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Set-weighted games and their application to the cover problem

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Set-weighted games and their application to the cover problem

The cover of a transport, social, or communication network is a computationally complex problem. To deal with it, this paper introduces a special class of simple games in which the set of minimal winning coalitions coincides with the set of least covers. A distinctive feature of such a game is that it has a weighted form, in which weights and quota are sets rather than real numbers. This game class is termed set-weighted games. A real-life network has a large number of least covers, therefore this paper develops methods for analyzing set-weighted games in which the weighted form is taken into account. The necessary and sufficient conditions for a simple game to be a set-weighted game were found. The vertex cover game (Gusev, 2020) was shown to belong to the set-weighted game class, and its weighted form was found. The set-weighted game class has proven to be closed under operations of union and intersection, which is not the case for weighted games. The sample object is the transport network of a district in Petrozavodsk, Russia. A method is suggested for efficiently deploying surveillance cameras at crossroads so that all transport network covers are taken into account.

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1 Introduction

1.1 The cover problem

The cover problem is a classical issue in many areas of science, and is formulated as follows. Suppose there exists a finite set of objects U and a family of its subsets $S = \{s_1, s_2, ..., s_n\}$. The task is to find a set $K \subseteq N, N = \{1, 2, ..., n\}$ such that $U \subseteq \cup_{i \in K} s_i$ and K consist of the least number of elements. The set K is called a minimum cover of the set U. We know that finding K is an NP-complete problem. Let $U = \{a, b, c, d, e\}, s_1 = \{a, b\}, s_2 = \{b, c, d\}, s_3 = \{c, e\}, s_4 = \{a, d\}$. Then, we have several minimum covers $K_1 = \{1, 2, 3\}, K_2 = \{1, 3, 4\}, K_3 = \{2, 3, 4\}$.

Let us list some applications of the cover problem:

- Staff recruitment. Let N be the set of agents and U be the set of tasks. The set $s_i, i \in N$ are the tasks that agent i is able to fulfill. There is a manager who runs job interviews with agents. The manager needs to choose agents from N to ensure that all tasks from U are fulfilled. Remuneration is the same to each agent. In this case, it is more profitable to hire agents from a minimum cover, since all tasks will be fulfilled by a minimal number of agents.

- Information systems. Let $N = U$ be a set of data points. The set $s_i, i \in N$ consists of the data points to which i can send a message. It takes one step for information from point i to reach all points in s_i. The manager needs to notify all points in N of an event. Personally transmitting the information to each individual point in N may take a long time. The efficient option in this case would be to inform only the points constituting a minimum cover, which ensures that the message reaches all data points in N in one step.

- Transport networks. Let N be the set of intersections, U be the set of roads, and s_i be the set of roads running through intersection $i, i \in N$. If surveillance cameras are deployed at intersections from the minimum cover, then all roads with be monitored and the number of cameras will be minimized.

- Social networks. Let N be the set of agents, $U \subseteq \{\{i, j\} | i, j \in N\}$. A conflict exists between two different agents i and j iff $\{i, j\} \in U$. The set s_i consists of conflicting agent pairs, implying that agent i is a party in the conflict. Removing agents from a minimum cover, we are left with a conflict-free group with a greatest possible number of members.

The classical cover problem is defined by the triplet (N, U, S). Solving this problem means finding a minimum cover. Yet, a minimum cover can be inferior to a non-minimum cover in some ways, for example, if new roads or intersections appear in the transport network, then a minimum cover is more likely to stop being a cover. The same applies to other real-life situations. A minimum cover is not stable to a change in U or S. Decision-making based on minimum cover alone may have negative implications for the future. If an upcoming change is known in advance, the minimum as well as other covers should be taken into account.

This paper introduces a simple cooperative game (N, v), which takes into account all covers of the set U. The value of the coalition $K, K \subseteq N$ is 1 if K is a cover of the set U, i.e. $U \subseteq \cup_{i \in K} s_i$. If K is not a cover, then the value of the coalition is 0. The set s_i is the weight of player $i, i \in N$, and U is the quota. The simple cooperative game (N, v) is called a set-weighted game.

Consider the distribution problem of surveillance cameras on a transport network. It is known that the cameras are ranked by quality. Usually, surveillance cameras are located at every intersection of the main roads of the city's transport network. If we have the number of cameras equal to the number of intersections, then at which intersection should we put the best quality camera? Finding the minimum coverage will not be sufficient to answer this question, because the cameras are ranked and distributed to all intersections. For the distribution problem of cameras on a network, we use the classical idea of the centrality of vertices in a graph [21]. All intersections can be ranked relative to some measure of centrality. Next, the rankings of intersections and cameras are compared and we get a solution. The question arises as to how to construct a measure of centrality. It is known that the cameras should cover the entire transport network. If some cameras stop working, other cameras can compensate for them. This is due to the fact that the section of road between two intersections is viewed by cameras installed at these intersections. Therefore, the centrality measure should take into account the coverage of the transport network. Earlier, we verbally defined a set-weighted cooperative game that takes into account all coverages of a given set. Hence, the centrality measure of vertices can be defined as the power index of the set-weighted game constructed for the transport network under consideration.

Power indices are axiomatized in the literature. If the axiom system has its own interpretation in the set-weighted game, then the index can be used as a measure of centrality to solve the cover problem. For example, the symmetry axiom in the set-weighted game says that vertices are symmetric in all coverings. In such vertexes, we can put cameras of the same quality. A similar interpretation can be given to other axioms. It may turn out that some power index of the set-weighted game is identical to some numerical characteristic of the graph that we want to calculate.

The paper shows that the Shapley-Shubik and Banzaf-Coleman indices have their own interpretation in the set-weighted game. For a transport network, a vertex cover game is constructed and the vertex rankings relative to the Banzaf-Coleman index are found. The new ranking is compared with the previously obtained vertex ranking relative to the Shapley-Shubik index of the same transport network. It turned out that the rankings are almost identical.

1.2 Literature review

The cover problem occurs in many areas of science, and both its theoretical features and practical applications are of interest. This review deals with the cover problem and the game-theoretic methods for solving it. My search for literature on set-weighted games was fruitless. This is because researchers are studying games with numerical player weights.

We know that finding a minimum cover in the general case is an NP-complete problem. NP-completeness is a feature of many related problems. As demonstrated in [27], the search for a connected vertex cover for some regular graphs is NP-complete. The large number of computing operations calls for approximated methods of finding optimal covers, for example, minimum vertex cover approximation methods, studied in [3].

The principal application of minimum covers is related to the allocation of resources and objects [22], and to network covers [9, 38]. A cover model taking into account distances between objects was suggested in [34]. [32] examines a cover problem with hybrid uncertainty, namely randomness and fuzziness.

The resource allocation process can involve agents with individual goals. The presence of agents enables the application of game-theoretic models for solving the cover problem. For a non-cooperative game, the main goal is to find an equilibrium. [6, 20] introduce a class of non-cooperative games dealing with combinatorial cover problems. The game is solved by integer programming methods. The price of anarchy in such games was studied.

Cooperative cover games were suggested in [10]. The existence of c-core conditions and a balanced state of a vertex cover game from this class were studied in [11, 35]. The characteristic function in this class of cooperative games depends on the graph. The value of a coalition can be equal to the flux between vertices, number of arcs, and so on.

[19] introduces a simple cooperative vertex cover game, in which the set of minimal winning coalitions coincides with the set of the graph's least vertex covers. The decomposition theorem was proved, permitting the game to be represented in an analysis-friendly form. The necessary and sufficient conditions for a simple game to be a vertex cover game for a graph were found. The cooperative game from [19] is a particular case of the game explored here.

Simple cooperative games can be used to model corporate networks [8], social systems [31], to solve the secretary problem [28]. The fact of a simple game having a weighted form permits using generating functions to calculate some power indexes [1, 2, 7]. Probability methods and binary diagrams for calculating the indexes were suggested in [25] and [4], respectively. The simple game class studied here also has a weighted form, but players' weights are sets rather than real numbers. It is demonstrated below that the weighted game and set-weighted game classes intersect but do not coincide.

1.3 Main results

This study produced the following major results:

- The necessary and sufficient conditions for a simple game to be a set-weighted game were found (Theorem 1).
- The set-weighted game class was proven to be closed under operations of union and intersection (Theorem 2).
- The decomposition lemma was proved, permitting the set-weighted game to be represented in the form of an intersection of simpler set-weighted games (Lemma 1).
- The vertex cover game from [19] was shown to be a set-weighted game. Its weighted form was constructed (Proposition 3).
- Cooperative generating functions were found for calculating the number of swings and the Shapley-Shubik index in a set-weighted game (Propositions 9 and 10).

The necessary and sufficient conditions for a simple game to be a weighted game were found in [37]. The proof is based on the enumeration of sequences consisting of the winning and the losing coalitions. Theorem 1 employs a different approach. A simple game is set-weighted iff it can be represented in the form of an intersection of special simple games. This form is called the canonical form. The canonical form has a physical meaning, and is used to prove some statements.

Theorem 2 states that a union and an intersection of set-weighted games is a set-weighted game. This result is important for the transformation of games and the calculation of some indexes.

If a vertex cover game is constructed for a transport network, there will be a large number of minimal winning coalitions, and finding them is a computationally complex problem. Proposition 3 states that there is no need to find all the least covers. To formulate the game, it suffices to determine the players' weights and the set covered. Knowing the weighted form of the game and applying the decomposition lemma, one can analytically calculate the power indexes (see e.g. Propositions 7 and 8).

Power indexes are estimated using generating functions. This paper introduces a definition of the cooperative generating function which takes into account the fact that players' weights are sets. Propositions 9 and 10 find the cooperative generating functions for some classical indexes.

