \right) = \sum_{\substack{K \subseteq N \\ K \neq \emptyset}} \lambda_K(v) \cdot \varphi_i(v_K) = \sum_{\substack{K \subseteq N \\ i \in K}} f(K) \lambda_K(v).$$

Then, the payoff of the player $i, i \in N$ in the cooperative congestion game has the following form,

$$h_i(s) = \sum_{v \in s_i} \varphi_i(N_v(s), v) = \sum_{v \in s_i} \sum_{\substack{K \subseteq N_v(s) \\ i \in K}} f(K) \lambda_K(v).$$

Let us demonstrate that $P(s)$ is a potential function for the cooperative congestion game in question. Let us simplify the difference of the players' payoff functions,

$$h_i(s_i, s_{-i}) - h_i(s_i', s_{-i}) = \sum_{v \in s_i} \sum_{\substack{K \subseteq N_v(s_i, s_{-i}) \\ i \in K}} f(K) \lambda_K(v) - \sum_{v \in s_i'} \sum_{\substack{K \subseteq N_v(s_i', s_{-i}) \\ i \in K}} f(K) \lambda_K(v)$$

$$= \sum_{v \in s_i \setminus s_i'} \sum_{\substack{K \subseteq N_v(s_i, s_{-i}) \\ i \in K}} f(K) \lambda_K(v) - \sum_{v \in s_i' \setminus s_i} \sum_{\substack{K \subseteq N_v(s_i', s_{-i}) \\ i \in K}} f(K) \lambda_K(v).$$

Then, the following property is used to simplify the difference of potential functions. Let $s_i, s_i' \in S_i$. If $v \in s_i \setminus s_i'$, then $N_v(s_i', s_{-i}) = N_v(s_i, s_{-i}) \setminus \{i\}$ and if $v \in s_i' \setminus s_i$, then $N_v(s_i, s_{-i}) = N_v(s_i', s_{-i}) \setminus \{i\}$.

It is assumed in the following that $K \neq \emptyset$. In this case, the sequence of equalities is true,

$$P(s_i, s_{-i}) - P(s_i', s_{-i}) = \sum_{v \in s_i \cup \bigcup_{j \in N \setminus \{i\}} s_j} \sum_{K \subseteq N_v(s_i, s_{-i})} f(K) \lambda_K(v) - \sum_{v \in s_i' \cup \bigcup_{j \in N \setminus \{i\}} s_j} \sum_{K \subseteq N_v(s_i', s_{-i})} f(K) \lambda_K(v)$$

$$= \sum_{v \in s_i \setminus s_i'} \left(\sum_{K \subseteq N_v(s_i, s_{-i})} f(K) \lambda_K(v) - \sum_{K \subseteq N_v(s_i', s_{-i})} f(K) \lambda_K(v) \right)$$

$$- \sum_{v \in s_i' \setminus s_i} \left(\sum_{K \subseteq N_v(s_i', s_{-i})} f(K) \lambda_K(v) - \sum_{K \subseteq N_v(s_i, s_{-i})} f(K) \lambda_K(v) \right)$$

$$= \sum_{v \in s_i \setminus s_i'} \left(\sum_{K \subseteq N_v(s_i, s_{-i})} f(K) \lambda_K(v) - \sum_{K \subseteq N_v(s_i, s_{-i}) \setminus \{i\}} f(K) \lambda_K(v) \right)$$

$$\sum_{v \in s_i' \setminus s_i} \left(\sum_{K \subseteq N_v(s_i', s_{-i})} f(K)\lambda_K(v) - \sum_{K \subseteq N_v(s_i', s_{-i}) \setminus \{i\}} f(K)\lambda_K(v) \right)$$

$$= \sum_{v \in s_i \setminus s_i'} \sum_{\substack{K \subseteq N_v(s_i, s_{-i}) \\ i \in K}} f(K)\lambda_K(v) - \sum_{v \in s_i' \setminus s_i} \sum_{\substack{K \subseteq N_v(s_i', s_{-i}) \\ i \in K}} f(K)\lambda_K(v)$$

$$= h_i(s_i, s_{-i}) - h_i(s_i', s_{-i}) \ \forall s_i, s_i' \in S_i, \ \forall s_{-i} \in S_{-i}.$$

Hence, $P(s)$ is a potential function for the given cooperative congestion game.