1.4 The article structure

Section 2 gives the key notations and definitions. Section 3 introduces the set-weighted game and studies its properties. Section 4 demonstrates that simple cover games are a subclass of set-weighted games. Section 5 describes a game with singleton weights and calculates some power indexes. Section 6 introduces the definition of the cooperative generating function. The cooperative generating functions are found for some classical indexes. Section 7 demonstrates the efficiency of surveillance camera distribution over a transport network in proportion to the Banzhaf-Coleman index for the vertex cover game. The Banzhaf-Coleman index was calculated for a specific network and compared with the previously determined Shapley-Shubik index. Section 8 describes the main conclusions. Novel theoretical results regarding set-weighted games are given in Table 1. The Appendix provides proofs of propositions.

2 Basic definitions of cooperative game theory

Let $N = \{1, 2, ..., n\}$ be a set of players and 2^N be the set of all subsets of the set N. Consider a cooperative game (N, v), where v is a characteristic function, $v : 2^N \to R, v(\emptyset) = 0$. A game (N, v) is a simple game when 1. $\forall S \subseteq N : v(S) = 0$ or $v(S) = 1$; 2. $v(N) = 1$; 3. $\forall S, T \subseteq N : S \subseteq T \Rightarrow v(S) \leq v(T)$. A characteristic function v is called superadditive if $\forall K, L \subseteq N, K \cap L = \emptyset : v(K \cup L) \geq v(K) + v(L)$.

A coalition S is winning if $v(S) = 1$ and losing otherwise. The set of winning coalitions is denoted by $W(v)$. K is called a minimal winning coalition if $v(K) = 1$ and $\forall i \in K : v(K \setminus \{i\}) = 0$. The set of minimal winning coalitions is denoted by $W^m(v)$. A pair (N, v_S) is an unanimity game if $W^m(v_S) = \{S\}$.

The *union* (*intersection*) of the simple games (N, v) and (N, w) is the game $(N, v \vee w)$ ($(N, v \wedge w)$) in which the set of winning coalitions is the union (intersection) of the sets of winning coalitions for (N, v) and (N, w).

Let φ be the value of the simple game. The value φ has the transfer property if

$$\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w).$$

A simple game (N, v) is a weighted game if there are non-negative real numbers $[q; w_1, ..., w_n]$ such that

$$v(K) = \begin{cases} 1, & \sum_{i \in K} w_i \geq q \\ 0, & \sum_{i \in K} w_i < q \end{cases}, \forall K \subseteq N.$$

The number w_i is the weight of player $i, i \in N$, and q is the quota.

The Shapley-Shubik index of the player $i, i \in N$ in a simple game (N, v) is calculated as:

$$\phi_i(v) = \sum_{\substack{K \in W(v): \\ K \setminus \{i\} \notin W(v)}} \frac{(|K|-1)!(|N|-|K|)!}{|N|!} \text{ [12].}$$

A swing for player i is a pair of coalitions $(S, S \cup \{i\})$ such that $S \cup i$ is winning and S is not. Denote by $\eta_i(v)$ the number of swings for $i, i \in N$ in the game (N, v).

The Banzaf-Coleman index $\delta_i(v)$ for player i in a simple game (N, v) is defined as:

$$\delta_i(v) = \frac{\eta_i(v)}{\sum_{i \in N} \eta_i(v)} \text{ [33].}$$

3 Set-weighted games

3.1 Game definition

Let $N = \{1, 2, ..., n\}$ be the set of players and $U, s_1, s_2, ..., s_n$ be the set of objects, supposing $U \subseteq \bigcup_{i \in N} s_i$. Let us define a set-weighted game.

Definition 1. *The set-weighted game is a simple game (N, v) for which there exists an array of sets $[U; s_1, s_2, ..., s_n]$ such that*

$$v(K) = \begin{cases} 1, & U \subseteq \bigcup_{i \in K} s_i; \\ 0, & \text{otherwise.} \end{cases}$$

The array of sets $[U; s_1, s_2, ..., s_n]$ will be called the weighted form of a set-weighted game (N, v).

The characteristic function of a set-weighted game is monotone, and takes the values 0 and 1. Since $U \subseteq \bigcup_{i \in N} s_i$, then $v(N) = 1$. Let $SW(N)$ denote the space of all set-weighted games with players from N.

The coalition K is a winning one in a set-weighted game iff K is a cover of the set U. If the set U is made up of many elements, then finding all the covers and their analysis is computationally challenging. However, the weighted form of a set-weighted game carries information about all covers of the set U. To analyze all the covers it suffices to work out the methods to analyze the weighted form.

Weights in a weighted game being real numbers, the players can be arranged by their weights. If $w_i > w_j$, then player i is said to be more powerful than player j. Such a comparison of weights is impossible in a set-weighted game. For example, let $N = \{1, 2, 3\}$ and $[\{a, b, c\}; \{a\}, \{a, b\}, \{c\}]$ be the weighted form of a set-weighted game. The weights of players 1 and 3, 2 and 3 in such a game cannot be compared without extra information. Regarding the operation of set inclusion, one can say that player 2 is more powerful than player 1.

Let me give an example of a simple game that is not a weighted game but is a set-weighted game. This is the game $(N, v), N = \{1, 2, 3, 4\}, W^m(v) = \{\{1, 2\}, \{3, 4\}\}$. Suppose $[q; w_1, w_2, w_3, w_4]$ is the weighted form of the (N, v). Then we have a contradiction,

$$\{1, 2\}, \{3, 4\} \in W(v) \Rightarrow w_1 + w_2 \geq q, w_3 + w_4 \geq q \Rightarrow \sum_{i \in N} w_i \geq 2q,$$

$$\{1, 3\}, \{2, 4\} \notin W(v) \Rightarrow w_1 + w_3 < q, w_2 + w_4 < q \Rightarrow \sum_{i \in N} w_i < 2q.$$

Therefore, this game has no real weights for players and a quota. The game, however, does have a weighted form with players' weights in the form of sets,

$$[U; s_1, s_2, s_3, s_4],$$

$$U = \{a, b, c, d\}, s_1 = \{a, b\}, s_2 = \{c, d\}, s_3 = \{a, d\}, s_4 = \{b, c\}.$$

In the following, we are interested in the properties of set-weighted games and their applications. The next section finds the necessary and sufficient conditions for a simple game to be a set-weighted game.

3.2 The decomposition of set-weighted games

If the number of minimal winning coalitions in a simple game is large, this can complicate its analysis. Using the operations of union and intersection, the original game can be decomposed into simpler games. This will reduce the number of minimal winning coalitions or simplify the original game. Consider an example. Let (N, v) and (N, w) be simple games, and $N = N_v \cup N_w, N_v \cap N_w = \emptyset$. Suppose the following conditions hold:

$$\forall A \in W^m(v)\ \forall i \in N_w : i \notin A,$$

$$\forall A \in W^m(w)\ \forall i \in N_v : i \notin A.$$

Such conditions indicate that players from the sets N_v, N_w are null players in the games w, v, respectively. Denote $|W^m(v)| = a, |W^m(w)| = b$. Then, $|W^m(v \vee w)| = a + b, |W^m(v \wedge w)| = a \cdot b$. Since the identity $v \wedge w = v + w - v \vee w$ holds for simple games, the analysis of the game (N, v) can be reduced to a parallel analysis of the games $(N, v), (N, w), (N, v \vee w)$. The total number of minimal winning coalitions in the three new games is $2(a + b)$. If $ab > 2(a + b)$, then the decomposition is efficient.

Lemma 1 shows how a set-weighted game can be represented as an intersection of simpler set-weighted games.

Lemma 1. *Let (N, v) be a set-weighted game with the weighted form $[U; s_1, s_2, ..., s_n]$. The set U is expressed in the form $U = U_1 \cup U_2 \cup ... \cup U_r, U_j \neq \emptyset \; \forall j \in \{1, 2, ..., r\}$. Let (N, v_j) denote a set-weighted game with the weighted form $[U_j; s_1, s_2, ..., s_n]$. Then*

$$v = v_1 \wedge v_2 \wedge ... \wedge v_r.$$

The proof is in the Appendix.

Suppose we need to reduce the number of minimal winning coalitions in a set-weighted game. One of the ways to solve this problem is to apply Lemma 1. It is convenient to express the set U as a union $\cup_{j=1}^{m} U_j$. For example, U_1 can coincide with the weight of player i, i.e. $U_1 = s_i$. Then the game $[s_i; s_1, s_2, ..., s_n]$ will have a minimal winning coalition $\{i\}$. Other minimal winning coalitions cannot contain the player i. Depending on the players' weights and on U, other decomposition methods are possible.

Lemma 1 is used to prove the following theorem.

Theorem 1. *The simple game (N, v) is a set-weighted game iff there exist simple games $(N, v_1), (N, v_2), ..., (N, v_r)$ for which*

$$v = v_1 \wedge v_2 \wedge ... \wedge v_r$$

holds and any minimal winning coalition of the game $(N, v_j), j = 1, 2, ..., r$ consists of one player, i.e. $\forall j \in \{1, 2, ..., r\} \; \forall A \in W^m(v_j) : |A| = 1$.

The proof is in the Appendix.

Let us clarify the meaning of the functions $v_j, j = 1, 2, ..., r$ described in Theorem 1. Suppose U is the set of objects to be covered. Note that each minimal winning coalition in the game v_j consists of one player. Hence, $W^m(v_j)$ consists of the players that can cover the object $j, j \in U$.

A set-weighted game can be set simply by defining U and the players' weights. Theorem 1 shows another way of introducing a set-weighted game. Let there be an array of simple games, supposing that a minimal winning coalition in any game consists of one player. Then an intersection of such games is a set-weighted game. Theorem 1 permits a set-weighted game to be introduced without knowing the players' weights and the set U in advance.

There is a variety of ways to represent the set U in the form of a union of sets. Each such representation has a matching decomposition of the characteristic function v in the form of an intersection of simpler characteristic functions. Let us find the special decomposition among all such representations and call it the canonical form.

Definition 2. *Let $[U; s_1, s_2, ..., s_n]$ be the weighted form of the set-weighted game (N, v). The canonical form of the game (N, v) is*

$$v = \bigwedge_{x \in U} v_x,$$

where $(N, v_x), x \in U$ is a simple game and any minimal winning coalition of the game (N, v_x) consists of one player.