□

Proof of Corollary 1. Let φ be the Shapley value. Then φ possesses the linearity property, and $\varphi_i(v_T) = \begin{cases} \frac{1}{|T|}, & i \in T; \\ 0, & i \notin T. \end{cases}$. The conditions of Theorem 1 are fulfilled, so the assertion being proved is true. The payoff of the player i in the cooperative congestion game with the Shapley value, and the potential function have the following form,

$$h_i(s) = \sum_{v \in s_i} \sum_{\substack{K \subseteq N_v(s) \\ i \in K}} \frac{\lambda_K(v)}{|K|}, \quad P(s) = \sum_{v \in \bigcup\limits_{i \in N} s_i} \sum_{\substack{K \subseteq N_v(s) \\ K \neq \emptyset}} \frac{\lambda_K(v)}{|K|}$$

□

Proof of Corollary 2. Let φ be the Banzhaf value. Then φ possesses the linearity property, and $\varphi_i(v_T) = \begin{cases} \frac{1}{2^{|T|-1}}, & i \in T; \\ 0, & i \notin T. \end{cases}$. The conditions of Theorem 1 are fulfilled, so the assertion being proved is true. The payoff of the player i in the cooperative congestion game with the Banzhaf value, and the potential function have the following form,

$$h_i(s) = \sum_{v \in s_i} \sum_{\substack{K \subseteq N_v(s) \\ i \in K}} \frac{\lambda_K(v)}{2^{|K|-1}}, \quad P(s) = \sum_{v \in \bigcup\limits_{i \in N} s_i} \sum_{\substack{K \subseteq N_v(s) \\ K \neq \emptyset}} \frac{\lambda_K(v)}{2^{|K|-1}}$$

□

Proof of Proposition 1. Consider a cooperative congestion game with the Shapley value, $V = \{v_1, v_2, ..., v_m\}, v_j(K) = |K| \cdot c_j(|K|), j \in \{1, 2, ..., m\}$. Denote by $I(s_i), i \in N$ the set of indexes of cooperative games from s_i. E.g., if $s_i = \{v_2, v_3, v_4\}$, then $I(s_i) = \{2, 3, 4\}$. All players in any cooperative game from V are symmetrical. In this case, the value $h_i(s)$ can be simplified as follows,

$$h_i(s) = \sum_{v \in s_i} \varphi_i(N_v(s), v) = \sum_{j \in I(s_i)} c_j(|N_{v_j}(s)|) = \sum_{j \in I(s_i)} c_j(k_j(s)).$$

Hence, any congestion game can be represented as a cooperative congestion game with a special form of characteristic functions.

□

Proof of Theorem 2. To prove the theorem, we introduce the definition of a potential cooperative congestion game with a vector of coalition structures. The game (N, V, X, H) is called a potential game if there exists a potential function $P : X \to R$ and $\forall i \in N, \forall \pi \in X, \forall \pi^j \in Y_i(\pi)$ the following equalities are valid,

$$H_i(\pi^j, \pi_{-j}) - H_i(\rho^j, \pi_{-j}) = P(\pi^j, \pi_{-j}) - P(\rho^j, \pi_{-j}) \ \forall \rho^j \in D_i(\pi^j),$$

$$H_i(\pi) - H_i(\rho) = P(\pi) - P(\rho) \ \forall \rho \in M_i(\pi).$$

.

The vector of coalition structures that maximizes the potential function is Nash-stable. Since φ satisfies the conditions of Theorem 1,

$$H_i(\pi) = \sum_{\pi^j \in Y_i(\pi)} \sum_{\substack{K \subseteq \pi^j(i) \\ i \in K}} f(K)\lambda_K(v_j), i \in N.$$

We demonstrate that for the game in question $P(\pi)$ is the potential function,

$$P(\pi) = \sum_{j=1}^{m} \sum_{B \in \pi^j} \sum_{\substack{K \subseteq B \\ K \neq \emptyset}} f(K)\lambda_K(v_j).$$