It follows from the proof of Theorem 1 that any set-weighted game has a canonical form. In view of Definition 2, a simple game can be said to be set-weighted iff it has a canonical form. The following example shows how to find the canonical form knowing the players' weights and the set U.

Example 1. Let $N = \{1, 2, ..., 6\}$ and $[U; s_1, s_2, s_3, s_4, s_5, s_6]$

$$= [\{a, b, c, d, e\}; \{a, b, c\}, \{a, d, c\}, \{a, b, e\}, \{b, d\}, \{c, e\}, \{d, e\}]$$

be the weighted form of the game (N, v). The set of minimal winning coalitions has the form

$$W^m(v) = \{\{1, 6\}, \{2, 3\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 4, 5\}, \{2, 4, 5\}, \{2, 4, 6\}, \{3, 4, 5\}\}.$$

Let us demonstrate the way to express a set-weighted game in the canonical form. Consider the simple games $(N, v_j), j \in U, W^m(v_j) = \{\{i\} | i \in N, j \in s_i\}$. Player i forms a minimal winning coalition $\{i\}$ in the game (N, v_j) if $j \in s_i$. Then

$$W^m(v_a) = \{\{1\}, \{2\}, \{3\}\}, W^m(v_b) = \{\{1\}, \{3\}, \{4\}\}, W^m(v_c) = \{\{1\}, \{2\}, \{5\}\},$$

$$W^m(v_d) = \{\{2\}, \{4\}, \{6\}\}, W^m(v_e) = \{\{3\}, \{5\}, \{6\}\}.$$

The canonical form of the set-weighted game considered here has the form

$$v = \bigwedge_{x \in U} v_x = v_a \wedge v_b \wedge v_c \wedge v_d \wedge v_e.$$

Knowing the players' weights and U, it is always possible to find the set of minimal winning coalitions and write down the canonical form of the game. Now consider the inverse problem. Let the set of minimal winning coalitions be defined, and the task is to determine whether the game is set-weighted. The following example offers a solution for this problem.

Example 2. Let $N = \{1, 2, 3, 4, 5\}$ and the set of minimal winning coalitions of the simple game (N, v) have the form

$$W^m(v) = \{\{1, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}.$$

The task is to determine whether the game is set-weighted. We assign a Boolean variable x_i to each player $i, i \in N$ and define the function

$$f(W^m(v), x) = \underset{A \in W^m(v)}{\vee} \left(\underset{i \in A}{\wedge} x_i \right).$$

In this case we get

$$f(W^m(v), x) = (x_1 \wedge x_4) \vee (x_1 \wedge x_2 \wedge x_5) \vee (x_1 \wedge x_3 \wedge x_4) \vee (x_1 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4) \vee (x_2 \wedge x_3 \wedge x_5).$$

We write down the minimal conjunctive normal form (MCNF) for the Boolean function f:

$$f(W^m(v), x) = (x_1 \vee x_2) \wedge (x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_3) \wedge (x_4 \vee x_5).$$

The MCNF has four multipliers, so let U consist of four elements, $U = \{a, b, c, d\}$. Each multiplier is paired with an element from U. They are a for $(x_1 \vee x_2)$, b for $(x_2 \vee x_3 \vee x_4)$, etc. We introduce the simple games $(N, v_j), j \in U$. If the variable x_i belongs to the multiplier with the element $j, j \in U$, then $\{i\} \in W^m(v_j)$. The sets $W^m(v_j), j = a, b, c, d$ will take the following form:

$$W^m(v_a) = \{\{1\}, \{2\}\}, W^m(v_b) = \{\{2\}, \{3\}, \{4\}\}, W^m(v_c) = \{\{1\}, \{3\}\}, W^m(v_d) = \{\{4\}, \{5\}\}.$$

Considering the MCNF, the canonical form of the simple game is $v = v_a \wedge v_b \wedge v_c \wedge v_d$. It follows from the proof of Theorem 1 that the players' weights can be written down as $s_i = \{j | j \in U, \{i\} \in W^m(v_j)\}$. Hence,

$$[\{a, b, c, d\}; \{a, c\}, \{a, b\}, \{b, c\}, \{b, d\}, \{d\}]$$

is the weighted form of the game (N, v). This method of finding weights from a given set of minimal winning coalitions is applicable only if each Boolean variable in the MCNF is written down without negation. The methods of constructing Boolean functions can be found in [26, 30].

3.3 Completeness and dimensionality

Let us consider the preference ratio between players in a simple game. We write $i \succ j$ if $v(S \cup \{i\}) \geq v(S \cup \{j\})\ \forall S \subseteq N \setminus \{i, j\}$. The game (N, v) is said to be complete if for any two players $i, j \in N, i \neq j$ $i \succ j$ or $j \succ i$ is true. If a simple game is complete, then the parametrization theorem applies [15]. We know that any weighted game is complete. The question arises about the completeness of set-weighted games.

Let us demonstrate that not all set-weighted games are complete. Let $N = \{1, 2, 3, 4\}$ and the weighted form of the game (N, v) have the form

$$[\{a, b, c\}; \{a\}, \{c\}, \{a, b\}, \{b, c\}].$$

If $S = \{1\}$, then $v(S \cup \{4\}) > v(S \cup \{3\})$. Now let $S = \{2\}$, then $v(S \cup \{4\}) < v(S \cup \{3\})$. Hence, (N, v) is not complete. The next proposition finds the sufficient conditions for a set-weighted game to be complete.

Proposition 1. *Let $[U; s_1, s_2, ..., s_n]$ be the weighted form of the set-weighted game (N, v). If $\forall i, j \in N : s_i \subseteq s_j$ or $s_j \subseteq s_i$, then (N, v) is a complete game.*

Proof. Let i, j be two players, $i \neq j, s_j \subseteq s_i, S \subseteq N \setminus \{i, j\}$. Then

$$s_j \subseteq s_i \Rightarrow s_j \cup \bigcup_{k \in S} s_k \subseteq s_i \cup \bigcup_{k \in S} s_k \Rightarrow v(S \cup \{i\}) \geq v(S \cup \{j\}).$$

Hence, (N, v) is a complete game. □

The dimensionality of (N, v) is the least l such that there exists a weighted majority of games $(N, v_1), ..., (N, v_l)$ for which

$$W(v) = W(v_1) \cap ... \cap W(v_l)\ [36].$$

We denote the dimensionality of the game (N, v) by $\dim(v)$. Some dimensionality properties of simple games were investigated in [16, 17].

Let $v = v_1 \wedge v_2 \wedge ... \wedge v_r$ be the canonical form of a set-weighted game. Since $\forall j \in \{1, 2, ..., r\}$ $\forall A \in W^m(v_j) : |A| = 1$, the game (N, v_j) is a weighted game, and can be expressed in the form $[1; w_1, w_2, ..., w_n]$,

$$w_i = \begin{cases} 1, & \{i\} \in W^m(v_j); \\ 0, & \text{otherwise.} \end{cases} \quad \forall i \in N.$$

Hence, $\dim(v) \leq r$. This estimate is a special case of the following proposition.

Proposition 2. *Let $[U; s_1, s_2, ..., s_n]$ be the weighted form of the set-weighted game (N, v). Consider the set-weighted game (N, v_j) with the weighted form $[U_j; s_1, s_2, ..., s_n]$, $j \in \{1, 2, ..., r\}, \bigcup_{j=1}^r U_j = U$. Then*

$$\dim(v) \leq \sum_{j=1}^{r} \dim(v_j)$$

Proof. This statement is a consequence of Lemma 1. Since $\bigcup_{j=1}^r U_j = U$, then $v = v_1 \wedge v_2 \wedge ... \wedge v_r$. Therefore, $\dim(v) \leq \sum_{j=1}^r \dim(v_j)$. □

3.4 Theorem of the union and intersection of set-weighted games

Union and intersection are classical operations on simple games. Such operations are often used to transform games. The question of interest is the following. Let there be a class of simple games, and (N, v) and (N, w) be two arbitrary games from this class. Do the games $(N, v \vee w)$ and $(N, v \wedge w)$ belong to this class? The answer for weighted games is negative. For example, let $N = \{1, 2, 3, 4\}, W^m(v) = \{\{1, 2\}\}, W^m(w) = \{\{3, 4\}\}$. The weighted forms of the games v and w have the form $[2; 1, 1, 0, 0]$ and $[2; 0, 0, 1, 1]$, respectively. Then, $W^m(v \vee w) = \{\{1, 2\}, \{3, 4\}\}$. The game $(N, v \vee w)$ is not a weighted game. Now, let N still consist of four players, and $W^m(v) = \{\{1\}, \{2\}\}, W^m(w) = \{\{3\}, \{4\}\}$. The weighted forms of the games (N, v) and (N, w) have the form $[1; 1, 1, 0, 0]$ and $[1; 0, 0, 1, 1]$, respectively. Then, $W^m(v \wedge w) = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$. The game $(N, v \wedge w)$ is not a weighted game.

The following result was obtained for set-weighted games.

Theorem 2. *The set $SW(N)$ is closed under the operations of union and intersection.*

The proof is in the Appendix.

Theorem 2 declares the following. Let $(N, v) \in SW(N)$ and $(N, w) \in SW(N)$. Then $(N, v \vee w) \in SW(N)$ and $(N, v \wedge w) \in SW(N)$. Let us show how this result can be used. Suppose we need to find the value $\varphi(v)$ of the set-weighted game (N, v), such that it has the transfer property. The characteristic function v can be decomposed using Lemma 1 to simpler set-weighted games, e.g. $v = v_1 \wedge v_2$. Then,

$$\varphi(v) = \varphi(v_1 \wedge v_2) = \varphi(v_1) + \varphi(v_2) - \varphi(v_1 \vee v_2).$$

Since (N, v_1) and (N, v_2) are set-weighted games, it follows from Theorem 2 that $(N, v_1 \vee v_2)$ is also a set-weighted game. Such a transformation of $\varphi(v)$ is useful if the weighted form of the game $(N, v_1 \vee v_2)$ is simpler than $(N, v_1 \wedge v_2)$. Lemma 1 and Theorem 2 form the basis for calculating the Shapley-Shubik index in the game with singleton weights described in Section 5.2.