Let $\pi \in X, \pi = (\pi^j, \pi_{-j})$ and $\rho^j \in D_i(\pi^j), \rho^j = \{\pi^j(i) \setminus \{i\}, A \cup \{i\}, \pi^j_{-\pi^j(i),A}\}$. Then,

$$H_i(\pi^j, \pi_{-j}) - H_i(\rho^j, \pi_{-j})$$

$$= \left(\sum_{\substack{K \subseteq \pi^j(i) \\ i \in K}} f(K)\lambda_K(v_j) + \sum_{\pi^g \in Y_i(\pi) \setminus \{\pi^j\}} \sum_{\substack{K \subseteq \pi^g(i) \\ i \in K}} f(K)\lambda_K(v_g) \right)$$

$$- \left(\sum_{\substack{K \subseteq A \cup \{i\} \\ i \in K}} f(K)\lambda_K(v_j) + \sum_{\pi^g \in Y_i(\pi) \setminus \{\pi^j\}} \sum_{\substack{K \subseteq \pi^g(i) \\ i \in K}} f(K)\lambda_K(v_g) \right)$$

$$= \sum_{\substack{K \subseteq \pi^j(i) \\ i \in K}} f(K)\lambda_K(v_j) - \sum_{\substack{K \subseteq A \cup \{i\} \\ i \in K}} f(K)\lambda_K(v_j).$$

Simplifying the difference of potential functions,

$$P(\pi) - P(\rho)$$

$$= \left(\sum_{B \in \pi^j} \sum_{\substack{K \subseteq B \\ K \neq \emptyset}} f(K)\lambda_K(v_j) + \sum_{\substack{g=1 \\ g \neq j}}^{m} \sum_{B \in \pi^g} \sum_{\substack{K \subseteq B \\ K \neq \emptyset}} f(K)\lambda_K(v_g) \right)$$

$$-\left(\sum_{B\in\rho^j}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_j)+\sum_{\substack{g=1\\g\neq j}}^m\sum_{B\in\pi^g}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_g)\right)$$

$$=\left(\sum_{\substack{K\subseteq\pi^j(i)\\K\neq\emptyset}} f(K)\lambda_K(v_j)+\sum_{\substack{K\subseteq A\\K\neq\emptyset}} f(K)\lambda_K(v_j)+\sum_{B\in\pi^j_{-\pi^j(i),A}}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_g)\right)$$

$$-\left(\sum_{\substack{K\subseteq\pi^j(i)\setminus\{i\}\\K\neq\emptyset}} f(K)\lambda_K(v_j)+\sum_{\substack{K\subseteq A\cup\{i\}\\K\neq\emptyset}} f(K)\lambda_K(v_j)+\sum_{B\in\pi^j_{-\pi^j(i),A}}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_g)\right)$$

$$=\left(\sum_{\substack{K\subseteq\pi^j(i)\\K\neq\emptyset}} f(K)\lambda_K(v_j)-\sum_{\substack{K\subseteq\pi^j(i)\setminus\{i\}\\K\neq\emptyset}} f(K)\lambda_K(v_j)\right)$$

$$-\left(\sum_{\substack{K\subseteq A\cup\{i\}\\K\neq\emptyset}} f(K)\lambda_K(v_j)-\sum_{\substack{K\subseteq A\\K\neq\emptyset}} f(K)\lambda_K(v_j)\right)$$

$$=\sum_{\substack{K\subseteq\pi^j(i)\\i\in K}} f(K)\lambda_K(v_j)-\sum_{\substack{K\subseteq A\cup\{i\}\\i\in K}} f(K)\lambda_K(v_j)=H_i(\pi^j,\pi_{-j})-H_i(\rho^j,\pi_{-j}).$$

Now let $\pi\in X,\pi=(\pi^j,\pi^l,\pi_{-j,l}),\left\{\pi^j-i,\{A\cup\{i\},\pi^l_{-A}\},\pi_{-j,l}\right\}$. The difference of payoff functions has the following form,

$$H_i(\pi)-H_i(\rho)=\sum_{\substack{K\subseteq\pi^j(i)\\i\in K}} f(K)\lambda_K(v_j)-\sum_{\substack{K\subseteq A\cup\{i\}\\i\in K}} f(K)\lambda_K(v_l).$$