Example 3. This example illustrates the process of making the weighted form for the union and the intersection of set-weighted games. Let the weighted forms of the games $(N, v), (N, w)$ have the form

$$[\{a, b, c, d\}; \{a, b, c\}, \{a, d\}, \{c, d\}, \{b, c, d\}] \text{ and } [\{a, b, c, d\}; \{a, b\}, \{b, c\}, \{a, d\}, \{d, c\}],$$

respectively. Let us find the weighted forms of the games $(N, v \wedge w)$ and $(N, v \vee w)$.

To do so, we write down the canonical forms for (N, v) and (N, w) :

$$v = v_a \wedge v_b \wedge v_c \wedge v_d,\ w = w_a \wedge w_b \wedge w_c \wedge w_d,$$
$$W^m(v_a) = \{\{1\}, \{2\}\}, W^m(v_b) = \{\{1\}, \{4\}\}, W^m(v_c) = \{\{1\}, \{3\}, \{4\}\},$$
$$W^m(v_d) = \{\{2\}, \{3\}, \{4\}\}, W^m(w_a) = \{\{1\}, \{3\}\}, W^m(w_b) = \{\{1\}, \{2\}\},$$
$$W^m(w_c) = \{\{2\}, \{4\}\}, W^m(w_d) = \{\{3\}, \{4\}\}.$$

Then,

$$v \wedge w = (v_a \wedge v_b \wedge v_c \wedge v_d) \wedge (w_a \wedge w_b \wedge w_c \wedge w_d)$$
$$= u_e \wedge u_f \wedge u_g \wedge u_h \wedge u_j,$$
$$W^m(u_e) = \{\{1\}, \{4\}\}, W^m(u_f) = \{\{1\}, \{3\}\}, W^m(u_g) = \{\{1\}, \{2\}\},$$
$$W^m(u_h) = \{\{2\}, \{4\}\}, W^m(u_j) = \{\{3\}, \{4\}\}.$$

The weighted form of the game $(N, v \wedge w)$ has the form

$$[\{e, f, g, h, j\}; \{e, f, g\}, \{g, h\}, \{f, j\}, \{e, h, j\}]$$

. Let us now transform $v \vee w$ as follows:

$$v \vee w = (v_a \wedge v_b \wedge v_c \wedge v_d) \vee (w_a \wedge w_b \wedge w_c \wedge w_d)$$
$$= z_e \wedge z_f \wedge z_g,$$
$$W^m(z_e) = \{\{1\}, \{2\}\}, W^m(z_f) = \{\{1\}, \{3\}, \{4\}\}, W^m(z_g) = \{\{2\}, \{3\}, \{4\}\}.$$

The weighted form of the game $(N, v \vee w)$ has the form

$$[\{e, f, g\}; \{e, f\}, \{e, g\}, \{f, g\}, \{f, g\}].$$

In Example 3, elements of players' weights in the games (N, v) and (N, w) differ from elements of players' weights in the games $(N, v \vee w)$ and $(N, v \wedge w)$. Depending on the application, elements of weights can coincide, e.g. $a = e, b = f$.

To find the weighted form of the games $(N, v \vee w)$ and $(N, v \wedge w)$ it suffices to know the weighted forms of (N, v) and (N, w). There is no need to find $W^m(v)$ and $W^m(w)$.

4 Simple cover games

4.1 Vertex, dominating, and edge cover games

Let $H = (V, E)$ be an undirected graph, where V is the set of vertices and E is the set of edges, $E \subseteq \{\{i,j\} | i,j \in V, i \neq j\}$. A vertex cover S of an undirected graph H is a subset of V such that $\forall \{i,j\} \in E : i \in S$ or $j \in S$. Let $M(H)$ be the set of least vertex covers of the graph H. The vertex cover game of H is a simple game (N, v) if $W^m(v) = M(H)$. In the vertex cover game, a player is a vertex of the graph, and $N = V$. The definition of the vertex cover game was suggested in [19]. Dominating and edge cover games will be introduced in a similar way.

The dominating set of the graph H is the subset S of the set V, such that $\forall i \in V \setminus S \ \exists j \in S : \{i,j\} \in E$. We use $D(H)$ to denote the set consisting of all least dominating sets of the graph H. The dominating cover game of H is the simple game (N, v) if $W^m(v) = D(H)$. In the dominating cover game, a player is a vertex of the graph, and $N = V$.

The edge cover of the graph H is the set of edges C, such that each vertex of the graph is incident upon at least one edge from C, i.e. $\forall i \in V \ \exists \{i,j\} \in E : \{i,j\} \in C$. Let $R(H)$ denote the set of all least edge covers. The edge cover game of H is the simple game (N, v) if $W^m(v) = R(H)$. In an edge cover game, a player is an edge, that is, $N = E$.

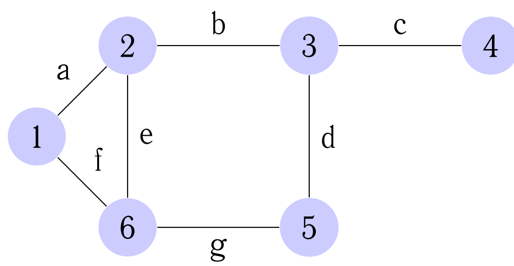

Fig. 1. Graph H with 6 vertices.

Consider the graph H in Figure 1, $N = \{1,2,3,4,5,6\}, E = \{a,b,c,d,e,f,g\}$. Then

$$M(H) = \{\{1,3,6\}, \{2,3,6\}, \{1,2,3,5\}, \{1,2,4,5\}, \{2,4,5,6\}\},$$

$$D(H) = \{\{1,3\}, \{2,3\}, \{3,6\}, \{4,6\}, \{1,4,5\}, \{2,4,5\}\},$$

$$R(H) = \{\{a,c,g\}, \{a,c,d,e\}, \{a,c,d,f\}, \{b,c,d,f\}, \{b,c,f,g\}, \{c,f,e,d\}, \{c,f,e,g\}\}.$$

In vertex and dominating cover games, players form coalitions to cover the entire set of edges. In edge cover games, players cover vertices. The calculation of the power index in these games shows the influence of a player (vertex or edge) in the network taking all least covers into account. If the game deals with a transport or communication network, then finding $W^m(v)$ is a computationally complex problem. The next section demonstrates that vertex, dominating, and edge cover games are set-weighted games. The existence of the weighted form makes the calculation of players' indexes easier, and eliminates the need to find $W^m(v)$.

4.2 The weighted forms of simple cover games

To find the set-weighted form of the cover games from the previous section, we introduce some notation. Let $E_i, i \in V$ denote the set of adjacent edges of the vertex i in the graph $H, E_i = \{\{i,j\}|\{i,j\} \in E\}$ and $V_i = \{i\} \cup \{j|\{i,j\} \in E\}$ be the set of adjacent vertices of the vertex i, including i. Let us enumerate edges in the set $E, E = \{e_1, e_2, ..., e_{|E|}\}$.

Consider the following theorem.

Theorem [19]. *A simple game (N, v) is a vertex cover game of H iff there exist simple games $(N, v_l), l \in \{1, 2, ..., r\}, W^m(v_l) = \{\{i_l\}, \{k_l\}\}$ for which the equality*

$$v = v_1 \wedge v_2 \wedge ... \wedge v_r$$

holds, and $H = (N, E), E = \{\{i_l, k_l\}|1 \leq l \leq r\}$.

According to this theorem, the characteristic function of a vertex cover game can be expressed as the intersection of simple games $v_j, j = 1, 2, ..., r$, and any minimal winning coalition of the game (N, v_j) consists of one player. It follows from Theorem 1 that the vertex cover game is a set-weighted game. Its weighted form is found in Proposition 3.

Proposition 3. *The vertex cover game is a set-weighted game, and its weighted form is:*

$$[U; s_1, s_2, ..., s_n] = [E; E_1, E_2, ..., E_n].$$

The weight of player i in a vertex cover game is the set of edges incident upon the vertex i in the graph H.

The set U consists of all the edges of a graph. Finding a minimum vertex cover is an NP-complete problem, therefore finding of the set of least vertex covers is associated with computation complexities. Hence, the easiest way is to define the vertex cover game using the weighted form. The propositions below find the weighted forms of the dominating and edge cover games.

Proposition 4. *The dominating cover game is a set-weighted game, and its weighted form is:*

$$[U; s_1, s_2, ..., s_n] = [V; V_1, V_2, ..., V_n].$$

Proposition 5. *The edge cover game is a set-weighted game, and its weighted form is:*

$$[U; s_1, s_2, ..., s_{|E|}] = [N; e_1, e_2, ..., e_{|E|}]$$

The proofs of Propositions 3-5 are given in the Appendix. A player's weight in these cover games is the set of adjacent vertices or edges, depending on the type of cover. If the graph H is given by an incidence matrix, then finding players' weights is easier than enumerating all minimal winning coalitions.

The weighted forms of the games of vertex, dominating, and edge covers for the graph H (Fig. 1) have the form:

$$[U; s_1, s_2, ..., s_n] = [\{a, b, c, d, e, f, g\}; \{a, f\}, \{a, b, e\}, \{b, c, d\}, \{c\}, \{d, g\}, \{e, f, g\}],$$

$$[U; s_1, s_2, ..., s_n] = [\{1,2,3,4,5,6\}; \{1,2,6\}, \{1,2,3,6\}, \{2,3,4,5\}, \{3,4\}, \{3,5,6\}, \{1,2,5,6\}],$$

$$[U; s_1, s_2, ..., s_{|E|}] = [\{1,2,3,4,5,6\}; \{1,2\}, \{2,3\}, \{3,4\}, \{3,5\}, \{2,6\}, \{1,6\}, \{5,6\}],$$

respectively.

Let the game (N, v) be a cover game on the graph H. If the dimensionality of the graph H is substantial, it would be convenient to decompose it into simpler subgraphs $H_1, H_2, ..., H_r$ and introduce the corresponding cover games $(N, v_j), j = 1, 2, ..., r$. Then, the study of the game (N, v) is reduced to analyzing the games composed of the unions and intersections of the new cover games. It follows from Theorem 2 that the new games are set-weighted.

5 Power indexes for cover problems

5.1 Shapley-Shubik and Banzhaf-Coleman indexes

Before applying one or another cooperative game value to solve an applied problem, one needs to demonstrate its efficiency. This is usually done by building an axiomatic system. Certain properties are identified, and it is demonstrated that only one value of the cooperative game has these properties. If the properties have a physical meaning, the value of the game can be used to solve the problem. The axiomatic system of the Shapley-Shubik index on a set of superadditive functions was suggested in [14, 24]. However, not all characteristic functions of set-weighted games are superadditive. Take, for example, the dominating cover game (N, v) on the graph H shown in Fig. 1. The game (N, v) is not superadditive since $v(\{1,3\}) + v(\{4,6\}) > v(\{1,3,4,6\})$. Yet, the next statement demonstrates that the axiomatics of the Shapley-Shubik index apply to set-weighted functions as well as to superadditive ones.

Proposition 6. *Let $\phi : SW(N) \to R^n$. The only ϕ that satisfies efficiency, null player, symmetry, and transfer is the Shapley-Shubik index.*

The proof is in the Appendix.

The properties that uniquely define the Shapley-Shubik index on the set $SW(N)$ are of applied value for cover games. The efficiency axiom implies that the sum of indexes is a constant number. Owing to the monotonicity property of set-weighted games, the index of each player is a number from 0 to 1. If a player's weight and the set of objects covered are independent, the player is a null player. It follows from the null player axiom that his payoff is 0. There may be other reasons for a player to be a null player, e.g. if any coalition $K, K \subseteq N$ united with player $i, i \in N, i \notin K$ covers the same set of objects as the coalition K. Hence, no resources will be allocated to player i in the cover game. Symmetric players get identical payoffs, as follows from the symmetry axiom. The transfer axiom suggests that if a change happens in the game, the players' payoffs can change only because of the new change, but not because of any other factors. This interpretation of the axiom was suggested in [13]. A new object to be covered can appear in the game, or, vice versa, an object can disappear. In this case, changes in the players' powers will be associated only with the emergence or vanishing of objects.

Let us clarify the physical meaning of the Banzhaf-Coleman index in the set-weighted game $[U; s_1, s_2, ..., s_n]$. Let K be a cover of the set U, that is, $K \subseteq N, U \subseteq \cup_{j \in K} s_j$. To cover U, a certain amount of resources is allocated to each player in K. Suppose all resources allocated to $i, i \in K$ have been spent or lost their quality over time. If $K \setminus \{i\}$ is not a cover, then more resources need to be allocated to player i. The power of players needs to be determined, taking into account the possible depreciation of the resources. This is done by finding the number of all covers K that stop being a cover without i. Averaging the resultant values, we get the Banzhaf-Coleman index for the set-weighted game. A high-quality resource should preferably be allocated to a player with a high Banzhaf-Coleman index.

5.2 Set-weighted games with singleton weights

Let $[U; s_1, s_2, ..., s_n]$ be the weighted form of the game $(N, v), U = \{a_1, a_2, ..., a_m\}$ and the weight of player i be an element of the set U, that is, $s_i \in U \; \forall i \in N$. We express the set N as a union of pairwise non-intersecting sets, $N = K_1 \cup K_2 \cup ... \cup K_m$, in which $\forall j = 1, 2, ..., m \; \forall i \in K_j : s_i = \{a_j\}$. It is safe to say that U is the set of objects to be covered, and that each player can cover only one object from U. Let $|K_j| = k_j \; \forall j = 1, 2, ..., m$. Since $K_j \cap K_l = \emptyset \; \forall j, l, j \neq l$, then $\sum_{j=1}^{m} k_j = n$. We shall call this set-weighted game a game with singleton weights.

Example 4. Let $N = \{1, 2, 3, 4, 5, 6\}, U = \{a_1, a_2, a_3\}$ and the weighted form of the singleton game (N, v) have the form

$$[\{a_1, a_2, a_3\}; \{a_1\}, \{a_2\}, \{a_2\}, \{a_3\}, \{a_3\}, \{a_3\}].$$

For players in this example to get a non-zero payoff, the coalition must have player 1. It is enough for player 1 to form a coalition with two players whose combined weight is $\{a_2, a_3\}$. We can find the players' power by calculating the Shapley-Shubik and Banzhaf-Coleman indexes.

Proposition 7. *Let $[U; s_1, s_2, ..., s_n], U = \{a_1, a_2, ..., a_m\}$ be the weighted form of the singleton game (N, v). Then, the Shapley-Shubik index for player i has the following form:*

$$\phi_i(v) = \int_0^1 x^{k_j - 1} \prod_{\substack{l=1 \\ l \neq j}}^{m} \left(1 - x^{k_l}\right) dx, i \in K_j, k_j = |K_j|,$$

where K_j is the set of players with weight $\{a_j\}, j = 1, 2, ..., m$.

The proof is in the Appendix.

The Shapley-Shubik index for each player in the game from Example 4 is calculated as follows:

$$\phi_1(v) = \int_0^1 x^0(1 - x^2)(1 - x^3)dx = \frac{7}{12} \approx 0.583,$$

$$\phi_i(v) = \int_0^1 x^1(1 - x^1)(1 - x^3)dx = \frac{2}{15} \approx 0.133, i \in \{2, 3\},$$

$$\phi_i(v) = \int_0^1 x^2(1 - x^1)(1 - x^2)dx = \frac{1}{20} = 0.05, i \in \{4, 5, 6\}.$$

Proposition 8. *Let $[U; s_1, s_2, ..., s_n], U = \{a_1, a_2, ..., a_m\}$ be the weighted form of the singleton game (N, v). Then, the Banzhaf-Coleman index for player i has the following form:*

$$\delta_i(v) = \frac{1}{2^{k_j}-1} \cdot \frac{1}{\sum_{l=1}^{m} \frac{k_l}{2^{k_l}-1}}, i \in K_j, k_j = |K_j|,$$

where K_j is the set of players with weight $\{a_j\}, j = 1, 2, ..., m$.

The proof is in the Appendix.

The Banzhaf-Coleman index for each player in the game from Example 4 is calculated as follows:

$$\delta_1(v) = \frac{1}{1+\frac{2}{3}+\frac{3}{7}} = \frac{21}{44},\ \delta_i(v) = \frac{1}{3} \cdot \frac{1}{1+\frac{2}{3}+\frac{3}{7}} = \frac{7}{44}\ \forall i \in \{2,3\},$$

$$\delta_i(v) = \frac{1}{7} \cdot \frac{1}{1+\frac{2}{3}+\frac{3}{7}} = \frac{3}{44}\ \forall i \in \{4,5,6\}.$$

6 Special generating functions for set-weighted games

6.1 The cooperative generating function

Generating functions permit calculating many combinatorial values and power indexes. The generating functions for the Shapley-Shubik index and the number of swings were found in [29, 5] for weighted game and have the form:

$$G_i(x) = \prod_{j \neq i}(1 + x^{w_j}), \quad G_i(x,z) = \prod_{j \neq i}(1 + zx^{w_j}),$$

respectively.

Knowing players' weights, we can write down the generating function, calculate its coefficients, and find the corresponding power indexes. In a set-weighted game, however, players' weights are sets, not real numbers. Further, the definition of the generating function is adapted to suit set-weighted games.

Definition 3. *Let U be a finite set, and $f : 2^U \to R$. The cooperative generating function for the number array $\{f(S)\}_{S \subseteq U}$ is defined as follows:*

$$G(T) = \sum_{\substack{S \subseteq U \\ S \neq \emptyset}} f(S) v_S(T), T \subseteq U,$$

where (U, v_S) is a unanimity game.

Classical generating functions with an infinite number of members generate a formal Taylor series. The cooperative generating function is a decomposition of a cooperative game with respect to the basis. In cooperative game theory, a game is usually given in the form of unanimity games in order to build an axiomatic system. We do not, however, consider $G(T)$ as a game – what matters for us is that $G(T)$ is a special generating function for an unordered array of numbers.

6.2 The cooperative generating function for the number of swings and the Shapley-Shubik index

In this section, we find the cooperative generating functions for some power indexes in set-weighted games.

Let $\eta_i(v)$ be the number of swings for player i in the set-weighted game (N, v) with the weighted form $[U; s_1, s_2, ..., s_n]$. Then, $\eta_i(v)$ can be given as follows:

$$\eta_i(v) = \sum_{S \in L_i} b_i(S),$$

where $L_i = \{L \cup (U \setminus s_i) | L \subset s_i, L \neq s_i\}$ and $b_i(S)$ is the number of coalitions that do not include i with weight S. Any element S of the set L_i does not contain the set U, but $S \cup s_i$ contains U. For example, let $U = \{a, b, c, d\}, s_1 = \{c, d\}$. Then $L_1 = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}\}$.

Proposition 9. *Let $[U; s_1, s_2, ..., s_n]$ be the weighted form of the set-weighted game (N, v). The cooperative generating function for the number array $\{b_i(S)\}_{S \subseteq U}, i \in N$ has the form:*

$$G_i(T) = \prod_{j \neq i} (1 + v_{s_j}(T)).$$

The proof is in the Appendix.

The cooperative generating function can be simplified using the property of unanimity games, for example, $v_S \cdot v_R = v_{S \cup R} \; \forall S, R \subseteq N$.