We transform the difference of potential functions,

$$P(\pi)-P(\rho)$$

$$=\left(\sum_{B\in\pi^j}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_j)+\sum_{B\in\pi^l}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_l)+\sum_{\substack{g=1\\g\neq j,l}}^m\sum_{B\in\pi^g}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_g)\right)$$

$$-\left(\sum_{B\in\pi^j-i}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_j)+\sum_{B\in\{A\cup\{i\},\pi^l_{-A}\}}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_l)+\sum_{\substack{g=1\\g\neq j,l}}^m\sum_{B\in\pi^g}\sum_{\substack{K\subseteq B\\K\neq\emptyset}} f(K)\lambda_K(v_g)\right)$$

$$=\left(\sum_{\substack{K\subseteq\pi^j(i)\\K\neq\emptyset}} f(K)\lambda_K(v_j)-\sum_{\substack{K\subseteq\pi^j(i)\setminus\{i\}\\K\neq\emptyset}} f(K)\lambda_K(v_j)\right)$$

$$-\left(\sum_{\substack{K\subseteq A\cup\{i\}\\ K\neq\emptyset}} f(K)\lambda_K(v_l)-\sum_{\substack{K\subseteq A\\ K\neq\emptyset}} f(K)\lambda_K(v_l)\right)$$

$$=\sum_{\substack{K\subseteq\pi^j(i)\\ i\in K}} f(K)\lambda_K(v_j)-\sum_{\substack{K\subseteq A\cup\{i\}\\ i\in K}} f(K)\lambda_K(v_l)=H_i(\pi)-H_i(\rho)$$

By definition, P is a potential function, hence a stable vector of coalition structures exists in the game. □

Proof of Theorem 3. The payoff of a player in a cooperative congestion game with the Shapley value satisfies the axioms of the theorem. Let us demonstrate that the axioms uniquely define the players' payoffs. Let$s\in S$. Taking into account the axiom (D), the value $h_i(s)$ can be transformed as follows,

$$h_i(s)=\sum_{v\in\bigcup\limits_{i\in N}s_i} h_i(s_1^v,s_2^v,...,s_n^v),$$

where

$$s_i^v=\begin{cases}\{v\} & v\in s_i;\\ \{v_0\} & v\notin s_i.\end{cases}$$

We represent v as $v=\sum_{\substack{K\subseteq N\\ K\neq\emptyset}}\lambda_K(v)v_K$. Applying axiom (L), the value of $h_i(s_1^v,s_2^v,...,s_n^v)$ has the form

$$h_i(s_1^v,s_2^v,...,s_n^v)=\sum_{\substack{K\subseteq N\\ S\neq\emptyset}}\lambda_K(v)\cdot h_i(s_1^{v_K},s_2^{v_K},...,s_n^{v_K}),$$

where

$$s_i^{v_K}=\begin{cases}\{v_K\} & v\in s_i;\\ \{v_0\} & v\notin s_i.\end{cases}$$

Given the axioms (NP), (S), (E), the value $h_i(s_1^{v_K},s_2^{v_K},...,s_n^{v_K})$ is calculated as follows,

$$h_i(s_1^{v_K},s_2^{v_K},...,s_n^{v_K})=\begin{cases}\frac{1}{|K|} & i\in K, v\in s_i;\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, the following sequence of equations is true,

$$h_i(s)=\sum_{v\in\bigcup\limits_{i\in N}s_i} h_i(s_1^v,s_2^v,...,s_n^v)=\sum_{v\in\bigcup\limits_{i\in N}s_i}\sum_{\substack{K\subseteq N\\ S\neq\emptyset}}\lambda_K(v)\cdot h_i(s_1^{v_K},s_2^{v_K},...,s_n^{v_K})$$

$$=\sum_{v\in s_i}\sum_{\substack{K\subseteq N_v(s)\\ i\in K}}\frac{\lambda_K(v)}{|K|}$$

The game v can be represented in the form of a linear combination of unanimity games uniquely. The value $h_i(s_1^{v_K},s_2^{v_K},...,s_n^{v_K})$ $\forall i\in N$ $\forall K\subseteq N, K\neq\emptyset$ is also defined uniquely.

Hence, only payoffs of players of a cooperative congestion game with the Shapley value satisfy the above axioms.

□

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