Example 5. Let $[\{a, b, c, d\}; \{a, b\}, \{a, c\}, \{b, c, d\}, \{d\}]$ be the weighted form of the game $(N, v), N = \{1, 2, 3, 4\}$. To calculate the number of swings for the first player, we write the cooperative generating function:

$$G_1(T) = (1 + v_{\{a,c\}}(T)) \cdot (1 + v_{\{b,c,d\}}(T)) \cdot (1 + v_{\{d\}}(T))$$
$$= 1 + v_{\{d\}}(T) + v_{\{a,c,d\}}(T) + v_{\{a,c\}}(T) + 2v_{\{b,c,d\}}(T) + 2v_{\{a,b,c,d\}}(T).$$

In the equality above, the coefficient in front of $v_S(T)$ is the number $b_1(S)$. For example, the set-weighted game (N, v) has one coalition with weight $\{a, c\}$, excluding player 1. This is the coalition $\{2\}$. The number of coalitions without player 1 with the weight $\{b, c, d\}$ is two. These are the coalitions $\{3\}, \{3, 4\}$.

To find $\eta_1(v)$, the numbers $b_1(S), S \in L_1$ should be summed. The set L_1 has the form $L_1 = \{\{c, d\}, \{a, c, d\}, \{b, c, d\}\}$. Taking into account the coefficients of the cooperative generating function $G_1(T)$, we get $b_1(\{c, d\}) = 0, b_1(\{a, c, d\}) = 1, b_1(\{b, c, d\}) = 2$. In this case, $\eta_1(v) = 3$. For the second player, we have the following:

$$G_2 = (1 + v_{\{a,b\}})(1 + v_{\{b,c,d\}})(1 + v_{\{d\}})$$

$$= 1 + v_{\{d\}} + v_{\{a,b\}} + v_{\{a,b,d\}} + 2v_{\{b,c,d\}} + 2v_{\{a,b,c,d\}},$$

$$\eta_2(v) = \sum_{S \in L_2} b_2(S) = b_2(\{b, d\}) + b_2(\{a, b, d\}) + b_2(\{b, c, d\}) = 3.$$

For the third player:

$$G_3 = (1 + v_{\{a,b\}})(1 + v_{\{a,c\}})(1 + v_{\{d\}})$$

$$= 1 + v_{\{d\}} + v_{\{a,b\}} + v_{\{a,c\}} + v_{\{a,b,c\}} + v_{\{a,b,d\}} + v_{\{a,c,d\}} + v_{\{a,b,c,d\}},$$

$$\eta_3(v) = \sum_{S \in L_3} b_3(S) = b_3(\{a\}) + b_3(\{a,b\}) + b_3(\{a,c\}) + b_3(\{a,d\})$$
$$+ b_3(\{a,b,c\}) + b_3(\{a,b,d\}) + b_3(\{a,c,d\}) = 5.$$

For the fourth player:

$$G_4 = (1 + v_{\{a,b\}})(1 + v_{\{a,c\}})(1 + v_{\{b,c,d\}})$$

$$= 1 + v_{\{a,c\}} + v_{\{a,b\}} + v_{\{a,b,c\}} + v_{\{b,c,d\}} + 3v_{\{a,b,c,d\}},$$

$$\eta_4(v) = \sum_{S \in L_4} b_4(S) = b_4(\{a,b,c\}) = 1.$$

Having normalized the number of swings, we get the Banzhaf-Coleman index, $\left(\frac{3}{12}, \frac{3}{12}, \frac{5}{12}, \frac{1}{12}\right)$.

Let us now find the cooperative generating function of a set-weighted game for the Shapley-Shubik index.

Let $[U; s_1, s_2, ..., s_n]$ be the weighted form of the set-weighted game (N, v). Then, the Shapley-Shubik index can be expressed in the form:

$$\phi_i(v) = \sum_{k=0}^{n-1} \frac{k! \cdot (n-k-1)!}{n!} \cdot \left(\sum_{S \in L_i} A_i(k, S) \right),$$

where $L_i = \{L \cup (U \setminus s_i) | L \subset s_i, L \neq s_i\}$ and $A_i(k, S)$ is the number of coalitions $K, K \subseteq N \setminus \{i\}, k = |K|$ with weight $S, S \subseteq U$.

Proposition 10. *The cooperative generating function for the number array* $\{A_i(k, S)\}_{k \geq 0, S \subseteq U}$ *has the form:*

$$G_i(x, T) = \prod_{j \neq i} \left(1 + x \cdot v_{s_j}(T)\right),$$

where x is a real number.

The proof is in the Appendix.

To find a power index in a set-weighted game in the general case, all subsets of U have to be searched through. Hence, the calculation of the indexes is not a polynomial problem. Cooperative generating functions permit the indexes to be calculated analytically. This can be done by simplifying $G_i(T)$ and finding its coefficients.

7 Application to transport networks

The distribution of surveillance cameras at the intersections of a transport network is a problem of high relevance. Cameras need to be deployed at intersections in such a way that all roads are covered. The solution of the cover problem for a transport network requires a great amount of computing operations.

The following approach is used to distribute cameras across the network. Suppose K is the set of intersections that covers all roads. This means that if cameras are deployed at intersections from K, then all roads of the transport network will be covered. Let us deploy cameras at intersections from the set K. Now suppose surveillance cameras at intersection $i, i \in K$ stops working. If $K \setminus \{i\}$ is not a cover of the network, then intersection i is pivotal for the cover K, and the cameras at intersection i should be of high quality. Let us now find the number of all covers in which intersection i is pivotal. Normalizing the resultant values we find the Banzhaf-Coleman index for the vertex cover game on this graph. In this case, cameras can be distributed among the intersections of the transport network proportionately to the values of the Banzhaf-Coleman index. This approach makes allowances for the possibility that surveillance cameras can stop working. The more cameras deployed at intersections with a high Banzhaf-Coleman index, the lower the possibility that the network is left uncovered if some cameras stop working.

Consider the graph H, representing the main roads in Kukkovka district, Petrozavodsk, Russia (Fig. 2). A vertex in this graph is an intersection, an edge is a road. The graph is taken from [19].

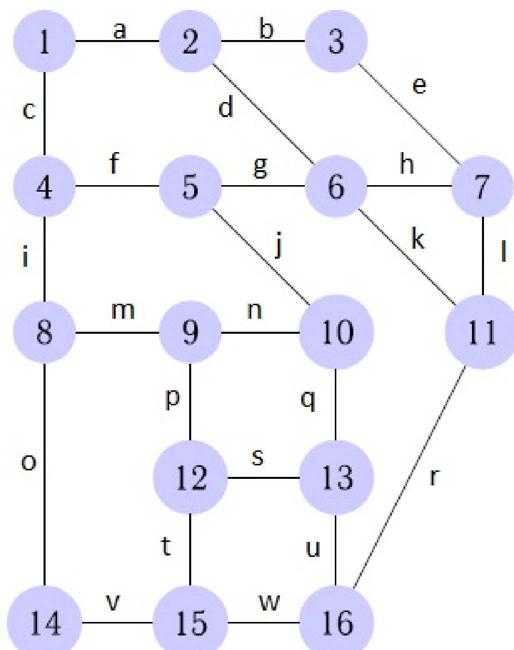

Fig. 2. Graph H. Transport network of the main roads of the Kukovka district, Petrozavodsk, Russia.

Let (N, v) be a vertex cover game on the graph H, where $N = \{1, 2, ..., 16\}$. According to Proposition 3, a vertex cover game is a set-weighed game. The weighted form of the game (N, v) has the form

$$[U; s_1, s_2, ..., s_{16}],$$

$$U = \{a, b, ..., w\}, s_1 = \{a, c\}, s_2 = \{a, b, d\}, ..., s_{16} = \{r, u, w\}.$$

The number of swings and the Banzhaf-Coleman index are given in Table 1. Table 1 also provides the values of the Shapley-Shubik index for the game (N, v) from [19].

Table 1: Power indices for the vertex cover game (N, v).

i	1	2	3	4	5	6	7	8
$\beta_i(v)$	442	680	424	678	576	754	674	668
$\delta_i(v)$	.0457	.0704	.0439	.0702	.0596	.0780	.0697	.0691
$\phi_i(v)$	.0453	.0674	.0447	.0675	.0617	.0822	.0672	.0671
i	9	10	11	12	13	14	15	16
$\beta_i(v)$	610	626	624	588	598	452	670	594
$\delta_i(v)$	.0631	.0648	.0646	.0608	.0619	.0468	.0693	.0615
$\phi_i(v)$	.0640	.0648	.0645	.0634	.0637	.0455	.0672	.0637

Arranging the players in the order of decreasing Banzhaf-Coleman and Shapley-Shubik indexes we get the vectors α_1 and α_2, respectively,

$$\alpha_1 = (6, 2, 4, 7, 15, 8, 10, 11, 9, 13, 16, 12, 5, 14, 1, 3),$$

$$\alpha_2 = (6, 4, 2, 7, 15, 8, 10, 11, 9, 13, 16, 12, 5, 14, 1, 3).$$

The numerical values of the Banzhaf-Coleman and Shapley-Shubik indexes differ, but players' rankings by index value decline almost fully coincide, apart from players 2 and 4.

The papers [18, 23] found the conditions under which players' rankings with respect to the Shapley-Shubik and Banzhaf-Coleman indexes coincide in weighted games with real weights. One of the sufficient properties is swap-robustness. A simple game (N, v) is swap-robust if any swap of two players between any two winning coalitions S and T is such that at least one of S' and T' is a winning coalition. Any weighted game is known to be swap-robust. For set-weighted games, however, this property is not necessarily valid. Take, for example, the game $[\{a, b, c, d\}; \{a, b\}, \{c, d\}, \{a, c\}, \{b, d\}]$. The coalitions $\{1, 2\}, \{3, 4\}$ are winning coalitions. If players 1 and 3 swap places, then the coalitions $\{3, 2\}, \{1, 4\}$ are the losing ones.

If the difference between α_1 and α_2 is ignored, then the suggested ranking possesses the properties of the Shapley-Shubik and Banzhaf-Coleman indexes. Suppose we only have 16 cameras to allocate, and they vary in quality and service life. A surveillance camera is supposed to be allocated to each intersection. In that case, the ranking α_1 can be used. The best camera should be deployed to crossroads 6, the second best to crossroads 2, etc.

8 Conclusions and future work

This paper examines a new subclass of simple games. It investigates the properties of the new subclass and applies them to cover games.

The main distinction between weighted and set-weighted games is the weights of players. In a weighted game, a sufficiently large number of players with small weights can make up for for one player with a large weight. For example, one player with a weight of 100 is equal to a coalition of 100 players each with a weight of 1. In set-weighted games, such parity is not always the case. Any number of players with weight $\{a\}$ cannot compensate for

Table 2: Statements for weighted and set-weighted games.

Characteristic	Weighted games	Set-weighted games (novel statements)
Weighted form	$[q; w_1, w_2, ..., w_n]$, q and player weights are real numbers	$[U; s_1 s_2, ..., s_n]$, U and player weights are sets
Necessary and sufficient conditions	A simple game (N, v) is a weighted game iff (N, v) is trade robust	A simple game (N, v) is a set-weighted game iff (N, v) has a canonical form
Completeness	Any weighted game is complete	Not every set-weighted game is complete
Swap robust	Any weighed game is swap robust	Not every set-weighted game is swap robust
Union and intersection	The union and intersection of weighted games is not always a weighted game.	The union and intersection of set-weighted games is a set-weighted game
Cover games	Not every cover game is a weighted game	Any cover game is a set-weighted game
(Cooperative) generating function for Shapley-Schubik index	$G_i(x, z) = \prod_{j \neq i}(1 + zx^{w_j})$	$G_i(x, T) = \prod_{j \neq i}(1 + xv_{s_j}(T))$
(Cooperative) generating function for the number of swings	$G_i(x) = \prod_{j \neq i}(1 + x^{w_j})$	$G_i(T) = \prod_{j \neq i}(1 + v_{s_j}(T))$

a player with weight $\{b\}$. This feature of set-weighted games highlights their application to cover problems. If the task is to cover the set $\{a, b\}$, but agents are only able to cover the object a, then their payoff is zero.

The results regarding set-weighted games are shown in Table 2. Many of the results build upon the decomposition lemma. What matters is that the decomposition lemma is not based on the set of minimal winning coalitions. If a cover game is constructed for a communication or a transport network, then $|W^m(v)|$ is large. There is no need to find the sets of minimal winning coalitions. It suffices to write down the weighted form and apply the decomposition lemma and Theorem 2 to simplify and analyze the game.

The class of weighted games with real weights is a special domain in cooperative game theory. Researchers have studied many properties of this class. Considering the application and results of the study, the set-weighted game class also has potential for becoming a separate research area.

For the future, the following questions are of interest. What conditions must be fulfilled for a simple game to be a weighted game and a set-weighted game at the same time? The answer to this question may be obvious – the trade-robustness property must be fulfilled and the game must have a canonical form – however, shared simplified conditions have not been found so far.

Theorem 2 deals with the closure of the set-weighted game class. Is the set of cover games closed? If so, what does the graph of a new game look like?

It would also be interesting to study the non-cooperative setup of cover games. A player's strategy is to choose a weight from the set of permissible weights. After players have chosen weights, a set-weighted game is formed. Each player gets a payoff from the set-weighted game (e.g. the Shapley value). Does an equilibrium exist in such a setup? This is an important question to be answered when analyzing network stability.

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Appendix

Proof of Lemma 1. Consider 2 possible cases.

1. Let $K \in W(v)$. Then $v(K) = 1$, $U \subseteq \underset{i\in K}{\cup} s_i$. Since $U = U_1 \cup U_2 \cup ... \cup U_r$, then $U_j \subseteq U \subseteq \underset{i\in K}{\cup} s_i$ $\forall j \in \{1, 2, ..., r\}$. Hence, $v_j(K) = 1$ $\forall j \in \{1, 2, ..., r\}$ and $(v_1 \wedge v_2 \wedge ... \wedge v_r)(K) = 1$. We have $v(K) = (v_1 \wedge v_2 \wedge ... \wedge v_r)(K) = 1$.

2. Let $K \in 2^N$ and $K \notin W(v)$. Then $v(K) = 0$, $U \not\subseteq \underset{i\in K}{\cup} s_i$. Since $U = U_1 \cup U_2 \cup ... \cup U_r$, then $\exists j \in \{1, 2, ..., r\} : U_j \not\subseteq \underset{i\in K}{\cup} s_i$. Hence, $v_j(K) = 0$, $(v_1 \wedge v_2 \wedge ... \wedge v_r)(K) = 0$. We have $v(K) = (v_1 \wedge v_2 \wedge ... \wedge v_r)(K) = 0$.

In each of the cases, $v(K) = (v_1 \wedge v_2 \wedge ... \wedge v_r)(K)$ holds, which proves the lemma. □

Proof of Theorem 1. Let $(N, v) \in SW(N)$. We show that there exists a set of simple games described in the condition of the theorem. Since $(N, v) \in SW(N)$, then there is a weighted form $[U; s_1, s_2, ..., s_n]$. Let $U = \{a_1, a_2, ..., a_r\}, U_j = \{a_j\}$ $\forall j \in \{1, 2, ..., r\}$. Consider the set-weighted game (N, v_j) with the weighted form $[U_j; s_1, s_2, ..., s_n]$. Since $U_j = \{a_j\}$ $\forall j \in \{1, 2, ..., r\}$, then $W^m(v_j) = \{\{i\} | \{a_j\} \subseteq s_i\}$. So $\forall j \in \{1, 2, ..., r\}$ $\forall A \in W^m(v_j) : |A| = 1$. Since $U = U_1 \cup U_2 \cup ... \cup U_r$, then by Lemma 1, the equality $v = v_1 \wedge v_2 \wedge ... \wedge v_r$ is true.

Let $v = v_1 \wedge v_2 \wedge ... \wedge v_r$ and $\forall j \in \{1, 2, ..., r\}$ $\forall A \in W^m(v_j) : |A| = 1$. To show that $(N, v) \in SW(N)$ we find the set U and the weights of the players $s_i, i \in N$. We compare the game (N, v_j) set $U_j = \{a_j\}, j \in \{1, 2, ..., r\}$. Let $U = \{a_1, a_2, ..., a_j\}, s_i = \{a_j | j \in \{1, 2, ..., r\}, \{i\} \in W^m(v_j)\}$. The game (N, v_j) is a set-weighted game with the weighted form $[U_j; s_1, s_2, ..., s_r]$. Consider the set-weighted game (N, v') with the weighted form $[U; s_1, s_2, ...s_n]$. Since $U = U_1 \cup U_2 \cup ... \cup U_r$, then Lemma 1 holds $v' = v_1 \wedge v_2 \wedge ... \wedge v_r$. But $v = v_1 \wedge v_2 \wedge ... \wedge v_r$, means $v' = v$. Hence, $(N, v) \in SW(N)$. □

Proof of Theorem 2. Consider two arbitrary games (N, v) and (N, w) from $SW(N)$. Let $v = v_1 \wedge v_2 \wedge ... \wedge v_r$ and $w = w_1 \wedge w_2 \wedge ... \wedge w_l$ are the canonical forms of the games (N, v) and (N, w), respectively. We show that $(N, v \wedge w)$ this is a set-weighted game. The following equality is true:

$$v \wedge w = v_1 \wedge v_2 \wedge ... \wedge v_r \wedge w_1 \wedge w_2 \wedge ... \wedge w_l.$$

The characteristic function $v \wedge w$ is represented as the intersection of simple games and $\forall A \in W^m(v_j) \cup W^m(w_g) : |A| = 1, j \in \{1, 2, ..., r\}, g \in \{1, 2, ..., l\}$. The conditions of Theorem 1 are true, hence $(N, v \wedge w)$ is a set-weighted game.

We show that $(N, v \vee w)$ is a set-weighted game. The following sequence of equalities is true,

$$v \vee w = (v_1 \wedge v_2 \wedge ... \wedge v_r) \vee (w_1 \wedge w_2 \wedge ... \wedge w_r) = \bigwedge_{j=1}^{r} \bigwedge_{g=1}^{l} (v_j \vee w_g)$$

where

$$W^m(v_j \vee w_g) = W^m(v_j) \cup W^m(w_g) \; \forall j \in \{1, 2, ..., r\}, \; \forall g \in \{1, 2, ..., l\}.$$

The characteristic function $v \vee w$ is represented as an intersection of simple games, and any minimal winning coalition of the game $(N, v_j \vee w_g), \forall j \in \{1, 2, ..., r\}, \; \forall g \in \{1, 2, ..., l\}$ consists of a single element. Hence, the conditions of Theorem 1 are true and the game $(N, v \vee w)$ is a set-weighted game.

□

Proof of Proposition 3. Let (N, v) be a vertex cover game of $H = (N, E)$. There are simple games $(N, v_j), j \in \{1, 2, ..., r\}$ for which $v = v_1 \wedge v_2 \wedge ... \wedge v_r$ is true, $W^m(v_j) = \{\{a_j\}, \{b_j\}\}, E = \{\{a_j, b_j\} | 1 \leq j \leq r\}$. From Theorem 1 it follows that the vertex cover game is a multiple-weight game and $U = \{a_1, a_2, ..., a_r\}, s_i = \{a_j | j \in \{1, 2, ..., r\}, \{i\} \in W^m(v_j)\}$. Players unite in a coalition to cover the set E then $U = E$. Since $W^m(v_j) = \{\{a_j\}, \{b_j\}\}$ and $\{a_j, b_j\}$ is an edge of the graph H, then $s_i = E_i \; \forall i \in N$. □

Proof of Proposition 4. Let (N, v) be a dominanting cover game of $H = (N, E)$. Then v can be represented as $v = v_1 \wedge v_2 \wedge ... \wedge v_n$, where $W^m(v_j) = \{\{j\}\} \cup \{\{i\} | i \in N, \{i, j\} \in E\}, j \in N, n = |N|$. Since the conditions of Theorem 1 are satisfied, it means that the dominanting cover game is a set-weight game. Players form coalitions to cover the $V set$. Hence, $U = V$. From the proof of Theorem 1, it follows that $s_i = \{a_j | j \in N, \{i\} \in W^m(v_j)\}$, means $s_i = V_i \; \forall i \in N$. □

Proof of Proposition 5. Let (E, v) be a edge cover game of $H = (N, E)$. Then v can be represented as $v = v_1 \wedge v_2 \wedge ... \wedge v_n$, where $W^m(v_j) = \{\{e\} | e = \{i, j\}, e \in E, i \in N\}, j \in N, n = |N|$. The set $W^m(v_j)$ consists of elements $\{e\}$, where e is an edge of the graph that is incident to j. Since the conditions of Theorem 1 are satisfied, it means that the edge cover game is a set-weight game. Players form coalitions to cover the N. Hence, $U = N$. From the proof of Theorem 1, it follows that $s_i = \{a_j | j \in N, \{i\} \in W^m(v_j)\}$, means $s_i = e_i \; \forall i \in N$. □

Proof of Proposition 6. Formally, the axioms are written as follows.

Efficiency. For all $(N, v) \in SW(N)$,

$$\sum_{i=1}^{n} \phi_i(v) = 1.$$

Null player. For any $(N, v) \in SW(N)$ if $v(\{i\} \cup S) = v(S) \; \forall S \subseteq N \setminus \{i\}, i \in N$, then

$$\phi_i(v) = 0.$$

Symmetry. For any $(N, v) \in SW(N)$ if $v(\{i\} \cup S) = v(\{j\} \cup S) \; \forall S \subseteq N \setminus \{i, j\}, \{i, j\} \subseteq N$, then

$$\phi_i(v) = \phi_j(v).$$

Transfer. For any $(N, v), (N, w) \in SW(N)$,

$$\phi(v) + \phi(w) = \phi(v \vee w) + \phi(v \wedge w).$$

Usually, the transfer property specifies that $v \vee w$ belongs to the class of games in question. In this case, such a condition can be omitted, since the union and intersection of set-weighted games is always a set-weighted game (Theorem 2).

Let $(N, v) \in SW(N)$ and the canonical form of the game (N, v) have the form $v = v_1 \wedge v_2 \wedge ... \wedge v_r$. Applying the transfer property to the canonical form, we get

$$\phi(v) = \phi(v_1 \wedge v_2 \wedge ... \wedge v_r) = \sum_{\substack{L \subseteq \{1,2,..,r\} \\ L \neq \emptyset}} (-1)^{|L|-1} \phi(\underset{l \in L}{\vee} v_l).$$

If $\{i\} \notin W^m(\underset{l \in L}{\vee} v_l)$, then the player i is a null player and $\phi_i(v) = 0$. All non-null players in the game $\underset{l \in L}{\vee} v_l$ are symmetric and according to the axiom of symmetry get the same payoff,

$$\sum_{i \in N} \phi_i(\underset{l \in L}{\vee} v_l) = \sum_{\{i\} \in W^m(\underset{l \in L}{\vee} v_l)} = \phi_i(\underset{l \in L}{\vee} v_l) \cdot |W^m(\underset{l \in L}{\vee} v_l)| = 1,$$

$$\phi_i(\underset{l \in L}{\vee} v_l) = \begin{cases} \frac{1}{|W^m(\underset{l \in L}{\vee} v_l)|}, & \{i\} \in W^m(\underset{l \in L}{\vee} v_l), \\ 0, & \text{otherwise.} \end{cases}$$

The value $\phi_i(\underset{l \in L}{\vee} v_l)$ is determined by the axioms uniquely and coincides with the Shapley-Shubik index of the game $(N, \underset{l \in L}{\vee} v_l)$. Then the value of $\phi_i(v), i \in N$ is the Shapley-Shubik index, which can be written as

$$\phi_i(v) = \sum_{L \in Y_i} \frac{(-1)^{|L|-1}}{\left| \bigcup_{l \in L} \bigcup_{j \in W^m(v_l)} \{j\} \right|},$$

where $Y_i = \{L | L \subseteq \{1, 2, ..., r\}, \exists j \in L : \{i\} \in W^m(v_j)\}$.

□

Proof of Proposition 7. Let $U = U_1 \cup U_2 \cup ... \cup U_m$, where $U_j = \{a_j\}, j = 1, 2, ..., m$. Consider the set-weighted game (N, v_j) with the weighted form $[U_j; s_1, s_2, ..., s_n], j = 1, 2, ..., m$. Then, using Lemma 1, the characteristic function v can be represented as $v = v_1 \wedge v_2 \wedge ... \wedge v_m$. By applying the transfer property several times, the Shapley-Shubik index can be represented as follows,

$$\phi(v) = \phi(v_1 \wedge v_2 \wedge ... \wedge v_m) = \sum_{L \subseteq \{1,2,...,r\}} (-1)^{|L|-1} \phi\left(\underset{l \in L}{\vee} v_l\right).$$

In the game $(N, \underset{l \in L}{\vee} v_l)$ the set of minimal winning coalitions has the form

$$W^m(\underset{l \in L}{\vee} v_l) = \{\{i\} | \{i\} \in W^m(v_l), j \in L\}.$$

If $\{i\} \notin W^m(\underset{l\in L}{\vee} v_l)$ then i is a null player and its Shapley-Shubik index is 0. The number of non-null players in the game $(N, \underset{l\in L}{\vee} v_l)$ is equal to $\sum_{l\in L} k_l$. Since non-null players are symmetric, the Shapley-Shubik index of each player can be calculated as follows,

$$\phi_i(\underset{l\in L}{\vee} v_l) = \begin{cases} \frac{1}{\sum_{l\in L} k_l}, & \{i\} \in W^m(\underset{l\in L}{\vee} v_l); \\ 0, & \text{otherwise.} \end{cases}$$

Let $i \in K_j$. This means that $s_i = \{a_j\}$. Then the following sequence of equalities is true

$$\phi_i(v) = \sum_{L\subseteq\{1,2,\ldots,r\}} (-1)^{|L|-1}\phi_i\left(\underset{l\in L}{\vee} v_l\right) = \sum_{\substack{L\subseteq\{1,2,\ldots,m\} \\ \{i\}\in W^m(\underset{l\in L}{\vee} v_l)}} \frac{(-1)^{|L|-1}}{\sum_{l\in L} k_l}$$

$$= \sum_{L\subseteq\{1,2,\ldots,r\}\setminus\{j\}} \frac{(-1)^{|L|}}{k_j + \sum_{l\in L} k_l} = \sum_{L\subseteq\{1,2,\ldots,r\}\setminus\{j\}} (-1)^{|L|} \int_0^1 x^{k_j-1+\sum_{l\in L} k_l} dx$$

$$= \int_0^1 x^{k_j-1}\left(\sum_{L\subseteq\{1,2,\ldots,r\}\setminus\{j\}} (-1)^{|L|} x^{\sum_{l\in L} k_l}\right) dx = \int_0^1 x^{k_j-1} \prod_{\substack{l=1 \\ l\neq j}}^{m} \left(1 - x^{k_l}\right) dx,$$

which was exactly what I needed to prove. □

Proof of Proposition 8. Let's fix the player i and let $s_i = \{a_j\}, j \in \{1, 2, ..., m\}$. Calculate the number of switches for the player i using the following formula,

$$\beta_i(v) = \sum_{K\subseteq N} (v(K\cup i) - v(K)) = \prod_{\substack{l=1 \\ l\neq j}}^{m} \left(2^{k_l} - 1\right).$$

Let's average the number of switches and we get the Banzaf-Coleman index,

$$\delta_i(v) = \frac{\beta_i(v)}{\sum_{l\in N} \beta_l(v)} = \frac{\prod_{\substack{l=1 \\ l\neq j}}^{m} \left(2^{k_l} - 1\right)}{\sum_{l=1}^{m} k_l \prod_{\substack{g=1 \\ g\neq l}}^{m} (2^{k_g} - 1)} = \frac{1}{2^{k_j} - 1} \cdot \frac{1}{\sum_{l=1}^{m} \frac{k_l}{2^{k_l}-1}}.$$

□

Proof of Proposition 9. Consider the product $(1 + v_{s_1}(T)) \cdot (1 + v_{s_2}(T)) \cdot ... \cdot (1 + v_{s_n}(T))$.

Let's do the multiplication and considering that $v_s(T) \cdot v_l(T) = v_{s\cup l}(T),\ \forall s, l, T \subseteq U$, we have

$$\begin{aligned} G(T) &= (1 + v_{s_1}(T)) \cdot (1 + v_{s_2}(T)) \cdot ... \cdot (1 + v_{s_n}(T)) \\ &= \sum_{L\subseteq N} \prod_{j\in L} v_{s_j}(T) = \sum_{L\subseteq N} v_{\underset{j\in L}{\cup} s_j}(T) = \sum_{S\subseteq U} b(S) v_S(T), \end{aligned}$$

where $b(S)$ is the number of coalitions with weight S. To get a cooperative generating function for the numbers $b_i(S)$, we remove the multiplier $(1 + v_{s_i})$.

□

Proof of Proposition 10. Consider the product $(1+x\cdot v_{s_1}(T))\cdot(1+x\cdot v_{s_2}(T))\cdot\ldots\cdot(1+x\cdot v_{s_n}(T))$. Let's do the multiplication and considering that $v_s(T)\cdot v_l(T)=v_{s\cup l}(T)$, $\forall s,l,T\subseteq U$, we have

$$G(T)=(1+x\cdot v_{s_1}(T))\cdot(1+x\cdot v_{s_2}(T))\cdot\ldots\cdot(1+x\cdot v_{s_n}(T))$$
$$=\sum_{K\subseteq N} x^{|K|}\cdot v_{\bigcup_{j\in K} s_j}(T)=\sum_{S\subseteq U}\sum_{k\geq 0} A(k,S)\cdot x^k\cdot v_S(T),$$

where $A(k,S)$ is the number of coalitions of k players with a weight of S. To get a cooperative generating function for the numbers $A_i(k,S)$, we remove the multiplier $(1+x\cdot v_{s_i}(T))$. □

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